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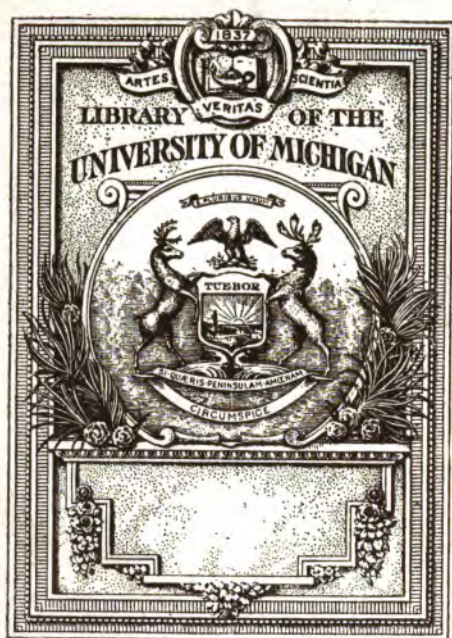
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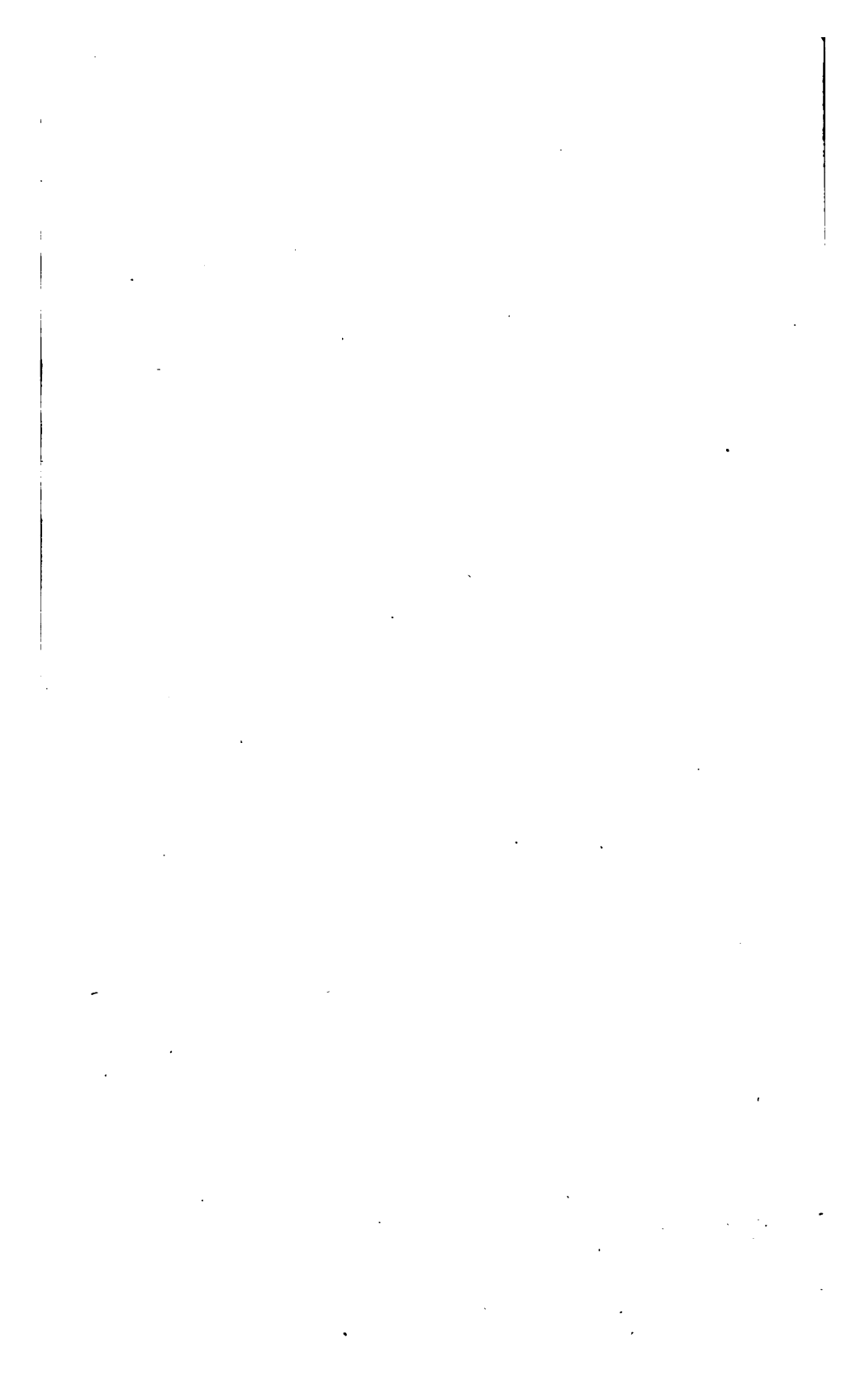


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1. The first part of the document is a list of names and addresses, which appears to be a directory or a list of contacts. The names are written in a cursive script, and the addresses are listed below them. The list includes names such as "Mr. J. H. Smith", "Mr. W. H. Jones", and "Mr. A. B. Brown".

THE  
MATHEMATICAL QUESTIONS,  
PROPOSED IN THE  
**LADIES' DIARY,**  
AND THEIR ORIGINAL ANSWERS,  
*TOGETHER WITH SOME NEW SOLUTIONS,*

FROM ITS COMMENCEMENT IN THE YEAR

1704 TO 1816.

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IN FOUR VOLUMES.

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BY THOMAS LEYBOURN,

OF THE ROYAL MILITARY COLLEGE.

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VOL. III.

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1817.



MATHEMATICAL QUESTIONS  
PROPOSED IN THE  
**LADIES' DIARY,**  
AND THEIR ANSWERS.



*Questions proposed in 1776, and answered in 1777.*

I. QUESTION 697, *by Mr. John Shadgett,*

Was proposed by mistake, it being the same with the 1st question for 1772.

II. QUESTION 698, *by Mr. William Reynolds.*

Of a cone and paraboloid, given the ratio of the diameters of their bases as 5 to 4, and the ratio of their axes or altitudes as 3 to 2; moreover the diff. of their solidities is 84823·2, and the diff. between the convex superficies and solidity of the paraboloid is 139392·3732: Required the dimensions of each?

*Answered by Mr. James Young.*

Let  $5x$  and  $3y$  = the diameter and height of the cone, and  $4x$  and  $2y$  = those of the paraboloid. Also put  $a = 84823·2$ ,  $b = 139392·3732$ , and  $c = 78539$ , &c. Then will  $25cyx$  and  $16yxx$  be the solidity of the cone and paraboloid respectively; hence per quest.  $9cyx = a$ ; therefore  $xx = a \div 9c$ . From the preceding it also appears that the two solidities are as 25 to 16, whose diff. is 9; therefore  $9 : 16 :: a : \frac{16}{9}a$  = the solidity of the paraboloid. Now, by p. 323, Hutton's Mensuration,  $((xx + 4yy)^{\frac{3}{2}} - x^3) \times 8cx \div 3yy$  = the convex surface of the paraboloid, and which is therefore  $= \frac{16}{9}a - b = 11404·4268 = d$  suppose. This equation reduces to  $768c^2x^6 + 3072c^2x^4y^2 + 4096c^2x^2y^4 = 9d^2y^2 + 48cdx^4$ . Substitute now in this equation the value of  $xx$  above found, and there will result  $110592ac^2y^5 - 2187cd^2y^4 + 9216a^2cy^3 - 144a^2cdy + 256a^3 = 0$ . From which we find  $y = 30$ . Hence  $x$  is found = 20. Then the diameter and altitude of the cone are 100 and 90, and those of the paraboloid are 80 and 60.

VOL. III.

## III. QUESTION 699, by Mr. Joseph James.

Six robbers, A, B, C, D, E, and F, comparing their several booties, found that the guineas respectively taken by A, B, and C were in arithmetrical progression, and that the guineas respectively taken by C, D, E, and F were in geometrical progression; it moreover appeared that C's money was equal to the square of A's, and E's equal to the square of C's; as also that F's money was  $= x^{2x-1} + 3$ , where  $x$  denotes A's money; to determine each person's number of guineas.

*Answered by Mr. Ralph Dees.*

Put  $x = A$ 's guineas. Then from the conditions of the question and the nature of proportion, the guineas of A, B, C, D, E, and F will be respectively  $x$ ,  $\frac{1}{2}(x + xx)$ ,  $x^2$ ,  $x^3$ ,  $x^4$ , and  $x^5$ . Then, by the question, we have this equation  $x^5 = x^{2x-1} + 3$ . From this equation, by double position, is found  $x = 2.994 = A$ 's guineas; then  $B$ 's  $= 5.979$ ,  $C$ 's  $= 8.964$ ,  $D$ 's  $= 26.838$ ,  $E$ 's  $= 80.353$ , and  $F$ 's  $= 240.576$ .

Mr. James, the proposer, remarks that he intended the sum of A's and F's guineas to be  $x^{2x-1} + 3 = x^5 + x$ . Hence is easily found  $x = 3$ . Then the numbers are 3, 6, 9, 27, 81, and 243.

Mr. Dening, and several others, supposing the 3 added to be a mistake, make  $x^5 = x^{2x-1}$ ; then, by equating the indexes,  $2x - 1 = 5$ , and  $x = 3$ . From which the numbers are found 3, 6, 9, 27, 81, and 243.

## IV. QUESTION 700, by Mr. Thomas Robinson.

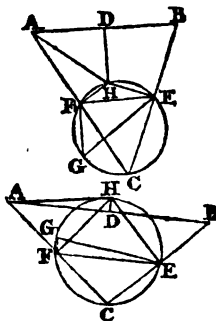
From a point, within a plane triangle, equally distant from the three angular points, there are drawn perpendiculars to the three sides, whose lengths are 1.736, 6.063, and 9.526: Required to find the sides and area of the triangle?

*Answered by Mr. John Aspland.*

Prior to the solution I shall demonstrate this

*Lemma.* ABC is any  $\Delta$ , H is the centre of its circumscribed circle, and HD, HE, HF are  $\perp$ s on the three sides. If another circle be described through H and any angular point, as C, and through the foot E of one of the  $\perp$ s on one of the sides about the  $\angle C$ ; then shall this circle pass thro' F the foot of the  $\perp$  on the other side about the  $\angle C$ , and its diameter EG shall always be  $= HA$  the radius of the circumscribed circle; also FE shall be  $\parallel$  to the 3d side AB, and FG  $=$  and  $\parallel$  to DH the third  $\perp$ ; the three chords EH, HF, FE ( $=$  the 3  $\perp$ s) com-

pleting the semicircle when the centre  $H$  is within the  $\Delta$ , as in fig. 1; but  $FG$  must be placed the contrary way to leave the semicircle  $GHE$  when  $H$  is without the  $\Delta ABC$ , as in fig. 2.—The circle will pass thro'  $F$ , because the opposite  $\angle$ s  $HFC$  and  $HEC$  are right ones; for the same reason the diameter of this circle will be  $=$  the distance  $HC$ , which is  $= HA$  or  $HB$ ; also  $FE$  is  $\parallel AB$ , because  $F$  and  $E$  are the middle points of  $AC$ ,  $BC$ . Moreover, since the right  $\angle$ s  $D$  and  $EFG$  are equal, as also  $\angle AHD = (\angle ACB$  in fig. 1  $=) \angle FGE$ , and the hypotenuse  $AH =$  the hypotenuse  $GE$ , therefore  $FG = DH$ ; they are also  $\parallel$ , because  $FE$  is  $\parallel AD$ . In fig. 2, the  $\angle AHD = \angle FHE = \angle FGE$ , and all the rest as in the 1st fig.



Hence, then, find (as in p. 95 and 230 of Sir I. Newton's Algebra) a circle  $FHECG$ , whose half is occupied by the 3 given  $\perp$ s  $EH$ ,  $HF$ , and  $FC$ , and draw  $DH =$  and  $\parallel$  to  $FG$ ; then thro'  $D$ ,  $E$ , and  $F$  draw the  $\perp$ s  $AB$ ,  $BC$ ,  $CA$ , forming the triangle required. By making the calculation as in p. 95 above-mentioned, there will be found  $GE = HA = 12.12$ , and the three sides of the  $\Delta ABC$  will be 15, 21, and 24.

*The same answered by Mr. Nathan Parnel, of Nuneaton.*

Put  $x =$  the radius  $HA$ , and the three  $\perp$ s  $DH = a$ ,  $EH = b$ , and  $FH = c$ . Then  $\sqrt{(xx - aa)} = AD = DB = FE$ ,  $\sqrt{(xx - bb)} = BE = EC$ , and  $\sqrt{(xx - cc)} = CF$ ; and since the rectangle of the diagonals  $FE$ ,  $HC$  is  $=$  the sum of the rectangles of the opposite sides, we have  $x\sqrt{(xx - aa)} = c\sqrt{(xx - bb)} + b\sqrt{(xx - cc)}$ . This equation squared, &c. gives  $x^6 - 2sx^4 + s^2x^2 = 4a^2b^2c^2$ ; and here extracting the root of each side,  $x^3 - sx = 2abc$ , where  $s$  is  $= a^2 + b^2 + c^2$ . Hence  $x = 12.125$ , and the three sides are 15, 21, and 24.

V. QUESTION 701, by Mr. George Eadon.

In a right-angled triangle, given the length of a line bisecting one of the acute angles, and terminating in the opposite leg,  $= 10$ ; to determine the triangle a maximum.

This question our correspondents generally remark, is not properly limited; which circumstance escaped our notice when we proposed it, as the author's solution had much the appearance of a proper one. The maximum of the triangle will be infinite; the hypotenuse becoming parallel to the base, the triangle degenerates into an infinite parallelo-

gramic space. Many contributors solved it on the supposition that the two legs are equal,——Or the triangle might easily be determined a given quantity instead of a maximum, but not geometrically.

VI. QUESTION 702, by Mr. Alex. Rowe.

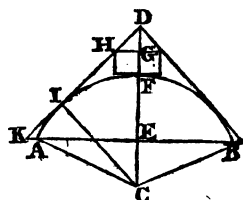
Given the radius of a sphere = 25, and the ratio of the chord to the versed sine, or altitude of a segment of it, as 3 to 1; now if the said segment be circumscribed by the least possible hollow cone, required the dimensions of the greatest cylinder that can be inscribed in the space within the cone above the segment, and having its base standing on the vertex of the said segment.

*Answered by Mr. Joseph James.*

From the common property of the circle, viz.  $(AE^2 + EF^2) \div 2EF =$  the radius  $CF$ , we easily find that the radius of a circle is  $1\frac{5}{8}$  when the versed sine is 1 and its chord 3; then, by sim. figures,  $1\frac{5}{8} : 25 :: 1 : 15\frac{5}{3} = EF :: 3 : 46\frac{2}{3} = AB$ . Hence  $CE = CF - FE = 9\frac{1}{3}$ .

By Cor. 1, quest. 685,  $ED$  is  $= \sqrt{(3CF^2 + CE^2)} - 2CE = 25.12524$ , and  $CD = 34.74062$ ; hence  $DI = \sqrt{(DC^2 - CI^2)} = 24.122$ .

Now  $DF = DC - CF = 9.74062$ , and (by Theorem 19, p. 209, Simpson's Geom.)  $FG = \frac{1}{3}DF = 3.24687$  the altitude of the cylinder; and  $DG = 2GF = 6.49374$ ; hence  $DI : IC :: DG : GH = 6.73$  the radius of the cylinder's base.

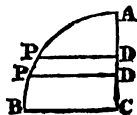


VII. QUESTION 703, by Mr. Stephen Roberts.

To find a point in the circumference of a quadrant of a circle whose radius is 10, so that a perpendicular being let fall from that point upon one of the perpendicular radii, it shall divide the area of the quadrant in extreme and mean ratio.

*Answered by Mr. Mark Elstob.*

Put  $a = 78.53982$  the area of the quadrant, and  $x =$  the less part; then  $a - x =$  the greater, and  $x : a - x :: a - x : a$ ; hence  $ax = (a - x)^2$ , and  $x = \frac{1}{2}a \times (3 - \sqrt{5}) = 29.99954$ . Then  $a - x = 48.54028$  the other part, either of which may be the semi-segment ADP. Then, having the radius AC and the area ADP, the versed sine AD will easily be found  $= 4.918$  or  $6.952$ , by p. 108 of Hutton's Mensura-



tion. Hence  $DP \perp AC$  determines the double point  $P$ .—Or  $CD = 5.082$  or  $3.046$  the sines of  $30^\circ 33'$  or  $17^\circ 45'$ , the two values of the arcs  $BP$ .

*The same answered by Mr. William Reynolds.*

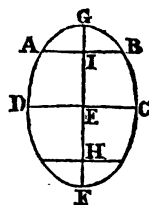
He makes  $10^3 \times .7854 = 78.54 =$  the area of the quadrant. Then, by the same equation, as in the former solution, are found 30 and 48.54 the two parts nearly: Then, by rule 5, p. 105, Hutton's Mensuration, the area  $ADF$  is  $\frac{2}{3}v\sqrt{(dv - \frac{2}{3}vv)} = 30$ , where  $v =$  the versed sine  $AD$ , and  $d = 20$  the diameter. Hence  $v' - 33\frac{1}{3}v' = -3375$ . The root of which is  $v = 4.915 = AD$ . And the semi-chord  $DP$  is  $8.6115 =$  the sine of  $59^\circ 27'$  the arc  $AP$ . Its complement  $BP$  is  $30^\circ 33'$ .

VIII. QUESTION 704, by Mr. Wm. Spicer.

A cask, of the form of the middle frustum of a spheroid, and having its head diameter, bung diameter, and length in a geometrical progression whose ratio is 2, when filled with water, and standing on-end, will empty itself thro' a circular hole in the bottom of one inch diameter in 27.570223 minutes. Required the dimensions and content of the cask?

*Answered by Mr. John Fatherly.*

Put  $z = AB$ ; then  $2z = DC$ , and  $4z = HI = 2IE$ . Then, by the nature of the ellipse,  $\sqrt{(DC^2 - AB^2)} : IH :: DE : GE$ , that is  $z\sqrt{3} : 4z :: z : 4z \div \sqrt{3} = GE$  the fixed semi-axe. Now, this is a particular example of Case 1, p. 15, Hutton's Miscellanea Mathematica, in which the values of the quantities are thus:  $n = .7854$ ,  $m = 386$ ,  $p = 4n$ ,  $r = z$ ,  $q = 2z$ , and  $x = 4z$ ; which being substituted, give for the time of evacuation  $(52zx \div 5)\sqrt{(z \div m)} = 27.570223 \times 60$  seconds  $= t$  suppose; hence  $z = \sqrt{(25mtt \div 52 \times 52)} = 25 = AB$  the head diameter; then  $DC = 50$  the bung diameter, and  $IH = 100$  the length. The content is  $522\frac{1}{2}$  ale gallons.



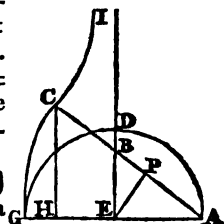
IX. QUESTION 705, by Mr. David Cunningham.

Suppose  $EB$  perpendicular to  $AG$  from the given point  $E$  in the given line  $AG$ , and let an infinite number of right-lines  $ABC$  be drawn, so that  $AB \times BC = AE^2$ ; it is required to determine the nature and area of the curve  $GC$  described by the point  $c$ .

*Answered by Mr. John Lynn.*

Put  $GE = EA = a$ , the abscissa  $GH = x$ , and ordinate  $HC = y$ .

Then  $EH = a - x$ ,  $AH = 2a - x$ , and  $AC = \sqrt{((2a - x)^2 + y^2)}$ ; also, by sim.  $\Delta s$ ,  $AH : AC :: AE : AB = (a \div (2a - x)) \sqrt{((2a - x)^2 + y^2)} :: EH : BC = ((a - x) \div (2a - x)) \sqrt{((2a - x)^2 + y^2)}$ . But, by the question,  $AE^2 = AB \times BC$ , or  $a^2 = a(a - x) ((2a - x)^2 + y^2) \div (2a - x)^2$ . Hence  $y = (2a - x) \sqrt{(x \div (a - x))}$  the equation defining the nature of the curve.



The fluxion of the area  $GHC$  is  $y\dot{x} = (2a - x) \times \dot{x} \times \sqrt{(x \div (a - x))}$ . And the fluent or area itself is  $\frac{1}{2} \times ((2x - 5a) \sqrt{(ax - xx)} + 10s)$ , where  $s$  is the circular sector, whose radius is  $a$  ( $GE$ ) and sine of its arc  $\sqrt{ax}$ . And when  $H$  coincides with  $E$ , or  $x = a$ , the whole area  $GEI$  (of an infinite length) is  $\frac{2}{3}$  of the quadrant  $GDE$ , whose radius is  $a$  or  $GE$ .

This curve, in its figure, much resembles the conchoid of Nicomedes. Its asymptote is  $IE$  infinitely extended both ways. And, by making the second fluxion of  $y$  or of  $(2a - x) \sqrt{(x \div (a - x))} = 0$  to nothing, we find  $x = \frac{2}{3}a$ , or  $GH = \frac{2}{3}GE$  when  $c$  is the point of contrary flexure.

The curve is readily constructed thus : On any ray  $AC$  let fall the perpendicular  $EP$ , and make always  $BC = AP$ , so shall  $c$  be a point in the curve.

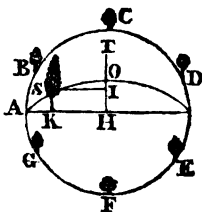
#### X. QUESTION 706, by Mr. Henry Clarke.

A gentleman having in his garden a circular grass-plot, which is exactly level, and upon the area a tall fir-tree, and in the mound of it 6 oaks at equal distances from each other, was desirous of having a tumulus terreus raised on it, with an obelisk, 17 yards high, on the top or vertex of it. He agreed with the workmen at 1d. a solid yard ; but now, the work being done, they are at a loss to determine the solidity and value. From the observations they have made, it appears that the tumulus is an equilateral hyperboloid, with its vertex exactly over the centre of the base, and its semi-transverse equal to the height of the obelisk ; also that the present foot of the fir is 20 yards below the level of the foot of the obelisk, and the two oaks nearest the fir-tree are equally distant from it ; and moreover the gentleman himself remembers that the continual product of the distances of the fir from the 6 oaks is 16883942000 yards. From hence the workmen hope, by the assistance of the Diarians, to know what is due to them?

*Answered by the Proposer Mr. Henry Clarke.*

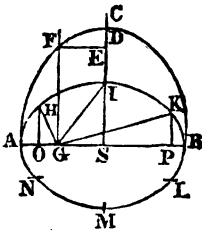
Let  $B, c, D$ , &c. represent the oaks in the mound of the tumulus ; and let  $ASO$  be a section of it through the obelisk  $ro$  and fir-tree  $s$ . Then, since  $ro$  is = the semi-transverse or semi-conjugate axis of the hyperbola  $ASO$  by the question, we have, per conics,  $\sqrt{((2ro$

+ 01)  $\times$  01) = 81 = KH = 32.86, the horizontal distance of the fir from the obelisk. And since KG = KB, and BC = CD = DE, &c. by the question, it will manifestly be, KC = KF, and KD = KE, therefore (by Simpson's Math. Essays, p. 117) KB  $\times$  KC  $\times$  KD, &c. = AH<sup>3</sup> + KH<sup>3</sup>; hence, AH =  $\sqrt[3]{(16883942000 - 32.86^3)} = 50$ . And, again by conics, OH =  $\sqrt{(AH^2 + TO^2)} - TO = 35.81$ . Hence (by cor. 3, p. 385, Hutton's Mensuration) 3.14159, &c.  $\times \frac{1}{2}$  OH  $\times (AH^2 - \frac{1}{2} OH^2) = 116581.27$  yards = the solid content of the tumulus, which, at one penny each, amount to 485l. 15s. 1 $\frac{1}{4}$ d. the sum the workmen have to receive.



*The same answered by Mr. Lynn.*

Let ADB represent the hyperboloid; H, I, K, L, M, N the six oaks; FG the fir; and CD the obelisk, whose height 17 yards is = the semi-transverse, or semi-conjugate, or semi-parameter of the hyperbola. Draw the rest of the lines as they appear in the figure. Then DE = 20, and  $as^2 = FE^2 = (2CD + DE) \times DE = 1080 = a^2$  suppose. Again, putting the radius as or SI = z, since AH = BK is an arc of 30°, its sine HO = KP is =  $\frac{1}{2}z$ , and co-sine OS = SP =  $\frac{1}{2}z\sqrt{3}$ ; hence GO =  $\frac{1}{2}z\sqrt{3} - a$ , and GP =  $\frac{1}{2}z\sqrt{3} + a$ ; and, by right-angled  $\Delta$ s,  $GH^2 = \frac{1}{4}z^2 + (\frac{1}{2}z\sqrt{3} - a)^2$ ,  $GI^2 = z^2 + a^2$ , and  $GK^2 = \frac{1}{4}z^2 + (\frac{1}{2}z\sqrt{3} + a)^2$ ; and the continual product of the squares of these three lines being equal to that of the six distances of the trees, because GL, GM, GN are respectively = GK, GI, GH, we have;  $GM^2 \times GI^2 \times GK^2 =$  (by actually multiplying)  $a^6 + z^6 = 16884712000$  (not as printed) =  $b^6$  suppose; hence we find  $z = \sqrt[6]{(b^6 - a^6)} = 50$  the radius of the grass-plot, or of the base of the hyperboloid. Then, by the nature of the figure  $DS = \sqrt{(CD^2 + AS^2)} - CD = 35.81098$ . And lastly, as above, the solid content is then easily found = 116583.47 yards, which amount to 485l. 15s. 3 $\frac{1}{4}$ d. nearly.



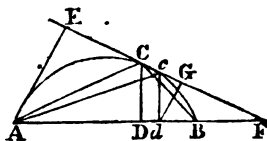
XI. QUESTION 707, by Mr. John Hellins.

In a given arc ACB of a circle, to determine a point c such, that, cb being drawn perpendicular to the chord AB, and A, c joined, the triangle ADC shall be the greatest possible.

*Answered by the Proposer, Mr. John Hellins.*

If EF be a tangent to the circle in the point c, cutting AB produced

in the point  $F$ ; it will always be  $FD : DC :: A'D : D'C$ . But when the right-angled  $\triangle ADC$  is the greatest possible, that is, when  $AD \times DC$  is the greatest possible, it is  $AD : DC :: A'D : D'C$ . Therefore, when  $\triangle ADC$  is the greatest possible, it will be  $FD : DC :: AD : DC$ , or  $FD = AD$ . Whence it is manifest that  $\angle DAC = \angle DFC$ , and that  $\angle ACE (= \angle DAC + \angle DFC) = 2\angle DAC$ , or that the arc  $AC$  is double the arc  $CB$ . Hence then the point  $c$  will always be determined by trisecting the given arc  $AB$ .



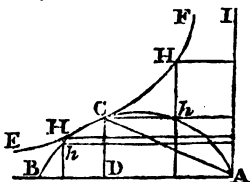
*Cor. 1.* If  $bc$  be drawn; it will very easily be found, by arguing as above, that  $bc = BF$ .

*Cor. 2.* It is evident that the problem may sometimes become a plane problem; as when the arc  $AB$  is  $=$  either 180, 216, 270, or 360 degrees; the chord  $bc$  being then  $=$  the radius of the given circle, the side of the inscribed pentagon, the side of the inscribed square, or side of the inscribed equilateral triangle, respectively.

*Scholium.* There are several other ways of demonstrating that the point  $c$  will be determined by trisecting the arc  $AB$ ; one of which I shall here add.—In the arc take  $c$  indefinitely near to  $c$ ; join  $ca$ , and draw  $cd \parallel CB$ . Then it is evident that the very little  $\triangle acc$  and parallelogram  $cd$ , having the common base  $cc$ , must, when the  $\triangle ADC$  is the greatest possible, be equal each other. If therefore their perpendicular altitudes  $AE$ ,  $DG$  be drawn, it is plain that  $AE$  must be  $= 2DG$ ; and thence, by sim.  $\triangle s$ ,  $AF = 2FD$ , or  $AD = DF$ .—The rest of the demonstration is the same as the above.

*The same answered by Mr. Alex. Rowe.*

Erect  $AI \perp AB$ , and to the asymptotes  $BA$ ,  $AI$  describe an hyperbola to touch the arc as in the point  $c$ . So shall  $c$  be the point required.—For, draw  $ca$ , also complete the parallelogram  $AC$ , and constitute any other parallelograms,  $AH$ ,  $Ah$  at the hyperbola and corresponding point of the circle. Then the  $\triangle ADC = \frac{1}{2}$  rect.  $AC = \frac{1}{2}$  rect.  $AH$  (by the nature of the hyperbola) is greater than  $\frac{1}{2}$  rect.  $Ah$ . That is,  $ADC$  is the greatest possible triangle.

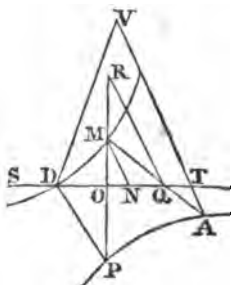


## XII. QUESTION 708, by Mr. Geo. Beck.

While a given right line moves parallel to itself in a given direction, if two other indefinite ones revolve about two given poles and continually pass through the extremities of the first line; it is required to find the locus of the intersection of the revolving lines?

*Answered by the Proposer, Mr. Geo. Beck.*

Let the end  $q$  of the given line  $QR$  move along the line  $st$ , given in position, while  $PMR$ ,  $AQM$ , revolving about the given poles  $P$  and  $A$ , always pass through  $R$  and  $q$  respectively, intersecting each other in  $M$ . Draw  $PD$ ,  $MN$ , and  $AT$  all parallel to  $QR$ ; and call  $QR$ ,  $m$ ;  $PD$ ,  $g$ ;  $AT$ ,  $d$ ;  $DT$ ,  $p$ ;  $DN$ ,  $x$ ; and  $NM$ ,  $y$ : Then,  $g + y : x :: g + m : dq = (gx + mx) \div (g + y)$ ; and  $d + y : p - x :: y : qN = (py - xy) \div (d + y)$ ; therefore  $(gx + mx) \div (g + y) = (py - xy) \div (d + y) = x$ , or  $dmx + (m + g - d)xy - (gp + py) \cdot y = 0$ ; or putting  $r = m + g - d$ , it is  $x = (pgy + pyy) \div (dm + ry)$ ; being a general equation to the common hyperbola, one of whose asymptotes is parallel to  $DT$  at the distance of  $dm \div r$  from it, towards  $P$ . Also if there be made  $TV = r$  and  $\parallel QR$ , and  $D$ ,  $V$  be joined, then  $DV$  is parallel to the other asymptote.



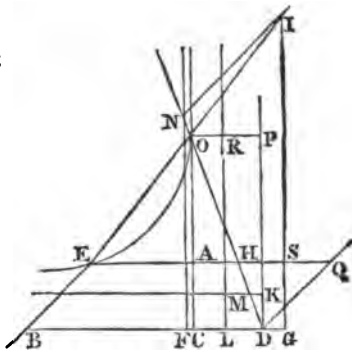
*Corollary 1.* If  $r = 0$ , then  $dmx = p \times (dy + y^2)$ , and the curve is a parabola.

2. If  $g = d$ , then ( $m$  being then  $= r$ )  $mx = py$ , and the locus is a right-line.

*The same answered by Mr. John Aspland.*

\* Let  $IN$  be the given line moving parallel to itself between  $GI$  and  $FN$ , and  $D$ ,  $E$  the given poles; draw  $DQ$ ,  $EB \parallel$  and  $= IN$ , and join  $E$ ,  $q$  and  $B$ ,  $D$ . Then it is evident that  $DB$  will be parallel to one asymptote of the curve, and  $GI$  parallel to the other. Draw  $DN$  and  $EI$  intersecting the curve in  $o$ ; also make  $oc$  and  $DF \parallel FN$ , and  $OF \parallel BD$ .

Put  $GS = a$ ,  $FD = n$ ,  $EH = r$ ,  $ES = m$ ,  $DF = CO = x$ , and  $DC = FO = y$ . Then, by sim. triangles,  $DCO$ ,  $DFN$ ,  $y : x :: n : nx \div y = FN = SI$  because of the parallels; and, by the sim.  $\Delta$ s,  $ESI$ ,  $EAO$ , it is  $ES : SI :: EA : AO$ , that is,  $m : nx \div y :: r - y : x - a$ ; hence this equation  $xy - may \div (m + n) - nrx \div (m + n) = 0$ . From which equation it is manifest that the required locus will be an hyperbola. And if we take as  $ES + FD : ES :: GS : DK$ , and  $ES + FD : FD :: EH : DL$ , and then draw  $KM$  and  $LM$  parallel to  $BD$  and  $GI$ , respectively, these lines will be the asymptotes intersecting in the centre  $M$  of the hyperbola: For  $DL = nr \div (m + n)$ , and  $DK = ma \div (m +$



$n$ ), therefore  $MR = x - ma \div (m + n)$ , and  $OR = y - nr \div (m + n)$ ; hence  $OR \times RM = xy - (may + nrx) \div (m + n) + mnra \div (m + n)^2$ ; but  $xy - (may + nrx) \div (m + n) = 0$  by our equation before obtained, therefore  $OR \times RM = mnra \div (m + n)^2$  a constant quantity; consequently the curve is an hyperbola.—After the same manner, if the revolving lines pass from  $D$  and  $E$  through  $I$  and  $X$  the two other ends of the line  $NI$ , it may be proved that the intersection will be in an hyperbola.

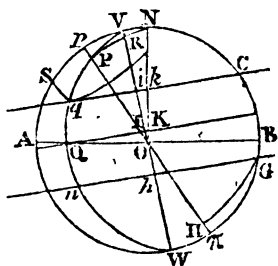
If the lines of direction pass through the poles  $D$  and  $E$ ; then  $m = n = r$ , and the equation becomes  $xy - \frac{1}{2}ay - \frac{1}{2}x \times EH = 0$ , in which case the asymptotes will pass through the middle of  $DI$  and  $HE$ . If the moving line move parallel to a line  $DE$  joining the poles, the locus will be the right line  $MR$ . For in this case  $a = 0$ , and the equation becomes  $y = nr \div (m + n)$ .

XIII. QUESTION 709, by Mr. Thomas Barker, of Westhall.

In the solar eclipse of September 5, 1774, given the sun's declination  $= 6^\circ 30' 17''$ , his semi-diameter  $= 15' 56''$ , that of the moon  $= 15' 21''$ , her horizontal parallax  $= 56' 19''$ , her latitude  $= 2' 15''$  N. increasing, and the angle of her visible way from the sun  $= 5^\circ 43' 30''$ . To determine the greatest latitude north and south, where this eclipse was visible.

*Answered by the Proposer, Mr. Thomas Barker, of Holton, near Halesworth.*

Let the circle  $ANBW$  represent the earth's enlightened disk at the time of the eclipse, and its radius, it is well known, will be equal to the difference of the horizontal parallaxes of the sun and moon. Draw  $AB$  to represent the ecliptic, and its axis  $ON$  at right angles to it: Moreover, draw  $vow$  making with  $ON$  an angle  $=$  to  $5^\circ 43' 30''$ , the moon's visible way from the sun; also  $po\pi$  to represent the terrestrial axis. Make  $OK = 2\frac{1}{2}$  the moon's latitude, and through  $K$ , at right angles to  $wv$ , draw  $xq$ , the moon's path, cutting its axis  $wv$  in  $i$ . Make  $ii = ih = 31' 17''$ , the sum of the semi-diameters of the sun and moon, and through  $i$  and  $h$  draw  $qc$  and  $na \parallel xq$ , which it is manifest, will be the limits beyond which the eclipse cannot be seen: consequently, if  $P$  be the north pole of the earth, and round it a parallel of latitude  $sqn$  be described to touch  $qc$ , it will be the parallel required to the northward; and its distance from the pole,  $P$ , will be measured by an arc of a great circle passing through  $P$  and the point  $q$ . Now, as this great circle passes through  $P$ , it will be perpen-



dicular to the lesser circle  $sqn$ , and of course to  $qc$  also, which by the nature of the orthographic projection, will represent a lesser circle described about the pole  $v$ , and consequently the great circle  $rq$ , will, if produced, intersect the primitive in  $v$  and  $w$ . Let now  $\Pi$  be the south pole of the world, and let arcs of great circles be described through  $r$  and  $n$ , and through  $\Pi$  and  $g$ : Then in the spherical  $\triangle rpn$  there is given the right  $\angle p$ , the hypotenuse  $pn = 23^\circ 27' 8''$ , the distance of the poles of the globe and ecliptic, also the leg  $rp = 6^\circ 30' 17''$  the sun's declination, to find  $pn = 22^\circ 35' 45''$ , which measures the  $\angle pon$ ; take from it the  $\angle von = 5^\circ 43' 30''$ , and there will remain  $16^\circ 52' 15''$  for the  $\angle vop$ , of which the arc  $vp$  is the measure; consequently there are now given the two legs  $vp, pp$  in the right-angled spherical triangle  $vpp$ , to find  $pv = 18^\circ 1' 50''$ . Again, having, in the right-angled plane triangle  $kro$ , the side  $ko = 2.25$ , and the  $\angle kor = 5^\circ 43' 30''$ ,  $ro$  will be found  $= 2.239$ ; add  $is = 31.283$ , the sum of the semi-diameters of the  $\odot$  and  $\text{J}$ , and we have  $oi = 33.522$ , which is the cosine of the arc  $vc$ , or its equal  $vq$ ; therefore, by the nature of the projection,  $56.317 (= on) : 33.522 :: 1 : \text{cosine of } 53^\circ 28' 14''$ , from which taking  $18^\circ 1' 50''$ , there will remain  $35^\circ 26' 24''$  for the complement of the latitude of the northern limit. And in this manner will the limit towards the enlightened pole be always found. But seeing that the penumbra continually approaches the obscured pole until it quits the disk at  $g$ , it is manifest that that point will be the nearest approach to the south pole in the present case. Now, from the semi-diameter  $ih$  of the penumbra,  $= 31.283$ , if  $ro = 2.239$  be taken, the remainder  $29.044$  will be  $=$  to  $oh$ , which is manifestly the cosine of the arc  $wg$  to the radius  $ow$ ; therefore  $56.317 : 29.044 :: 1 : \text{cosine of } 58^\circ 57' 15'' = wg'$ ; from it take  $w\pi = vp$ , and there will remain  $\pi g = 42^\circ 5' 00''$ : therefore in the spheric  $\triangle \Pi\pi g$ , right-angled at  $\pi$ , there are given  $\Pi\pi$ , and  $\pi g$ , to find  $\Pi g$ , the co-latitude of the southern limit, which comes out  $= 42^\circ 29' 25''$ .

*Scholium* The longitudes of these two places are also readily found. For  $\Pi\Pi$  being the meridian whereon the sun is centrally eclipsed, its longitude is known, and the  $\angle s org, o\Pi g$ , which are the longitudes of these places from that meridian, are found by a single proportion each in the right-angled  $\triangle s vrp, g\Pi\pi$ .

The whole might easily have been constructed by scale and compasses, or indeed solved near enough for common use by the globe itself; which methods I should have put down, were it not that I am afraid of exceeding the limits of your diary.

#### XIV. QUESTION 710, by Mr. Cullen O'Connor.

If, by experiments on the attraction of large masses of matter, it be found that the mean density of the whole earth is  $2\frac{1}{2}$  times that of the mean matter at the surface; it is required to find the density of

the matter at the centre of the earth, and the place or depth at which the density is equal to the said mean density, together with the law of the increase of density all the way from the surface to the centre: supposing that it is always reciprocally as some power of the difference between the whole diameter and the depth below the surface, and that the earth is truly spherical.

*Answered by Mr. Mic. Taylor.*

Put  $r = AC$  the radius of the earth, 1 = the density at the surface at A,  $c = 3.1416$ ,  $n$  = the required exponent of the power, and  $z = CB$  any distance from the centre.

Then, by the question,  $(2r)^{-n} : (r + z)^{-n} ::$

$1 : (2r \div (r + z))^n =$  the density at the place B.

But  $4cz^2$  is = the surface whose radius is  $CB$  or  $z$ ;

consequently  $4cz^2 \times (2r \div (r + z))^n = 2^{2+n}$

$cr^n z^2 \cdot (r + z)^{-n}$  is = the fluxion of the quanti-

ty of matter in the sphere BDE; or by putting  $2^{2+n} cr^n = a$ , the

same fluxion is  $az^2 \cdot (r + z)^{-n}$ . The fluent of this is  $\frac{a \cdot (r + z)^{1-n}}{3-n}$

$\times (z^3 - \frac{2rz}{2-n} + \frac{2rr}{2-n \cdot 1-n})$ . But when  $z = 0$ , this becomes

$= \frac{2ar^{3-n}}{3-n \cdot 2-n \cdot 1-n}$ ; therefore, the correct fluent is

$\frac{-2ar^{3-n}}{3-n \cdot 2-n \cdot 1-n} - \frac{a \cdot (r + z)^{1-n}}{3-n} \times (z^3 - \frac{2rz}{2-n} + \frac{2rr}{2-n \cdot 1-n})$  = the quantity of matter in the sphere whose

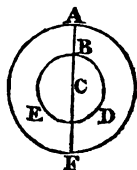
radius is  $z$ . And when  $z = r$ , the same becomes  $2^3 r^3 c \times$

$\frac{2-2^n-n+n^2}{3-n \cdot 2-n \cdot 1-n} =$  the matter in the whole earth; which be-

ing, by the question,  $= 2\frac{1}{2} \times 2^3 r^3 \times \frac{1}{6} c$ , from this equation we ob-

tain  $\frac{2-2^n-n+n^2}{3-n \cdot 2-n \cdot 1-n} = \frac{1}{12}$ ; from which equation the value of  $n$  comes out 5.

The general density being  $(2r \div (r + z))^n$ , if  $z$  be taken  $= 0$ , this expression becomes  $2^n = 2^5 = 32$  the density at the centre. Moreover, by taking  $(2r \div (r + z))^n = 2\frac{1}{2}$ , we obtain  $r - z = r \times (2 -$



$(64 \div 5)^{\frac{1}{5}} = .334935r = AB$ ; or the depth below the surface is nearly  $\frac{1}{3}$  of the radius at the place of the mean density.

*The same answered by Line.*

Put  $Ac = r$ , the circumference  $= c$ , and  $FB = z$ . Then  $\frac{2}{3}cr^3 =$  solidity of the globe. But  $c(z - r) \div r =$  the circumference BDE and  $2c \div r \times (z - r)^2 =$  the surface of this sphere; also as  $(2r)^{-n} : z^{-n} :: s$  (density at A) :  $s \cdot (2r)^n \cdot z^{-n} =$  the density at B; therefore the fluent of  $\frac{2cs}{r} \times (2r) z^{-n} \times (z - r)^2 \dot{z}$  or  $4csr^2 \times \frac{2^n - n^2 + n - 2}{n - 1 \cdot n - 2 \cdot n - 3}$  divided by the mass  $\frac{2}{3}cr^3$ , must be  $= \frac{2}{3}s$  by the question; hence  $n = 5$ .—The rest is easily found.

THE PRIZE QUESTION, *by* Peter Puzzlem.

To find how long a cylinder of heavy metal, one foot in diameter will be in motion, and how many revolutions it will make about its axis, in descending (by its gravity) down a plane 30 yards long, inclined to the horizon in an angle of  $60^\circ$ : admitting it known by experiment, that, if the plane's inclination were  $25^\circ$ , the friction would be just sufficient to cause the cylinder to roll down without sliding; and, that, in general, with respect to a body sliding along a plane, the friction is as the pressure against the plane.

*Answered by* Peter Puzzlem, *the Proposer.*

Let  $f$  denote the force accelerating the velocity of a point on the surface of the cylinder about the axis thereof at any instant during the motion down the plane, when its inclination is  $25^\circ$ . Then, putting  $x$  for the distance of any particle ( $p$ ) of the cylinder from the axis;  $r$  for the radius of the cylinder; and  $s$  for its content; we have  $r : x :: f : fx \div r$ , the force accelerating  $p$ : and  $fp \div r$  will be the motive force of  $p$ . Therefore, by the property of the lever,  $r : x :: fp \div r : fpx \div r^2$ , the motive force which acting at the surface of the cylinder will be equivalent to  $(fpx \div r)$  the motive force of  $p$ . Let  $c$  be put for  $(6.283)$  the circumference of the circle whose radius is 1; then, considering the length of the cylinder as unity, the ring of particles at the distance  $x$  from the axis will be denoted by  $cxx$ : consequently  $(\frac{1}{2} cfr^2)$  the fluent of  $cfx^2 \dot{x}$ , when  $x$  is therein taken  $= r$ , will be the whole motive force which must necessarily act at the surface

of the cylinder, that the accelerative force of a point there may be  $f$ . Which motive force, by substituting  $b$  for its equal  $\frac{1}{2}cr^2$ , becomes  $\frac{1}{2}fb$ . Now, by the question, this quantity  $\frac{1}{2}fb$  will be equal to the friction when the inclination of the plane is  $25^\circ$ : therefore  $gmb - \frac{1}{2}fb$  will then be the motive force, and  $gm - \frac{1}{2}f$  the accelerative force urging the axis of the cylinder downwards parallel to the plane,  $g$  denoting  $(32\frac{1}{2})$  the accelerative force of gravity, and  $m$  the sine of  $25^\circ$  to the radius 1. Which accelerative force  $(gm - \frac{1}{2}f)$  of the axis will be  $= f$ , the velocity of the axis being just equal to the velocity wherewith a point on the surface of the cylinder will be carried about the axis, when the friction keeps the cylinder from sliding. Therefore, from the equation  $gm - \frac{1}{2}f = f$ , we have  $f = \frac{2}{3}gm$ ; and consequently  $(\frac{1}{2}fb)$  the friction, when the plane's inclination is  $25^\circ$ , will be  $= \frac{1}{3}gmb$ .

Moreover, the cosine of  $60^\circ$  being  $= \frac{1}{2}$ , the pressure against the plane, when its inclination is  $60^\circ$ , will be  $= \frac{1}{2}gB$ ; and when the inclination is  $25^\circ$ , the pressure will be  $= gnB$ ,  $n$  denoting the cosine of  $25^\circ$ . Therefore, the friction being as the pressure, we have  $gnB : \frac{1}{2}gB :: \frac{1}{3}gmb : gmb \div 6n$ , the friction when the plane's inclination is  $60^\circ$ .

Which friction  $(gmb \div 6n)$ , being taken from  $3\frac{1}{2}gB \div 2$ , the whole motive force on the cylinder in a direction whose inclination is  $60^\circ$ , the remainder  $3\frac{1}{2}gB \div 2 - gmb \div 6n$  will be the motive force and  $3\frac{1}{2}g \div 2 - gm \div 6n$  the accelerative force, urging the axis of the cylinder downwards parallel to the plane when inclined in an angle of  $60^\circ$ .

The accelerative force being thus found, the required time of descent, by the well-known theorems relating to the motion of bodies uniformly accelerated, is readily found equal to  $\sqrt{(1080n \div (3\frac{1}{2}gn - gm))} = 2.66$  seconds; which will be the same, let  $r$  be what it will.

Furthermore,  $c$  being put for the accelerative force of a particle at the surface of the cylinder about the axis, when the plane's inclination is  $60^\circ$ , it appears by what is done above, that  $\frac{1}{2}cb$  will be  $= gmb \div 6n$ : whence  $c = gm \div 3n$ . It follows therefore from the

well-known theorems just now mentioned, that  $\frac{3\frac{1}{2}g}{2} - \frac{gm}{6n} : \frac{gm}{3n} ::$

$90 : \frac{180m}{3\frac{1}{2}n - m} = 17.7457$ , the space a point at the surface of the cy-

linder will be carried about the axis during the descent, when the inclination of the plane is  $60^\circ$ : which space  $(17.7457)$  being divided by  $(3.1416)$  the circumference of the cylinder, gives  $5.64$ , the required number of revolutions.

*Mr. David Kinnebrook, of Norwich, likewise answered it thus:*

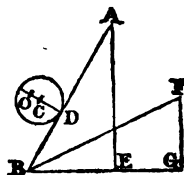
Let  $AB$  and  $BF$  represent the two inclined planes, each of 30 yards

or 90 feet long; AE and FG perpendiculars on the horizontal plane BEG; c the centre of gravity of the cylinder, and o its centre of oscillation with respect to the point of suspension D. Also put  $s = 32\frac{1}{2}$  feet.

Then, if the cylinder Dco be supposed to be upon the plane FB, and to descend freely, the velocity it acquires in 1'' is  $(GF \div FB) \times s$ ; but if it be supposed to roll down FB, we have  $oc :: 3 : 1 ::$  the relative weight of the cylinder : the friction just sufficient to hinder it from sliding  $:: (GF \div FB) \times s : (GF \div 3AB) \times s$  (because  $AB = BF$ ) the velocity which would be destroyed in 1'' by the said friction, which is also the proper measure of the friction; but the friction is supposed to be as the pressure against the plane, therefore  $BG : BE :: (FG \div 3AB) \times s : (BE \times FG \div 3AB \times BG) \times s$  the velocity destroyed in 1'' by the friction of the cylinder upon AB which taken from  $(AE \div AB) \times s$  the velocity generated by gravity in 1'' on AB, gives  $(AE \div AB - BE \times FG \div 3AB \times BG) \times s$  the velocity generated in 1'' by the centre of gravity of the cylinder in its motion down AB; whence  $\sqrt{\left(\frac{AE}{AB} - \frac{BE \cdot FG}{3AB \cdot BG}\right)}$

$\times s : \sqrt{AB} :: 1'' : AB \div \sqrt{\left(AB - \frac{BE \cdot FG}{3BG}\right)} \times s =$  (when AB is 30 yards or 90 feet) 2.664'' the time of the cylinder's descending down AB by a mixt motion of sliding and rolling.

To find the number of revolutions; we have as  $oc : CD :: 1 : 2 :: (BE \cdot FG \div 3AB \cdot BG) \times s : (2BE \cdot FG \div 3AB \cdot BG) \times s =$  the accelerating force at the surface of the cylinder to turn it about its axis, hence as  $(AE \div AB - BE \cdot FG \div 3AB \cdot BG) \times s : (2BE \cdot FG \div 3AB \cdot BG) \times s :: 90 \text{ (feet)} : AB : 2BE \cdot FG \cdot 90 \div (3AE \cdot BG - BE \cdot FG) = 17.7459$  the space rolled; which divided by 3.1416, we have 5.6487 for the number of revolutions.



*Questions proposed in 1777, and answered in 1778.*

I. QUESTION 712, by Mr. James Nicholson.

Equating gents, who love to hide  
The age and fortune of your bride,  
From what's below please to descry  
The values of  $x$ ,  $z$ , and  $y$ .

$$x^2 + y^4 + z^3 = a.$$

$$x^3y^3 + x^2z + y^2z = b,$$

$$x^3 + 3x^6 \cdot (y^3 + z) + 3x^3 \cdot (y^3 + z)^2 = c.$$

*Answered by Mr. Tho. Hall.*

By adding the first equation to the double of the second, the root of the sum is  $x^3 + y^3 + z = \sqrt{(a + 2b)} = d$ ; from the cube of which subtract the third equation, and the cube root of the remainder is  $y^3 + z = \sqrt[3]{(d^3 - c)}$ ; this taken from that above, and the cube root extracted, we have  $x = \sqrt[3]{(d - \sqrt[3]{(d^3 - c)})}$ ; these values of  $x$  and  $y^3 + z$  being used in the second equation, by transposing we have  $y^2z = b - x^3(y^3 + z) = b - \sqrt[3]{(d^3 - c)} \times (d - \sqrt[3]{(d^3 - c)}) = e$  suppose; having thus the sum and product of  $y^3$  and  $z$ , viz.  $y^3z = e$ , and  $y^3 + z = \sqrt[3]{(d^3 - c)} = f$  suppose, those letters themselves are easily found, viz.  $z = \frac{1}{2}f \mp \sqrt{(\frac{1}{4}f^2 - e)}$ , and  $y = \sqrt{(\frac{1}{2}f \pm \sqrt{(\frac{1}{4}f^2 - e)})}$ .

II. QUESTION 713, *by Mr. John Shadgett.*

Required the dimensions of a right-angled triangular piece of ground, whose base and perpendicular are in the ratio of 3 to 2, and the area of its inscribed circle equal to the superficial content of a wall 6 feet high inclosing the triangle?

*Answered by Mr. James Phillips.*

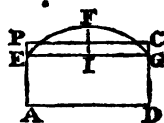
Put  $3x =$  the base of the triangle; then will  $2x =$  the perpendicular, and  $\sqrt{(9x^2 + 4x^2)} = x\sqrt{13} =$  the hypotenuse, also  $5x + x\sqrt{13} =$  the perimeter or length of the surrounding wall. It is a known property of the right-angled triangle, that the difference between the hypotenuse and the sum of the other two sides, is = the diameter of the inscribed circle; therefore that diameter is  $= 5x - x\sqrt{13}$ , and consequently its area  $= (5x - x\sqrt{13})^2 \times .7854$ , which must be  $= (5x + x\sqrt{13}) \times 6$  or the area of the wall: from hence  $x = \frac{5 + \sqrt{13}}{(5 - \sqrt{13})^2} \times \frac{6}{.7854} = \frac{(5 + \sqrt{13})^3}{24 \times .7854} = \frac{40 + 11\sqrt{13}}{3 \times .7854} = 33.8091$ . Hence the three sides of the triangle are 67.6182, 101.4273, and 121.9004.

III. QUESTION 714, *by Mr. Mark Elstob, of Shotton.*

Let a cylindrical vessel, of 2 feet diameter, be exactly filled with water, at the surface of the earth; and suppose it to descend gradually towards the earth's centre: It is required to find its distance from the centre when the extreme part of the water next the sides of the vessel is one hundredth part of an inch below the top of the vessel; or when it wants one hundredth part of an inch of being full; admitting the surface of the water at the commencement of the motion to be a real plane, the earth a perfect sphere, and its radius 21000000 feet?

*Answered by Mr. Stephen Williams.*

The surface of the water is always a part of the surface of a sphere whose centre is the centre of the earth, and radius = the distance to that centre; and therefore, by approaching the centre, the surface of the water becomes more and more convex, rising higher over the middle of the vessel, but descending down the sides; so that the spherical part  $EFG$  will always be = the cylindrical part  $PCGE$ . But the cylinder  $PCGE$  is  $= 24' \times .01 \times .7854$ ; and by rule 1, p. 202, Hutton's Mensuration, the seg.  $EFG$  is  $= \frac{2}{3}x \times .7854 \times (x^2 + 3 \times 12^2)$ , making  $x$  = the versed sine  $FI$ ; putting these equal, we have  $2.4' = \frac{2}{3}x^3 + 288x$ ; hence  $x^3 + 432x = 8.64$ , and  $x = .02$  very nearly =  $FI$ . Then the radius of the spherical surface is  $= (FI^2 + 12^2) \div 2FI = (.02^2 + 12^2) \div .04 = 3600$  inches = 300 feet, the distance of the surface of the water from the centre of the earth, as was required.



IV. QUESTION 715, by Mr. Ralph Dutton.

There is a well at Durham town,  
Whose bucket's by a rope let down  
Off a cylindric axle-tree,  
Whose length and girt below you see.  
The rope's involv'd from end to end;  
Each fold per margin does extend;  
Its length does fathom just the well;  
Therefore its depth to me pray tell.

$x^2 + z^2 + x + z = 23.75$ , and  $xz = 6$ ; where  $x$  represents the length, and  $z$  the girt or circumference in feet of the axle-tree. The rope is 1 inch thick, and the folds touch each other.

*Answered by Mr. John Jackson.*

First the dimensions of the roller may be found from the given equations by quadratics thus: To the 1st equation add the double of the 2d, and the sum is  $(x + z)^2 + x + z = 35.75$ ; then, by completing the square, &c. is found  $x + z = 5.5$ ; from the square of this take four times the 2d given equation, and the root of the remainder is  $x - z = 2.5$ , which being added and subtracted with the last one, &c. we find  $x = 4$ , and  $z = 1.5$  feet; or  $x = 48$  inches, and  $z = 18$  inches, the length and circumference.

To this 18 add  $3.1416$  the circumference of the rope, so shall the sum  $21.1416$  be the circumference passing thro' the centre or middle of the rope, which may be considered as the base of a right-angled triangle, of which the  $\perp$  is 1 inch, the distance of the folds, and its hypotenuse the length of the axis of the rope in one fold; consequently  $\sqrt{(1^2 + 21.1416^2)} = 21.1653$  is the length of each fold of the rope. And, if we count by the middle or axis of the rope, the num-

ber of folds will be 48. Hence then  $48 \times 21.1653 = 1015.934$  inches  $= 84.6612$  feet, the depth of the well required.

V. QUESTION 716, by Mr. Edward Boucher.

To determine a circular arc such, that its sine may be equal to  $a$  times the square of its cosine; and to find the value of  $a$  so, that both the sine and cosine may be rational numbers: the rad. being 1.

*Answered by Mr. John Fatherly.*

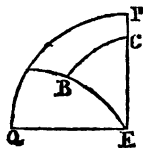
Let  $x$  be the sine of the required arc; then will  $1-x^2$  be the square of the cosine; and therefore  $x = a(1-x^2)$  per question, hence  $a = x \div (1-x^2)$ . But that the cosine may be rational,  $1-x^2$  must be a square; suppose it  $= (1-rx)^2$ , then will  $x = \frac{2r}{r^2+1}$  the sine; consequently  $\sqrt{1-x^2} = \frac{r^2-1}{r^2+1}$  the cosine, and  $a = \frac{x}{1-xx} = \frac{r^2+1}{r^2-1} \times \frac{2r}{r^2-1}$  the value of  $a$ . Which values are general, and  $r$  may be assumed  $=$  any number greater than 1.

VI. QUESTION 717, by Mr. Wm. King, of Lofthouse.

Suppose two ships, the one at the north pole and the other at the equator, to commence motion at the same time; the ship from the pole sails directly south towards that point of the equator from whence the other departs, depressing the pole uniformly one degree in a day, or twenty-four hours; the other from the equator sails on the arc of a great circle, which at the beginning of the motion bears W. N. W. uniformly two degrees a day: To find when they will be the nearest possible to each other?

*Answered by Mr. Ra. Thompson, of Witherley Bridge.*

Let  $B$  and  $C$  be any two contemporary positions of the two ships; and put  $c$  for the cosine of  $\angle BEC = 67^\circ 30'$ , and  $x$  and  $y$  for the sine and cosine of  $EC$ , passed over by the ship from the pole  $E$ , or  $x$  and  $y$  are the cosine and sine of  $CE$ : then, the velocity of the ship from the equator being double of that from the pole,  $2xy$  and  $1-2x^2$  will express the sine and cosine of  $EB$ , passed over by the ship from the equator. Hence, by spherics,  $x-2x^2+2cxy^2 = x-2x^3+2cx-2cx^3$  is the cosine of  $BC$ , which must be a maximum when  $BC$  is a minimum the fluxion of it therefore



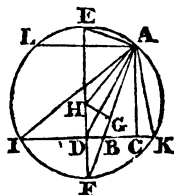
being made  $= 0$ , there results  $x = \sqrt{((1 + 2c) \div (6 + 6c))} = .4612968$ , which answers to  $27^\circ 28' 14'' = pc$ ; therefore  $EB = 54^\circ 56' 28''$ ; from whence is easily found  $BC = 57^\circ 6'$  their nearest distance, after sailing 27 days, 11 hours, 17 minutes, 36 seconds.

VII. QUESTION 718, by Mr. John Aspland.

In a plane triangle, given the line from the vertical angle to the middle of the base, the line bisecting the vertical angle and terminated by the base, and the difference of the angles at the base; to determine the triangle?

Answered by Mr. Mark Elstob.

**Construction.** Make a right-angled  $\triangle ABC$ , having its hypotenuse  $AB$  = the given line bisecting the vertical angle, and  $\angle BAC$  = half the given difference of the angles at the base; and produce  $BC$  indefinitely both ways; to which apply  $AD$  equal the given line bisecting the base; draw  $EDF \parallel AC$  meeting  $AB$  produced in  $F$ ; bisect  $AF$  with the  $\perp$   $GH$  meeting  $EF$  in  $H$ ; with the centre  $H$  and radius  $HF$  or  $HA$ , describe the circle  $EAKFI$ ; join  $AI$ ,  $AK$ ; so shall  $AIK$  be the triangle required.



**Demonstration.** First,  $ID = DK$  (Eucl. 3, 3.). Secondly,  $\angle IAB = \angle KAB$ , insisting on the equal arcs  $IF$ ,  $KF$ . And, lastly, the given difference of the angles is  $= 2\angle BAC$  (construction),  $= \angle K - \angle I$ , by article 40, Hutton's Mathematical Miscellany.

On the same principle (that the difference of the angles at the base is double the angle formed by the perpendicular and line bisecting the vertical angle) is the construction given by most of the other gentlemen who answered the problem, viz. Messrs. *Lynn*, *Parnel*, *Wildbore*, &c. only differing a little in the manner of finding the centre of the circle, after the  $\triangle ABC$  is described. Thus:

Mr. *Lynn* draws  $AE \perp AF$ , and then bisects  $EF$  in  $H$  the centre.

Mr. *Parnel* draws  $AH$  making the  $\angle HAF = \angle F$ , which equally gives the centre.

The Rev. Mr. *Wildbore* draws  $AL \parallel$  and  $= 2cn$ , and describes the circle thro' the three points  $I$ ,  $A$ ,  $F$ .—He also farther remarks that, since when  $AB$  and  $\angle BAC$  are given,  $AC$  is also given, this includes the 270th Diary question, and problem 19 of Simpson's Algebra.

VIII. QUESTION 719, by Mr. John Fatherley.

Supposing a round tapering tree to girt fourteen feet at the greater end, and two feet at the less, the length being thirty-two feet: Now

by the usual method of multiplying the length by the square of the mean quarter-girt, the content of this piece of timber is 128 feet; but by bisecting the length it is known that the two parts will then measure to the most possible by the same method; then if the lengths of these two parts be again bisected, and all those last parts bisected again, and so on, bisecting continually; it is required to find the limit to which the sum of the measures of all the parts approximates by the continual bisections ad infinitum, and also to find the least number of parts whose contents taken together shall make the sum just 151½ feet, still computing by the usual method first mentioned.

*Answered by Mr. John Lynn.*

Let the bisections be supposed to be made, and let the mean quarter-girts of the several parts be found by taking half the sum of those of the two ends of each piece of timber, and multiply the sum of their squares by the whole length divided by the number of parts, so shall the several successive contents be as below, viz. putting  $L = 32$ , the length;  $G = \frac{1}{2}$ , the greatest quarter-girt;  $g = \frac{1}{2}$ , or the least; and  $c$  = always the content; they will be

$$c = \frac{L}{1 \cdot 2^1} \times (G + g)^2, \text{ when the tree is whole, or in one part.}$$

$$c = \frac{L}{2 \cdot 4^1} \times : (3G + g)^2 + (G + 3g)^2, \text{ when in two parts.}$$

$$c = \frac{L}{4 \cdot 8^1} \times : \left\{ \begin{array}{l} (7G + g)^2 + (5G + 3g)^2 + (3G + 5g)^2 + (G + 7g)^2, \\ \text{for four parts.} \end{array} \right.$$

$$c = \frac{L}{8 \cdot 16^1} \times : \left\{ \begin{array}{l} (15G + g)^2 + (13G + 3g)^2 + (11G + 5g)^2 + (9G + 7g)^2 + (7G + 9g)^2 + (5G + 11g)^2 + (3G + 13g)^2 \\ + (G + 15g)^2, \text{ for eight parts.} \end{array} \right.$$

And, in general,

$$c = \frac{L}{x \cdot (2x)^1} \times : \left\{ \begin{array}{l} ((2x - 1) \cdot G + g)^2 + ((2x - 3) \cdot G + 3g)^2 + \\ ((2x - 5) \cdot G + 5g)^2 + \&c. \text{ for } x \text{ parts; that is,} \end{array} \right.$$

$$c = \frac{L}{4x^3} \times \left\{ \begin{array}{l} G^2 \times : (2x - 1)^2 + (2x - 3)^2 + (2x - 5)^2 + \&c. \\ g^2 \times : 1^2 + 3^2 + 5^2 + \dots \&c. \\ 2Gg \times : 1(2x - 1) + 3 \cdot (2x - 3) + 5 \cdot (2x - 5) + \&c. \end{array} \right.$$

But the series  $(2x - 1)^2 + (2x - 3)^2 + (2x - 5)^2$  &c. is the same with the series  $1^2 + 3^2 + 5^2$  &c. when inverted, or beginning with the last term first; and therefore

$$c = \frac{L}{4x^3} \times \left\{ \begin{array}{l} (G^2 + g^2) \times : 1^2 + 3^2 + 5^2 \&c. \\ + 4Gg \times : 1 + 3 + 5 \&c. \\ - 2Gg \times : 1^2 + 3^2 + 5^2 \&c. \end{array} \right.$$

$$\text{or } c = \frac{L}{4x^3} \times \left\{ \begin{array}{l} (G - g)^2 \times : 1^2 + 3^2 + 5^2 \&c. \\ + 4Gg \times : 1 + 3 + 5 \&c. \end{array} \right.$$

But  $1 + 3 + 5 + \&c.$  to  $x$  terms is  $= x^2$ ; and  $1^2 + 3^2 + 5^2 + \&c.$  is  $= \frac{4x^3 - x}{3}$ ;

therefore  $c = \frac{L}{4x^3} \times \left( \frac{(G - g)^2}{3} \times (4x^2 - x) + 4Ggx^2 \right) = \frac{1}{3}L \times (G^2 + Gg + g^2 - \left( \frac{G - g}{2x} \right)^2)$  which is the general value of the sum of the contents when divided into any number of parts denoted by  $x$ .

Here then it is evident that the quantity  $((G - g) \div 2x)^2$ , which in this general value is to be subtracted, decreasing as  $x$  increases, the general remainder  $G^2 + Gg + g^2 - ((G - g) \div 2x)^2$ , or  $c$ , always increases as  $x$  increases, and will be greatest when  $x$  is infinite; but when  $x$  is infinite then  $(G - g) \div 2x$  is nothing, and the limit or greatest quantity required is  $\frac{1}{3}L \times (G^2 + Gg + g^2)$ , which is exactly equal to the frustum of a square pyramid of the same length and girt, *i. e.* the side of the square end being  $=$  the quarter-girt of the round tree. In numbers this is 152 feet. Which is the first part of the answer.

To find  $x$ , or the number of parts to be cut into, when the sum of the contents  $c$  is a given quantity. Since it is always  $c = \frac{1}{3}L \times ((G^2 + Gg + g^2) - ((G - g) \div 2x)^2)$ , therefore  $x = (G - g) \div 2\sqrt{(G^2 + Gg + g^2 - 3c \div L)} = 8$  the number of parts required when  $c$  is  $= 151\frac{1}{8}$ .

*Corol. 1.* The least content, or that of the whole tree before the sections are made, by the usual method, is  $\frac{1}{4}L \times (G + g)^2 = \frac{1}{4}L \times (G^2 + 2Gg + g^2)$ . Therefore the least content is to the greatest, as  $3 \times (G^2 + 2Gg + g^2)$  or  $3 \times (G + g)^2$  to  $4 \times (G^2 + Gg + g^2)$ . And the difference between the least and the greatest contents is  $\frac{1}{4}L \times (G^2 - g^2)$ , which therefore is the greatest increase made to the content by the sections. In numbers this is 24; which added to 128, the least, the sum is 152  $=$  the greatest.

*Corol. 2.* By each successive bisection, the content is increased by one-fourth of the increase at the immediately preceding bisection. So that, if  $a$  be the increase by the first bisection, then the several single increases at the following bisections, are  $\frac{1}{4}a$ ,  $\frac{1}{16}a$ ,  $\frac{1}{64}a$ , &c. This is evident from the quantity  $((G - g) \div 2x)^2$  or  $(G - g)^2 \div 4xx$ ; for at each bisection,  $x$  is doubled, and therefore  $xx$  is quadrupled.

IX. QUESTION 720, by Mr. Thomas Moss.

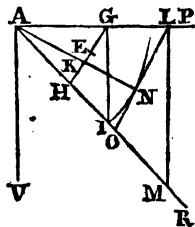
Two indefinite right lines AP and AR forming a given angle, and a point E between them being also given; it is proposed to draw a line thro' the given point E, meeting the said indefinite lines in G and H, so that GI being drawn parallel to a line AV given by position, and terminating in the other indefinite line in I, and AK perpendicular to GH

and meeting it (produced if necessary) in  $\kappa$ , the lines  $GI$  and  $AK$  shall be to each other in the given ratio of  $m$  to  $n$ .

*Answered by Mr. Tho. Wilson.*

Any where between the lines  $AP$ ,  $AR$ , forming the given angle, and  $\parallel$  to  $AV$  draw  $LM$ ; to which take another line in the given ratio of  $m$  to  $n$ , and with it as a radius, and the centre  $A$ , describe an arc, and draw  $LNO$  a tangent to it in  $N$ ; join  $AN$ , and draw  $GEKH \parallel$  to  $LO$ , and  $GI \parallel LM$ ; and the thing is done.

For  $AN$  is  $\perp$  to the tangent  $LN$ , and therefore to its  $\parallel$   $GH$ . And, because of the parallels,  $GI$  to  $AK$  is the same ratio as  $LM$  to  $AN$ , that is, the given ratio by the construction.



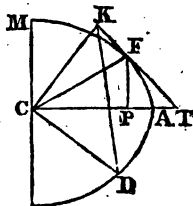
**X. QUESTION 721, by Mr. Mic. Taylor.**

To find the longest tangent that can be drawn to a given ellipse: the length of the tangent being determined by the point of contact and the perpendicular let fall on the tangent from the centre.

*Answered by the Rev. Mr. Wildbore.*

**Construction.** Let  $CA$ ,  $CM$  be the semi-axes of the given ellipsis; take the semi-conjugate  $CD$  a mean proportional to  $CA$ ,  $CM$ , and  $\parallel$  thereto draw the tangent  $KF$ , perpendicular to which let fall  $CK$ ; then is  $FK$  a maximum.

For by conics  $CA^2 + CM^2 = AM^2 = CD^2 + CF^2 = CD^2 + CK^2 + KF^2$  and is given, therefore  $KF^2 = CD^2 + CK^2 + KF^2$  and is given, therefore  $KF^2$  will be a maximum when  $CD^2 + CK^2$  is a minimum; or because, by conics,  $2CD \times CK = 2CA \times CM$ , which is given, therefore  $CD^2 + CK^2 = 2CD \times CK$  and consequently  $CD - CK$  is a minimum; therefore  $CD = CK$ , and  $CA \times CM = CD$ , and  $CA \times CM = CD \times CK = CD^2$  as per construction.



**Corollary.** If  $KF$ , instead of a maximum, be given; then  $CD^2 + CK^2 = KD^2$  is given; consequently  $CD^2 + CK^2 \mp 2CD \cdot CK$  is given, and therefore both  $CD - CK$  and  $CD + CK$  are given, and consequently both  $CD$  and  $CK$  are given.

*The same answered by Mr. Tho. White.*

Put  $CA = t$ ,  $CM = c$ , and  $z = CD$ . Then, per conics,  $tc \div z = CK^2$  and  $t^2 + c^2 - z^2 = CF^2$ ; therefore,  $KF^2 = CF^2 - CK^2 = t^2 + c^2 - z^2 - t^2 c^2 \div z^2$  a maximum, or  $z^2 + t^2 c^2 z^{-2}$  a maximum, this put in fluxions,

&c. we find  $z = \sqrt{tc} = CD$  a mean between the two axes ; which gives the position of the tangent.

*Corollary.* From this value of  $CD$ , the conjugate to  $CF$ , when the tangent is a maximum, are deduced the following curious values of the other lines in the figure ; where, besides the lines before-mentioned,  $FP$  is  $\perp CA$ , and  $T$  is the intersection of the tangent and axis.

$$1. CF = t \sqrt{\frac{t}{t+c}}.$$

$$2. PF = c \sqrt{\frac{c}{t+c}}.$$

$$3. TP = c \sqrt{\frac{t}{t+c}}.$$

$$4. CF = \sqrt{\frac{t^3 + c^3}{t+c}}.$$

$$5. CK = \sqrt{tc} = CD \text{ the conjugate to } CF.$$

$$6. FK = t - c, \text{ the difference between the axis.}$$

$$7. TF = c = CM. \text{ And } 8. TK = t = CA.$$

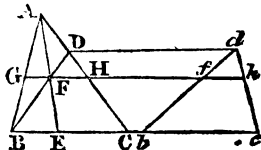
*N. B.* Many curious relations between hyperbolic and elliptic arcs and tangents, as discovered by Mr. *Larden*, may be seen in the Philosophical Transactions for the years 1771 and 1775.

#### XI. QUESTION 722, by the Rev. Mr. Lawson.

Having given two triangles on the same base, or on equal bases in the same right line, but of unequal altitudes ; it is required to draw a line parallel to the common base cutting the other sides of the two triangles so, that the parts of it intercepted between the said other two sides in each triangle may be in a given ratio.—*N. B.* This question has formerly been proposed elsewhere, but it is here re-proposed in order to receive different answers to it.

*Answered by Mr. Geo. Beck.*

Let  $ABC, DBC$  be the two triangles. Divide the common base  $BC$  in  $E$ , so that  $BE : EC :: P : Q$ , the given ratio ; join  $AE$ , cutting  $BD$  in  $F$  ; through which draw  $GH$  the  $\parallel$  to  $BC$  required.—*For*, by reason of the parallels,  $GH : FH :: BE : EC :: P : Q$ , by the construction.

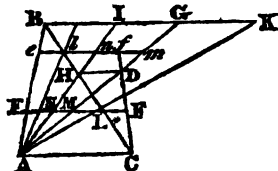


*Scholium.* If the less triangle have not the position  $DBC$ , but some other as  $dbc$  ; let this latter be reduced to the former by drawing  $db \parallel ac$ , and joining  $nb$  ; then proceed as

before. For every two sections  $fh, fh$ , in the same parallel, are equal, when the bases  $bc, bc$  are equal.

*The same answered by the Rev. Mr. Lawson.*

Let  $ABC$  and  $ADC$  be the two triangles; and suppose the thing done, viz.  $EMLF$  so drawn, that  $EL : MF :: R : s$ , the given ratio, or  $el : mf :: r : s$ . Let  $DH$  and  $BK$  be drawn  $\parallel$  to the common base  $AC$ , and let  $AH$  be drawn and produced to meet  $BK$  in  $I$ , and let it cut  $EF$  in  $N$ ; also, let  $AD$  be produced to meet the same  $BK$  in  $G$ , and  $AL$  meet it in  $K$ . Then because the triangles  $AHD$ ,  $CHD$  are upon a common base and between the same parallels,  $NM = LF$ , and either by adding in common  $ML$ , or subtracting the common part  $nf$ , we have  $NL = MF$ , therefore  $EL : NL :: EL : MF$ . But  $EL : NL :: BK : IK$ , therefore the ratio of  $BK : IK$  is given, and the points  $B$  and  $I$  are given, therefore the point  $K$  is given, and  $ALK$  will be given in position, and the point  $L$  will be given, and consequently  $EMLF$  will be given in position. Hence this



*Construction.* Through  $D$ , the vertex of the less triangle, draw a parallel to the base, and let it meet the nearest side of the greater triangle in  $H$ ; join  $AH$ , and produce it to meet an indefinite  $\parallel$  to the base drawn through the vertex of the greater triangle in  $I$ ; then on this indefinite  $\parallel$  take  $BK$  to  $KI$  in the given ratio, and  $AK$  being joined, its intersection with  $BC$  will determine the point  $L$  and the position of  $EMLF$ . The demonstration is evident from the analysis.

### XII. QUESTION 723, by Mr. Thomas Barker, of Holton.

To find when the three concentric hands of a clock make all equal angles with each other, or have such positions as to divide the circle into three equal parts (supposing them to begin to move all together from a conjunction), in each of these two cases, viz. 1st, supposing that for every time the first hand moves round, the second hand goes 10 times, and the third goes 100 times round; and 2ndly, for the case of the usual hour, minute, and second hands of a clock (if possible) wherein the first goes round in 12 hours, the second in one hour, and the third in one minute.

*Answered by Mr. Henry Clarke.*

Let 1,  $a$ , and  $b$ , denote the ratio of the velocities of the three hands, so that while the first goes once round, the second goes  $a$  times, and the third goes  $b$  times round; also, let 1 denote the whole circumference,  $x$  the part of it passed over by the first,  $y$  the entire or complete revolutions made by the second, and  $z$  those made by the third

hand; so shall  $y + \frac{1}{3} + x$  be the whole space moved over by the second hand, and  $z + \frac{2}{3} + x$  the space passed over by the third. But the three spaces passed over are in proportion as 1,  $a$  and  $b$ ; therefore we have

$$\left\{ \begin{array}{l} ax = y + \frac{1}{3} + x \\ bx = z + \frac{2}{3} + x \end{array} \right\}, \text{ hence } x = \frac{y + \frac{1}{3}}{a-1} = \frac{z + \frac{2}{3}}{b-1}; \text{ from this last}$$

equation is found  $z = \frac{b-1}{a-1} \times (y + \frac{1}{3}) - \frac{2}{3} = \frac{b-1}{a-1} \times \frac{3y+1}{3} - \frac{2}{3}$ , where both  $y$  and  $z$  must be whole numbers.

Now, in the first case, when  $a = 10$ , and  $b = 100$ , this general equation becomes  $z = \frac{99}{9} \times \frac{3y+1}{3} - \frac{2}{3} = 11 \times \frac{3y+1}{3} - \frac{2}{3} =$

$11y + 3$ ; where  $y$  may be either 0 or any whole number, and  $z$  will always be the whole number  $11y + 3$ ; and the three hands will come into the required position in some point of every revolution of the second of the hands.

The general value of  $x$  is  $\frac{y + \frac{1}{3}}{a-1}$  or  $\frac{3y+1}{27}$ ;

and taking  $y$  severally = to 0, 1, 2, 3, &c. the corresponding values of  $x$  will be  $\frac{1}{27}, \frac{4}{27}, \frac{7}{27}, \frac{10}{27}$ , &c. which are the whole spaces passed over by the first hand at the time of each of the proposed positions happening. So that, from each time till the next, the first hand will pass over  $\frac{3}{27}$  or  $\frac{1}{9}$  of the circle.

Again, for the second case, take  $a = 12$ , and  $b = 12 \times 60 = 720$ , so shall  $z (= \frac{b-1}{a-1} \times \frac{3y+1}{3} - \frac{2}{3})$  become  $\frac{719}{11} \times \frac{3y+1}{3} - \frac{2}{3} = \frac{2157y + 697}{33}$ , which must be a whole number when  $y$  is = 0,

or some integer. Now, by division,  $\frac{2157y + 697}{33}$  is =  $65y + 21 + \frac{12y+4}{33}$ ; therefore both  $\frac{12y+4}{33}$  and  $y$  must be integers.

To find  $\frac{12y+4}{33}$  an integer, put it =  $c$ , then is  $y = \frac{33c-4}{12} =$

$2c + \frac{9c-4}{12}$  an integer; therefore  $\frac{9c-4}{12} = d$  an integer, hence

$c = \frac{12d+4}{9} = d + \frac{3d+4}{9}$  an integer; therefore  $\frac{3d+4}{9} = e$  an

integer, hence  $d = \frac{9e-4}{3} = 3e - \frac{4}{3}$  an integer, which is impos-

sible, because 4 is not divisible by 3. Therefore  $\frac{2157y + 697}{33}$  can-

not be made a whole number when  $y$  is a whole number, and consequently in this case the three hands can never come into the proposed position.



= the logarithmic tangent of  $31^\circ 45' = \frac{1}{2}SL$ ,  $x$  = the logarithmic tangent of  $\frac{1}{2}SN$ ,  $c$  = the natural tangent of  $22^\circ 30'$  the course, and  $g$  = that of  $51^\circ 38' 9''$ : Then it is proved by writers on Loxodromics, that  $(10000c \div g) \times (\tau - x) = HK$  expressed in minutes of a degree; which reduced into time is the difference of time between the two meridians  $SL, SN$ ; and therefore being taken from  $7^h 30^m$  will give the time of sun-setting at  $N$ , or being taking from  $1^h 30^m$  will give the ascensional difference at  $N$  in time. Hence  $1^h 30^m \times 15 = 22^\circ 30'$ , and  $22^\circ 30' - (10000c \div 60g) \times (\tau - x) =$  the ascensional difference at  $N$  in degrees and minutes of motion. And if  $a$  = the tangent of  $23^\circ 28'$ , and  $t$  = that of  $NK$  the complement of  $SN$ , we have  $at$  = the sine of this same ascensional difference. Whence, by the well-known method of trial-and-error, the latitude  $NK$  comes out =  $35^\circ 35'$ . And as the cosine of  $22^\circ 30'$  the course : 545 minutes the difference of latitude :: radius : 590 geographical miles, the distance run, which is therefore at the rate of 41.16 miles an hour.

*The same answered by Mr. Mic. Taylor.*

First, as radius : sine of  $12^\circ 30'$  (= the ascensional difference or  $50^m$  time) :: co-tangent  $23^\circ 28'$  (declination) : tangent of  $26^\circ 30'$  the latitude at sun-rising. Put  $t$  = tangent of  $23^\circ 28'$ ;  $z$  = tangent of latitude at sun-set;  $a = 3437\frac{1}{4}$ , the reciprocal of an arc of  $1'$ ;  $b = 1649.9$  the meridional parts in  $26^\circ 30'$  the first latitude, and  $d$  = the tangent of  $22^\circ 30'$  the course. Then as  $1 \div t : z ::$  radius :  $tz$  the sine of the ascensional difference at sun-set. But the arc, whose sine is

$tz$ , is  $tz + \frac{t^2 z^3}{3.2} + \frac{1.3t^2 z^5}{5.2.4} + \frac{1.3.5t^7 z^7}{7.2.4.6}$  &c. therefore  $a \times : tz + \frac{t^2 z^3}{3.2}$ , &c. will express the number of minutes in the arc of ascensional

difference.——Again, the fluxion of the meridional parts in the latter latitude is known to be  $\frac{az}{\sqrt{(1+z^2)}} = az \times : 1 - \frac{z^2}{2} + \frac{1.3z^4}{2.4}$  &c.

whose fluent  $az \times : 1 - \frac{z^2}{3.2} + \frac{1.3z^4}{5.2.4}$  &c. is the meridional parts

in this latitude, and therefore  $-b + a \times : z - \frac{z^3}{3.2} + \frac{1.3z^5}{5.2.4} -$

$\frac{1.3.5z^7}{7.2.4.6}$  &c. expresses the meridional difference of latitude. But as

rad. : tang. course :: meridional diff. lat. : difference of longitude;

therefore  $-bd + ad \times : z - \frac{z^3}{3.2} + \frac{1.3z^5}{5.2.4}$  &c. is the difference

of longitude in minutes of a degree. Then the sum of these two series will be the minutes in  $22^\circ 30'$ , which answers to  $1^h 30^m = 7^h 30^m - 6^h$ ,

that is,  $22 \times 60 + 30 = 1350 = -bd + ad \times : z - \frac{z^3}{3 \cdot 2} + \frac{1 \cdot 3z^3}{5 \cdot 2 \cdot 4} \&c. + a \times : tz + \frac{t^3 z^3}{3 \cdot 2} + \frac{1 \cdot 3t^3 z^3}{5 \cdot 2 \cdot 4} \&c. \text{ Or } \frac{1350 + bd}{a}$   
 $= (t + d) \cdot z + (t^3 - d) \frac{z^3}{3 \cdot 2} + (t^3 + d) \cdot \frac{1 \cdot 3z^3}{5 \cdot 2 \cdot 4} + (t^3 - d) \frac{1 \cdot 3 \cdot 5z^5}{7 \cdot 2 \cdot 4 \cdot 6} \&c. \text{ Then either by reverting this series, or applying to it the method of trial-and-error, will be found } z = .7156 \text{ the tangent of } 35^\circ 35\frac{1}{4}' \text{ the latitude at sun-set.}$

Then, as radius to sec. of course  $:: 545\frac{1}{4}$  the difference of latitude  $: 590.17$  the distance run by the ship; which divided by  $14\frac{1}{2}$  hours, the time of sailing, gives  $41.175$  miles per hour, the rate required.

#### THE PRIZE QUESTION, by Peter Puzzlem.

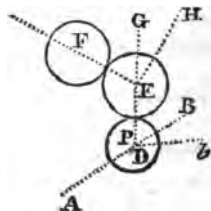
If the one of two given balls, touching each other and resting on an horizontal plane, be struck by a third given ball moving with a given velocity upon that plane in any given direction oblique to the line passing through the centres of the two quiescent balls; how will the three balls move after the stroke, supposing them all perfectly hard?

*Answered by Peter Puzzlem, the Proposer.*

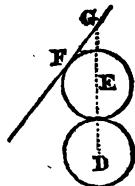
Let E, F be the two quiescent balls; EF the line joining their centres; D the ball going to strike E, in the direction DEG, with the velocity  $g$ ; and EH the direction of E after the stroke. Then denoting the sine and cosine of the given  $\angle FEG$  by  $s$  and  $c$  respectively, and the sine and cosine of the required  $\angle GEH$  by  $m$  and  $n$  respectively, radius being 1; if  $u, v$ , and  $w$  be respectively put for the velocities of D, E, and F after the stroke; we shall have  $nv = n, (cn - ms) \times v = w, Du + Env + Fcw = dg$ , and  $Env = Fsw$ ;  $cn - ms$  being the cosine of the  $\angle FEH$ : seeing that the velocity of E in the direction EG will be  $=u$ , and in the direction EF equal to  $w$ ; and that the quantity of motion in all the balls, in any certain direction, will be the same after the stroke as before.

Whence  $m = Fcs \div \sqrt{(E^2 + (F^2 + 2EF) \times s^2)}$ ,  $n = (E + Fs^2) \div \sqrt{(E^2 + (F^2 + 2EF) \times s^2)}$ ,  $u = dg \times (E + Fs^2) \div N$ ,  $v = dg \sqrt{(E^2 + (F^2 + 2EF) \times s^2)} \div N$ , and  $w = DEcg \div N$ ; the balls D and F moving after the stroke in the directions DEG and EF respectively, and N being put for the quantity  $DE + E^2 + EF + DFs^2$ .

The expression for the value of  $v$  admits of a maximum; D, E, F, and  $g$  being given, and the  $\angle FEG$  variable; provided  $E \times (DF + 3EF$



$+2e^2 + f^2) \div DF \times (2E + F)$  (the resulting value of  $s'$ ) be less than 1 : and it is particularly remarkable, that if  $F$  be infinite, *i.e.* if instead of the ball  $F$  there be an immovable plane  $FG$  perpendicular to the horizon ;  $D$ ,  $E$ , and  $g$  being given ;  $E$  after the stroke will have the greatest velocity possible when the cosine of the  $\angle EGF$  is  $= \sqrt{(E \div D)}$  ; its velocity then being  $\frac{1}{2}g\sqrt{(D \div E)}$ , with which it will proceed along continually touching the said plane  $FG$ .

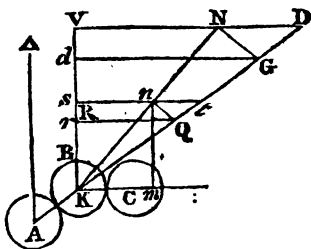


It may happen that  $D$ , after so striking  $E$ , may also strike  $F$ . When such second stroke will take place may be found by trigonometry ; and the consequence of it may be computed by a well known method, only two bodies being then concerned.

If the direction of the striking ball  $D$ , when it comes to impinge against  $E$  at the point  $F$ , be not  $DFG$ , but oblique to it, as  $ADB$  : Let the sine and cosine of the  $\angle BDE$ , to the radius 1, be  $p$  and  $q$ , respectively ; and  $f$  being the first velocity of  $D$  in that direction  $ADB$ , let  $g$  be taken  $= qf$  in the values of  $v$  and  $w$  above ; and then those velocities will be rightly assigned for this case. Moreover, drawing  $DB$  so that the co-tangent of the  $\angle BDE$  may be  $= (E + fs^2) dq \div np$ ,  $DB$  will be the direction in which  $D$  will move after the stroke, and its velocity will then be  $= (f \div N)\sqrt{(E + fs^2) \cdot D^2q^2 + N^2p^2}$ .

*The same answered by the Rev. Mr. Wildbore.*

Suppose  $B$  and  $C$  to be the two quiescent balls ; and that the body  $A$  in direction  $AD$ , with a velocity represented by the line  $KD$ , strikes  $B$  ; then the momentum in direction  $KD$  is  $KD \times A$ , and in the direction  $A\Delta \parallel$  to  $KV \perp$  to  $KC$ , it is  $KV \times A$ , which momentum cannot be altered by any action of  $C$  upon  $B$ , because  $C$  being impelled in direction  $KC$  can move in no other, and therefore can have no-momentum in direction  $KV \perp$  to  $KC$ . Also because  $B$  can only act on  $C$  in direction  $KC$ , their velocity in that direction must be the same ; and  $A$  impelling  $B$  in direction  $KD$ , their velocity in that direction must be the same. — Now  $A+B : A :: KV : KR$  the velocity of  $A$  and  $B$  after impulse in direction  $KV$  if  $C$  were away ; but since by the resistance of  $C$ ,  $B$  is turned out of the direction  $KD$ , and moves in one nearer to  $KV$  than before, if its velocity in direction  $KV$  be thereby increased, and consequently its momentum, those of  $A$  must be diminished, otherwise the momentum in that direction cannot continue the same : let therefore  $Ks$  be such a velocity of  $B$  ; then since the momentum  $Ks \times B$  gained by  $B$





*Questions proposed in 1778, and answered in 1779.*

## I. QUESTION 727, by Mr. Wm. Purver.

In Ely's fertile isle a nymph resides,  
 In whose accomplish'd mind each grace abides;  
 In her fair face kind nature's hand displays  
 Majestic sweetness, soft bewitching rays;  
 Love from her eye directs each pointed dart,  
 And melt's to fondness ev'ry feeling heart.

From hence, \* the name of this engaging fair.

In your learn'd page, kind Artists pray declare.

$$\begin{array}{rcl}
 * v^2y^3 - 2v^2y^3 + vy^4 = 28350 = a & & \\
 v^2y^3 - v^2y^3 = 135056250000 = b & & \\
 vx^2 + vy^2 = 4396 = c & & \\
 x^2y + vy^2 = 2155 = d & & \\
 u + x = 2w, \text{ and } w = z. & & 
 \end{array}$$

To find the values of  $u, v, w, x, y,$  and  $z$ , which denote the places of the letters in the alphabet that compose her name.

*Answered by Mr. J. Scott, Schoolmaster, Woolwich.*

Divide the square of the second equation by the first, and the quotients will be  $v^2y^4 = bb \div a$ ; therefore  $vy^4 = \sqrt[3]{(bb \div a)} = 350$ . Divide now the first given equation by this last, and the root of the quotient will give  $v - y = \sqrt[3]{(a' \div b)} = 9$ . Hence  $v = 9 + y$ ; this substituted for  $v$  in the former equation, gives  $y^3 + 9y^2 = 350$ ; hence  $y = 5$ , and therefore  $v = 9 + y = 14$ . These values of  $v$  and  $y$  being substituted in the 3d and 4th given equations, they become  $14w^2 + 350 = c$ , and  $5x^2 + 350 = d$ ; hence we easily find  $w = 17 = z$ , and  $x = 19$ . Consequently  $u = 2w - x = 15$ , and the fair lady's name is PORTER.

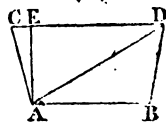
## II. QUESTION 728, by Mr. Wm. Reynolds.

The Cooper a conical tub made for Joan,  
 Whose dimensions and solid content may be known  
 From the data subjoin'd, by those who will try,  
 And discover the value of  $x$  and of  $y$ .

$x + y$  = greater diameter,  $\frac{1}{2}xy + 4$  = diagonal,  
 $x - y + 2$  = less diameter,  $\frac{1}{2}12x^2 + y^2 + y + 1.008$  = solidity in inches,  
 and the difference of the area of its two ends =  $452.3904$  inches.

*Answered by Mr. Wm. Francis, of Shinfield.*

From the given equation, we have  $x + 1 = DE$  = half the sum of the diameters, and  $y - 1 = CE$  = half their difference, and therefore  $\sqrt{((\frac{1}{2}xy + 4)^2 - (x + 1)^2)} = AE$  the depth; hence, by the question,  $((y - 1)^2 + 3(x + 1)^2) \sqrt{((\frac{1}{2}xy + 4)^2 - (x + 1)^2)}$



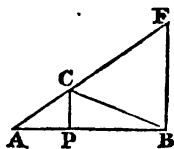
$-(x+1)^3 \times .2618 = (\text{the solidity} =) 12x^3 + y^3 + y + 1.008$ , and  $(x+y)^3 - (x-y+2)^3 = 4xy - 4x + 4y - 4 = 452.3904 \div .7854 = 576$ , or  $(x+1) \cdot (y-1) = 576 \div 4 = 144$ ; hence  $x+1 = 144 \div (y-1)$ , and  $x = (145-y) \div (y-1)$ , which being substituted in the equation of the solidity, there will result an equation, the solution of which gives these two values of  $y$ , viz. 5 and 8.3989. With the former of these two values the measures of the two diameters, the depth, and solidity, are 41, 32, 15, and 1533.0008 respectively. But from the latter value of  $y$  the same dimensions are found to be 26.8611, 12.0634, 29.1049, and 9075.7439.

### III. QUESTION 729, by Mr. Alex. Rowe.

In a plane triangle, given the sum of the base and longest side equal to 60, and the product of the cube of the base and square of the same side a maximum; to find the other side, so that the time of descent of a heavy body down it may be a minimum.

*Answered by Mr. Joseph James.*

When the product of any powers of any number of quantities is a maximum, their sum being given, then are those quantities directly as the exponents of their respective powers: (See Cor. 2, page 80, Hutton's *Miscellanea Mathematica*). Hence the base is to the longer side, as 3 is to 2, their sum being 60; consequently the base and that side are 36 and 24 respectively. Then the latter part is similar to question 468 in the Diary for 1760, and may be constructed in the same manner, viz. making AB and BF  $\perp$  to each other, and = the given base and side, joining AF, and applying to it  $BC = BF$ . For then ABC is the triangle. The same manner of calculation also gives the 3d side  $AC = (AB^3 - BC^3) \div \sqrt{(AB^3 + BC^3)} = 16.641$ .



*The same answered by Mr. John Willes, of Marsh.*

Put  $a = 60$ , and  $x =$  the base AB; then  $a - x = BC$ . Hence  $AB^3 \times BC^2 = a^3 x^3 - 2ax^4 + x^5 =$  a maximum, the fluxion of this made = 0, there results  $5x^3 - 8ax = 43x^3$ , the proper root of which is  $x = \frac{3}{2}a = 36 = AB$ . Consequently  $BC = 24$ . Put now  $b = 36 = AB$ ,  $c = 24 = BC$ , and  $z = \sin \angle ABC$ ; also draw the  $\perp$  CP. Then  $CP = cz$  and  $CA = \sqrt{(b^2 + c^2 - 2bc \sqrt{(1-z^2)})}$ . But the time of descent down CP is as  $\sqrt{cz}$ , and the time in CP is to the time in CA ::  $CP : CA$  therefore the time in CA is as  $\sqrt{(b^2 + c^2 - 2bc \sqrt{(1-z^2)}) \div cz}$ , which must therefore be a minimum, the fluxion of which being put = 0, we obtain  $z = (b^3 - c^3) \div (b^3 + c^3)$ . And hence  $AC = (b^3 - c^3) \div \sqrt{(b^3 + c^3)} = 16.641006$ .

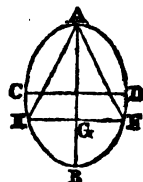
## IV. QUESTION 730, by Mr. Joseph James.

Given the revolving axe of a prolate spheroid equal 30, also the difference between its surface and solidity equal 19417.1735: Required the solidity of that inscribed cone whose convex superficies shall be a maximum?

*Answered by Mr. Tho. Robinson, of Biddick.*

Put  $r = CD = 30$ ,  $p = 3.1416$ ,  $x = AB$ . Then (by Rule 5, page 26C, Hutton's Mensuration)  $prx\sqrt{1-(x^2-r^2)} \div 3x^3$  is the surface of the spheroid, and  $\frac{1}{2}pr^2x$  is the solidity, therefore  $\frac{1}{2}pr^2x - prx\sqrt{1-(x^2-r^2)} \div 3x^3 = 19417.1735$ . Hence is found  $x = 50$  nearly  $= AB$ .

Again, put  $AB = 50 = t$ ,  $CD = 30 = c$ , and the height of the cone  $AG = z$ . Then  $BG = t - z$ , and  $t^2 : c^2 :: AG \times GB : GE^2 = (tz - z^2) \times c^2 \div t^2$ , also  $AE = \sqrt{(AG^2 + GE^2)} = \sqrt{(z^2 + (tz - z^2) \times c^2 \div t^2)}$ ; then  $p \times AE \times EG = (pc \div t^2) \times \sqrt{(tz - z^2) \times (tc^2z + t^2z^2 - c^2z^2)}$  is a maximum, the fluxion being equated to 0, there results  $(t^2 - c^2) \cdot 4z^2 - (t^2 - 2c^2) \cdot 3tz = 2t^2c^2$ , from which is found  $z = 35.9595$ ; and thence the solidity  $= 6844.5$ .



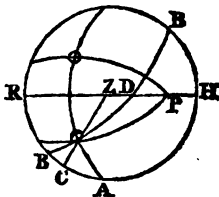
## V. QUESTION 731, by Mr. Tho. Barker, of Holton.

To find the position of a great circle passing through the moon's horns with respect to the horizon of a place in latitude  $18^\circ 46' 30''$  south, and longitude  $197^\circ 43'$  east from Greenwich, at 1 hour 11 minutes in the afternoon, by the time at that place, on the 13th of June 1774; and also the distance, in degrees, &c. of the nearest cusp from the lowest point of the moon's limb supposing the places of the luminaries to be rightly assigned in the Nautical Almanack.

*Answered by Nauticus.*

The longitude turned into time is  $13^h 11^m$ , which taken from the given time  $1^h 11^m$  leaves  $12^h$  for the time at Greenwich on the 12th day; at which time the R. A. of the  $\odot$  was  $124^\circ 14'$ , and her declination  $16^\circ 10' N.$ ; those of the  $\odot$  being  $81^\circ 13'$  and  $23^\circ 13' N.$ ; and hence the R. A. of the mid-heaven was  $98^\circ 58'$ .

Let then  $EBHA$  represent the horizon of the place,  $z$  its zenith, and  $BNM$  the meridian. Make  $zP$  = the semi-tangent of the co-latitude, and describe the hour circles  $P\odot$ ,  $P\zeta$  to make angles with the meridian



= those of the sun and moon. Make the arc  $P\odot$  = the sun's distance from the elevated pole, and  $P\mathcal{C}$  = that of the moon; through  $\odot$  and  $\mathcal{C}$  describe the great circle  $\odot(A$ , and through its pole  $\mathfrak{B}$  the great circle  $\mathfrak{B}(B$  cutting the horizon in  $\mathfrak{B}$ , which will manifestly pass through the cusps of the moon; and the angle at  $\mathfrak{B}$  will express its position with respect to the horizon of the place.

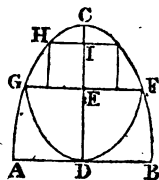
*Calculation.* In the  $\triangle P\odot\mathcal{C}$  are given  $P\odot$ ,  $P\mathcal{C}$ , and the  $\angle P$ , to find the  $\angle\mathcal{C}$ , whose complement is the  $\angle P(D$ . And, in the  $\triangle P\mathcal{C}(z$  are known  $P\mathcal{C}$ ,  $Pz$ , and the  $\angle P$ , to find  $\mathcal{C}z$  and the  $\angle\mathcal{C}$ ; from which taking the  $\angle P(D$ , there will remain the  $\angle Z(D$ , the distance of the nearest cusp of the lowest point from the moon's circumference. Lastly, in the right-angled  $\triangle(\mathfrak{B}c$ , will be given  $\mathcal{C}$  and the  $\angle\mathcal{C}$ ; and therefore as radius : cosine  $\mathcal{C} :: \sin \angle\mathcal{C} : \cosine \angle \mathfrak{B}$ , which is that made by the horizon and great circle passing through the moon's horns.

VI. QUESTION 732, by Mr. Ralph Thomson, of *Witherly Bridge*.

Required the difference of the areas between the greatest curve whose equation is  $px^3 = y^4$ , that can be inscribed in a curve whose bounding ordinate = 20, greatest abscissa = 30, and equation  $p^2x = y^3$  (the vertex of the former in the middle of the base of the latter); and the greatest parallelogram that can be inscribed between the base of the inscribed curve and the vertex of the circumscribed curve.

*Answered by Mr. Tho. White, of Hexham.*

From the given equation  $px^3 = y^4$ , is got  $y = \sqrt[4]{px^3}$ , and therefore  $yx = \sqrt[4]{px^3} \times x = \sqrt[4]{px^7}$ , whose fluent  $\frac{7}{4}\sqrt[4]{px^3} = \frac{7}{4}xy$  is half the area of the inscribed curve  $GDF$ , which therefore is as  $GE \times ED$ . But if  $a = 30 = CD$ ,  $b = 10 = AD$ , and  $x = CE$ ; then, by the nature of the circumscribed curve  $AGC$ , we have  $a : x :: b^3 : GE^3$ , hence  $GE = b^3 \sqrt[3]{(x \div a)}$ ; therefore  $GE \times ED = b^3 \sqrt[3]{(x \div a)} (a - x)$  is a maximum, its fluxion put = 0, gives  $x = \frac{1}{2}a = 5 = CE$ ; therefore  $ED = 25$ ,  $2GE = 2b^3 \sqrt[3]{\frac{1}{2}} = 13.97654 = GF$ , and  $\frac{1}{2}GF \times ED = \frac{3}{2}ab^3 \sqrt[3]{\frac{1}{2}} = 199.665 =$  the area  $GDFG$ .



Again, like as  $CE$  is  $\frac{1}{2}CD$  when  $GE \times ED$  is a maximum, so will  $CI = \frac{1}{2}CE$  when  $HI \times IE$  is a maximum, therefore  $IE = \frac{1}{2}CE = \frac{5}{2} = \frac{5}{2}$ , and  $HI = GE \sqrt[3]{\frac{1}{2}} = b^3 \sqrt[3]{\frac{1}{2}}$ ; hence  $2HI \times IE = \frac{5}{2}ab^3 \sqrt[3]{\frac{1}{2}} = 40.6966 =$  the area of the inscribed rectangle  $2HE$ . Therefore the difference required is  $158.968$ .

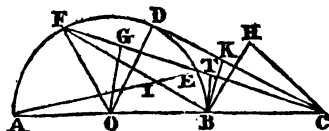
VII. QUESTION 733, by Mr. Henry Clarke.

From a given point  $c$  in the diameter  $AB$  of a semicircle produced, let a tangent  $cd$  be drawn, and another line from the point  $c$  cutting the

periphery in E and F, so that the sine of the angle BCE may be a fourth proportional to CB (considered as radius) and the sines of BCD and a given angle P. I say that the acute angle AIF, or BIE, formed by the diagonals AE, BF, is equal to the given angle P. Required the demonstration?

*Answered by Mr. Henry Clarke.*

Construct the figure as per question, and make the  $\angle BCH =$  the given  $\angle P$ ; draw BH, BK, BT  $\perp$  CH, C, CF; then supposing  $BC : BH :: BK : BT$ , we are to prove that the  $\angle BCH$  is  $= \angle BIE$  or AIF. In order to which,



Join OF and OD, and draw OG  $\perp$  CF, O being the centre of the circle.

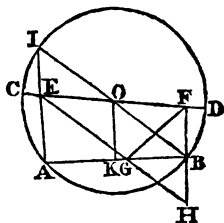
Now the angles at D and G being right as well as K and T, it will be, by similar triangles,  $CB : CO :: BK : OD$  (or)  $BT : OG$ , therefore by altern.  $BK : BT :: OF : OG ::$  (by construction)  $BC : BH$ ; but the  $\angle G = \angle H$ , therefore the triangles FGO, BCH are equiangular (Euc. VI. 7), and  $\angle OFG = \angle BCH$ ; but  $\angle OFB = \angle OBF = \angle AEF$  or IEF, therefore  $\angle IEF + \angle IFE = \angle OFE = \angle BIE$ , or  $\angle AIF = \angle BCH$ .

#### VIII. QUESTION 734, by Lieutenant Glenie, A. M.

If AB be any chord in a circle, from the extremes of which are drawn perpendiculars to it AE, BF, cutting any diameter CD in E and F; and if from E and F there be drawn EG, FG meeting in the chord AB in G, and making equal angles with it: Then shall the sum of EG and GF be equal to the diameter of the circle. Required the demonstration?

*Answered by Philarithmus.*

Draw the diameter BOI, which will meet AE produced in I because A is a right angle; and produce EG and FH to meet in H. Then the  $\angle BGH = \angle AGE = \angle BGF$ ; and, GB being  $\perp$  FH, the triangles GBF, GBH are equal in all respects, therefore  $BH = BF$ , and  $GH = GF$ ; and because of the parallels AI, FH, and  $OI = OB$ , the triangles OEI, OFB are mutually equal, and therefore  $OE = OF$ ; hence  $FO : FE (2FO) :: OB : EH (2OB)$ ; but  $CD = 2OB$ , therefore  $EH$  or  $EG + GF = CD$ .



*The same answered by Mr. Nathan Parnel.*

Draw the radius OB, produce EG and FB, to meet in H, and draw OK  $\perp$  AB. Then  $AK = KB$  (Euc. III. 3), and because of the parallels AE,

EO, BF, it is OE = OF; and since  $\angle EGH = AGE = BGF$ , and  $GB \perp FH$ , therefore  $GH = GF$ , and  $EH = BF$ ; consequently  $OB \parallel EH$ , and  $EG + GF = EG + GH = EH = 2OB = CD$ .

IX. QUESTION 735, by Mr. John Willis, of Marsk.

Walking leisurely one evening last summer at the rate of two and a half miles an hour, I took up a small inflexible rod of two and a half feet long, with a small ball fixed at one end, and caused it to revolve 20 times successively in a conical motion, and in describing each revolution it circumscribed the greatest cone possible. What distance did I walk during the 20 revolutions?

*Answered by Mr. Wm. Walton.*

Let  $a = 2\frac{1}{2}$  feet the length of the rod,  $b = 16\frac{1}{13}$ ,  $c = 3.1416$ , and  $x =$  the height of the cone. Then will  $a^2 - x^2 =$  the square of the radius of its base, and by the question,  $a^2x - x^3 =$  a maximum, this in fluxions, &c. gives  $x = a\sqrt{\frac{1}{3}}$  the height of the cone. Then by centripetal forces,  $\sqrt{b} : \sqrt{2x} :: c : c\sqrt{(2x \div b)} = c\sqrt{(2a \div b\sqrt{3})} = c\sqrt{(20\sqrt{3}) \div 193} = 1.330968''$  the time of one revolution; consequently as 1 hour :  $20 \times 1.330968 :: 2\frac{1}{2}$  miles :  $32.53477$  yards the distance that might be walked by a person while the rod described 20 conical revolutions.

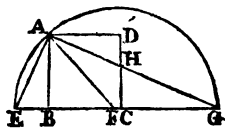
*Scholium.* This solution is given on the principle as if the person holding the pendulum stood still, and another person walked forward at the rate of  $2\frac{1}{2}$  miles an hour; for if the person carries the pendulum forward it will not describe a cone, contrary to the supposition, and it would be a quite different question.

X. QUESTION 736, by the Rev. Mr. Lawson.

Having given a square ABCD; it is required, from one of its angular points A, to draw a line AEG meeting the opposite sides DC in E and BC produced in G, in such a manner that the exterior triangle HCG, thereby formed, may have a given ratio to the square.

*Answered by the Rev. Mr. Lawson.*

*Construction.* Let  $R : s$  be the given ratio. To BC, produced if necessary, apply AF such that  $R : R + s :: BC : AF$ ; with F centre and FA radius describe the semicircle EAG meeting BC produced in E and G; then join AG and the thing is done.



*Demonstration.* Join AE. Now  $R : R + s :: BC : AF :: ABCD : AB \times AF$ ; but  $ABCD =$  trapezium AECH, for  $\triangle ABE = \triangle ADH$ ; and  $AB \times AF = \triangle AEG$ ;

therefore  $R : R + S :: \text{trapezium } AECH : \triangle AEG$ , and by division,  
 $R : S :: ABCD : \triangle CGH$ .

*The same by Mr. Parnel.*

By problem 1, of the Determinate Section, divide  $CD$  in  $H$ , so that  $CH^2 : CD \times DH :: 2s : R$ , and  $H$  will be the point through which  $AG$  must be drawn. For, by construction,  $CH^2 : CD \times DH :: 2s : R$ , and by similar triangles  $CG : AD = DC :: CH : HD$ , consequently  $\frac{1}{2}CH \times CG : CD^2 :: s : R$ .

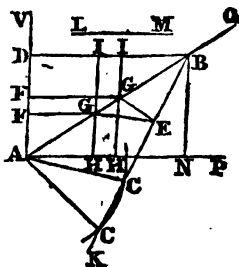
XI. QUESTION 737, by Mr. Thomas Moss.

Two right lines  $AP$  and  $Aq$  given by position, and also a point  $E$  between them; it is proposed to draw a right line from that given point  $E$ , meeting  $Aq$  in  $G$ , and drawing thence a line  $GH$  meeting  $AP$  in  $H$ , and parallel to another  $AV$  given by position, so that the sum of  $EG$  and  $GH$ , thus drawn, shall be the least possible.

*Answered by Mr. Tho. Moss.*

In geometrical problems of this kind, the solution will be most easily derived from the general construction of the problem; that is, by considering the thing sought to be a given quantity, instead of the greatest or least possible. Hence this

*Geometrical Analysis.* Conceive the thing done, when  $EG + GH$  is = a given line  $LM$ , instead of the least possible; and suppose, in  $HG$  produced,  $GI = GE$ , and  $AD \parallel AP$ , also  $AC \parallel GE$  meeting  $BE$  produced in  $C$ : It will then appear, by similar triangles, that  $AG : BA :: GI : AD$  and  $BG : BA :: GE : AC$ ; whence by equality  $GI : AD :: GE : AC$ ; but the antecedents are equal by hypothesis, therefore  $AC = AD = HG + GE$ . Whence, since the  $\angle ABC$ , and the sides  $AB$  and  $AC$  ( $AD$ ) become known, we have this easy



*Construction.* In  $AV$  take  $AD = LM$  the given sum, and draw  $DB \parallel AP$ ; draw  $BK$ , and with centre  $A$  and radius  $AD$  describe an arc meeting  $BK$  in  $C$ ; draw  $AC$  and  $EG \parallel AC$ , also  $GH \parallel AD$ , and the thing is done.

*Demonstration.* Draw  $GF \parallel DB$ , and produce  $HG$  to meet  $DB$  in  $I$ . Then, by similar triangles,  $AB : AD :: GB : GI$  ( $FD$ ), and  $AB : AD$  ( $AC$ )  $:: GB : GE$ ; hence, by equality,  $FD = GE$ , and therefore  $GH$  ( $AF$ )  $+ GE$  ( $FD$ )  $= AD =$  the given sum by construction.

*Scholium.* Since  $EG + GH$  is always taken in  $AV$ , therefore the said sum, or the radius of the circle described as above from the centre  $A$ ,

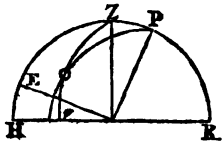
will be least when  $AC$  is perpendicular  $BK$ ; hence then we have only to find  $BEK$  so that drawing  $AC$  perpendicular to  $BK$  and equal to  $BN$  (parallel and equal  $AD$ ); which is a particular case of question 720 solved in the last Diary.

### XII. QUESTION 738, by Nauticus.

The usual method of finding the latitude at sea, is by observing the greatest altitude of the sun, and taking that for the meridian altitude: Now on March 23d, 1777, an observer, on board a ship, running N. N. W. at the rate of 10 miles an hour, made the greatest altitude of the sun's centre  $13^{\circ} 5' 15''$  on the south meridian: I would know the true latitude he was then in, and the time when he had completed his observation.

*Answered by Nauticus.*

Let  $HZPR$  be the meridian,  $P$  the pole,  $z$  the zenith,  $\odot$  the position of the sun when he had the greatest altitude; also let  $P\odot$  be a meridian, and  $z\odot$  a vertical circle. Now it is manifest that every one of the sides of the oblique  $\triangle z\odot P$  will be variable; the side  $P\odot$  being so on account of the sun's change in declination, which at the given time is  $59''$  in a minute of time;  $zP$  is variable on account of the ship's motion towards the north pole at the rate of  $10 \times \sin. 67^{\circ} 30' \div 60 = 9.238795''$  in 1 minute; and lastly,  $z\odot$  varies, not only on these two accounts, but also from the alteration of the  $\angle P$  at the rate of  $14.6884'$  in a minute of time, which is the difference between  $15'$  the measure of the diurnal rotation in a minute and  $0.3116'$  the ship's motion in long. in the same time, found from the given course with the given rate of sailing and approximate latitude. It is farther manifest that in the present instance this side is shortened by the first and last of these causes, and lengthened by the 2d; and therefore when the effect of this cause is = the sum of the effects of the other two, the sun will be the highest possible, and its altitude then must be = that given in the question.



Now by Simpson's Fluxions, Article 258,  $254$ ,  $P\odot$  ( $59'' = m$ ):  $z\odot :: \text{radius } (1) : \cosine \angle \odot$ , and  $Pz$  ( $9.23'' \&c. = n$ ):  $z\odot :: 1 : \cosine \angle z$ ; also (Article 256)  $\frac{1}{t}$  ( $881.304'' = q$ ):  $z\odot :: 1' : \sin P\odot \times \sin \angle \odot$ . Hence the alteration of  $z\odot$ , on account of the change of declination will be  $P\odot \times \cosine \angle \odot$ , and on account of the change in latitude  $Pz \times \cosine \angle z$ , and on account of the diurnal rotation and change in longitude  $\frac{1}{t} \times \sin P\odot \times \sin \angle \odot$ ; and consequently  $Pz \times \cosine z = P\odot \times \cosine \odot + \frac{1}{t} \times \sin P\odot \times \sin \odot$ . Or, putting  $s$  and  $a$  for the sines of  $P\odot$  and  $z\odot$ ,  $t$  for the cotang. of  $P\odot$ ,  $b$  for the cosine of  $z\odot$ , and  $y$  for the cosine of  $\angle \odot$ ,

we shall have  $\sqrt{(1 - yy) \div (at + by)}$  for the tangent, and therefore  $(at + by) \div \sqrt{((at + by)^2 - 1 + y^2)}$  for the cosine  $\angle z$ , and the above equation will be  $n \times (at + by) \div \sqrt{((at + by)^2 - 1 + y^2)} = my + qs \sqrt{(1 - y^2)}$ , which may be solved by any of the known methods.

But this equation being high, and its resolution, any way, very troublesome, the following approximation may be used, which will be found both expeditious and exact. The angles at  $z$  and  $\odot$ , tho' not absolutely equal, must always be both of them small, and as the cosines of small arcs differ but by small quantities, the cosines of these angles may be esteemed equal, and used for each other; and the original equation after transposing will be  $r'z \times \text{cosine } \odot - r' \odot \times \text{cosine } \odot = r' \times \text{sine } r \odot \times \text{sine } \odot$ , or  $(n - m) \times \text{cosine } \odot = qs \times \text{sine } \odot$ ; and hence  $(n - m) \div qs = \text{sine } \odot \div \text{cosine } \odot = \text{tangent } \odot = .0093886$  the tangent of  $0^\circ 32' 16''\frac{1}{2}$ . The true latitude will therefore be  $78^\circ 10' 32''$  N. and the time  $0h. 10' 12'' 16'''$  before noon, answering to the  $\angle r = 2^\circ 33' 4''$  found from the three sides now known.

*The same answered by Mr. Michael Taylor.*

Let  $z = \text{sine } rz$ ,  $x = \text{cosine } \angle r$ ,  $p = \text{sine } r \odot (88^\circ 43' 30'')$ , and  $q = \text{its cosine}$ , also  $m = \text{cosine } z \odot (76^\circ 54' 45'')$ . Then, by spherics,  $pxz + q \sqrt{(1 - z^2)} = m$ , which is the first equation, the fluxion of which must also be  $= 0$ , because  $z \odot$  was the least possible, and which gives this 2d equation  $pxz + xzp + pxz - (pp(1 - z^2) + q^2z) \div q \sqrt{(1 - z^2)} = 0$ , and in these two equations are only two unknown quantities  $x$  and  $z$ , for  $p$ ,  $q$ , and all the fluxions are known, as below.

From the approximate latitude  $78^\circ 11' 15''$ , with the given course  $22^\circ 30'$  and distance  $10'$ , we find the difference of longitude to be  $.3116'$  in a minute of time, and at this rate the  $\angle r$  increases by the ship's motion; but the same angle decreases  $15'$  per minute by the diurnal rotation; therefore  $\dot{r} = -.15 + .3116 = -.146884'$  (a) per minute. Also  $r' \odot = -.23' 36'' \div 24 \times 60 = -.59' \div 3600 = -.0155555'$ , &c. (b); and as radius (1) :  $\text{sine } 67^\circ 30' :: -.10' \div 60 :: r'z = -.1539799'$  (c). But the fluxion of an arc is to the fluxion of its sine, as radius to the cosine, hence  $\dot{z} = a \sqrt{(1 - x^2)}$ ,  $\dot{p} = -bq$ , and  $\dot{x} = -c \sqrt{(1 - z^2)}$ .

Now these values of the fluxions being substituted for them in the 2d equation, as well as the value of  $x = (m - q \sqrt{(1 - z^2)}) \div pz$  found from the first, after proper reduction there will result this equation  $a \sqrt{(z^2 + 2mq \sqrt{(1 - z^2)} - m^2 - q^2)} = bmq \div p - cq \div z + (cm \div z - b \div p) \sqrt{(1 - z^2)}$ , the root of which is  $z = .2049136 = \text{cosine of } 78^\circ 10' 32''$  the true latitude. And hence  $x = (m - q \sqrt{(1 - z^2)}) \div pz = .9990086 = \text{cosine of } 2^\circ 33' 5'' = \angle r$ , which in time is  $10' 12'' 20'''$ .

Therefore the latitude was  $78^{\circ} 10' 32''$  N. and the time  $10' 12''\frac{1}{4}$  before noon.

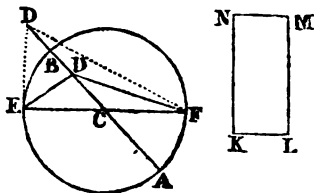
N. B. If the place differ much in longitude from Greenwich, then correct the declination according to the difference of longitude.

XIII. QUESTION 739, by the Rev. Mr. Crakelt.

On a given base to constitute a triangle, such, that the square of a line from the vertical angle to bisect the base may be to the rectangle of the sides containing the vertical angle in a given ratio; and moreover that, if a circle be described upon the given base as a diameter, cutting the bisecting line (produced if necessary) in two points, the square on the said bisecting line may also be to the rectangle under the two segments thereof, so intercepted, in another given ratio:

*Answered by Mr. Nathan Parnel.*

*Construction.* With a radius equal to half the given base describe a circle, and thro' the centre  $c$  draw in any direction the diameter  $AB$ , which (produced if necessary) cut in  $D$  (by problem 3, case 4 and 5 of Lawson's Determ. Sections) in such a manner, that  $nc^2$  may be to  $DA \times DB$  in the last given ratio: Also make the rectangle  $KLMN$  (by Simpson's Eucl. Data, prop. 58 and 88) so that its area may be to  $DC^2$  in the other given ratio, and that  $KL^2 + KN^2$  may be equal  $2bc^2 + 2cd^2$ . Apply then either of the sides as  $KL$  to the circumference at  $E$ ; lastly draw the diameter  $EF$ , and join  $D, F$ ; and  $EDF$  will be the triangle required,



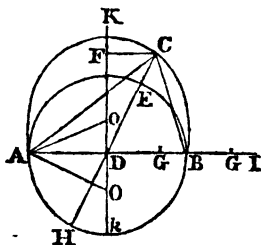
*Demonstration.* For  $ED^2 + DF^2 = 2bc^2 + 2cd^2 = KL^2 + KN^2$ , and  $KL = DE$  (construction), therefore  $DF = KN$  and consequently  $ED \times DF (= KL \times KN)$  is to  $DC^2$ , in the first given ratio. The rest is evident from the construction.

*The same by Mr. George Sanderson.*

*Construction.* Let  $s$  to  $m$  be the first given ratio, and  $s$  to  $n$  the other. In the indefinite line  $AI$  take  $AB =$  the given base, which bisect in  $D$  by the indefinite  $\perp KDK$ . Then (by prop. 6, epitag. 1. cases 3, and 4, or epitag. 3, cases 1 and 2, of Wales' Determ. Sect.) cut  $DI$  in  $G$ , so that  $DG^2$ : rectangle  $BGA :: s : n$ . On  $DK$  or  $DK$ , according as  $DG$  is greater or less than  $DB$  take  $DO$  such, that the distance  $AO : DO :: m : n$ . Then with centre  $O$  and radius  $OA$  describe the

segment ACB, to which apply  $DC = DG$ ; and lastly draw AC, BC; so shall ACB be the triangle sought.

**Demonstration.** On the diameter AB describe a circle cutting CD (produced if necessary) in E and H, and draw CF||AB. Because OD is perpendicular to and bisects AB, the circle AC passes through B; it is also manifest that CE = BG, CH = GA, and DF = the  $\perp$  height of the  $\triangle ABC$ . Whence (by the properties of plane triangles prop. 12, Simpson's Trigonometry)  $2OD \times DF = EG \times CH$ ; but (by construction)  $s : N :: DC^2 : ECH$ , and  $M : N :: AO : DO :: 2AO \times DF : 2DO \times DF$  (Eucl. 5, 15); hence of equality  $s : M :: DC^2 : 2AO \times DF = AC \times CB$  (Simpson's Geometry 3, 25).



**XIV. QUESTION 740, by the Rev. Mr. Wildbore.**

If the one of two bodies, connected together by an invariable line, be moved uniformly forward with a given velocity in a direction oblique to that line; required the path of the other body, its position when the velocity in a direction perpendicular to that of the first is a maximum, and the quadrature of the evolute of the path.

*Answered by C. Bumpkin.*

This gentleman, after remarking the general case of this problem, makes the following application to the case in hand. Let the string  $b'b''$  be stretched out on a horizontal plane, with the equal bodies,  $b'$ ,  $b''$  fastened to it, one at each end; and let  $b'$  be struck by a third body  $d$  in the direction  $b'c$  making a given angle with  $b'b''$ . Then will the middle point of the string be carried uniformly forward in a direction  $\parallel b'c$ , whilst the two bodies will revolve about that middle point with an uniform angular velocity, and therefore describing a cycloid, which if  $b'b''$  be not at first  $\perp b'b''$  will be prolate, the first position of  $b''$  being a point of contrary flexure in the path of that body. Which path will be that required in the question, if the velocity of  $b'$  and the direction  $b'b''$  be properly adapted. — If  $a$  be the velocity proposed to be given in the question,  $r$  the length of the string (or line), and  $c$  the cosine of the angle which the string at first makes with the direction of the body moved uniformly forward, to the radius  $r$ ; the velocity with which  $d$  must strike  $b'$  must be  $= ((2r^2 - c^2) \div r^2 + 2b' \div d) \times a$ , that  $b''$  may describe the path the question requires as appears by an easy computation founded on the principles of this kind of motion and collision, both  $b'$  and  $d$  being perfectly hard. And after such stroke the uniform angular velocity of  $b'$  or  $b''$  will be  $(a \div r) \sqrt{(r^2 - c^2)}$ .

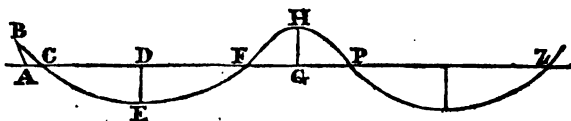


That the required path is a cycloid appears also by reasoning as follows. Let  $a$  be the velocity of  $A$  moved uniformly forward in the line  $AC$ ,  $v$  and  $u$  the velocity of  $B$  in direction  $\parallel$  and  $\perp$  to  $AC$  respectively,  $r$  the length  $AB$ ,  $x$  the cosine of the variable  $\angle BAE$  to the radius  $r$ . Then  $\sqrt{(r^2 - x^2)} : x :: u : v$ , and  $\sqrt{(r^2 - x^2)} : x :: a - v : u$ , therefore  $a - v : u :: u : v$ , and  $uv = av - v^2$ ; hence  $\frac{1}{2}u^2 = av - \frac{1}{2}v^2 + \kappa$ . But,  $c$  being the first value of  $x$ , the first values of  $v$  and  $u$  will be  $ac \div r^2$  and  $(ac \div r^2) \sqrt{(r^2 - c^2)}$  respectively; and it follows, by correction, that  $\kappa$  will be  $= -a^2 c^3 \div 2r^4$ , and  $u^2 = 2av - v^2 - a^2 c^3 \div r^4$ . Then from this equation and the proportion  $\sqrt{(r^2 - x^2)} : x :: a - v : u$ , is found  $u = (a \div r^2) \sqrt{(r^2 - c^2)} \times x$  : which, it is obvious, answers to the description of a cycloid; the centre of the generating circle being carried forward with the velocity  $a$ , whilst the describing point revolves about that centre with the velocity  $((a \div r) \sqrt{(r^2 - c^2)})$ , at the distance  $r$  from it.



*The same answered by Mathematicus.*

This gentleman demonstrates that the required path is a cycloid, after making the following remarks on the nature of the motions in question.



If  $AB$  be the first position of the line connecting the two bodies  $A$ ,  $B$ ; and  $AZ$  the rectilinear path along which  $A$  is carried with the given uniform velocity. It is evident that, by the tension of the line,  $B$  will be continually acted on by a force perpetually varying in its direction, and that therefore the motion of  $B$  will be continually varying, whether it be estimated in a direction  $\parallel$  or  $\perp$  to  $AZ$ . Thus, after the beginning of the motion,  $B$  being drawn by the line  $AB$  in directions between the  $\parallel$  and  $\perp$  to  $AZ$ , the velocity of that body will be continually accelerated in those two directions, till it arrive at and cross the path of  $A$  at the point  $c$ ; where it will have the same velocity with  $A$  in the direction  $cp$ ; but the velocity of  $B$  in the  $\perp$  direction will there be the greatest, because that as soon as  $B$  has passed to the lower side of  $AZ$ , the tension of the line then draws it upwards and diminishes its velocity continually again in the  $\perp$  direction till it be nothing in that direction or move in its path  $\parallel$  to  $AZ$ . This it is evident cannot happen 'till the connecting line come into the position  $BE \perp AZ$ , where the latter body overtakes and passes the former; which it will somewhere do, because that in  $c$  the velocities are equal in the direction  $AZ$  and from  $c$  to  $E$  the latter is always accelerated in that direction, and

it must therefore continually gain upon and at length overtake *A*. After passing *E*, the body *B* moving faster than *A*, it will be retarded by the tension of the line, and be made to approach *AZ* with an accelerated velocity, while the velocity  $\parallel$  to *AZ* decreases till *B* again cross *AZ* before the body *A*, their velocities being there again equal; and the motion of *A* continuing to gain on that of *B*, it will in its turn overtake *B* when the connecting line is in the position  $GH \perp AZ$ . After which *A* will again accelerate the motion of *B*, &c. as before, causing *B* to describe the curve *BCEBHP*, &c.

He then found the curve to be a cycloid agreeably to C. Bumpkin's answer.

THE PRIZE QUESTION, by the Rev. Mr. Wildbore.

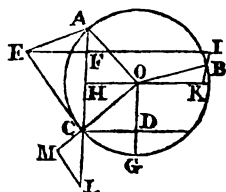
To find the position in which a given flexible line or string must be held, one end of which is fixed to an immoveable tack; so that a heavy body fastened to the other end, after descending freely from rest till the line become stretched, may ascend to the greatest height possible on the other side; also to find what that height is, together with the whole circular arch described, and the point where the body quits the circumference of the circle?

Answered by the Rev. Mr. Wildbore, the Proposer.

Let *OA* be the string fixed at the centre *O*, and the body *A* fall down the  $\perp$  *AC* and then describe the circular arc *cgb*. Draw the horizontal diameter *OH*, and *CD  $\parallel$  *OH*; draw also the tangent *CE* and *AE*  $\perp$  to it; and *EPI*  $\parallel$  and *OG* and *IBK*  $\perp$  *OH*.*

Now the velocity acquired in falling from *AC* is to the velocity in the direction of the circle at *c*, as *OC* to *OH*, or as *AC* to *EC*; but the velocity down *AC* is to the velocity down *EC* or down *FC*, as *AC* to *EC*; therefore the velocity in the circle at *c* is equal to that acquired in falling freely from *F* to *C*. Put now  $a = OA = OC$ , and  $x = OH = CD$ . Then  $AC = 2\sqrt{(a^2 - x^2)}$ ; and since  $CF : CA :: CE^2 : CA^2 :: OH^2 : OC^2$ , therefore  $CF = (2x^2 \div a^2) \sqrt{(a^2 - x^2)}$ ; consequently  $HF = CF - CH = (2x^2 \div a^2) \sqrt{(a^2 - x^2)} - \sqrt{(a^2 - x^2)} = a$  maximum, because  $CF + DG = HF + OG = a$  maximum, that the velocity at *G* may be a maximum or the body ascend to the greatest height, and *OG* is given; hence then  $a^2 \times HF = (2x^2 - a^2) \times \sqrt{(a^2 - x^2)}$  is a maximum, and therefore  $x = a\sqrt{\frac{2}{3}} = OH = CD$  the sine of *CG* =  $65^\circ 54' 18''$ . Hence  $AH = \sqrt{(a^2 - x^2)} = a\sqrt{\frac{1}{3}}$ , and  $FH = \frac{2}{3}a\sqrt{\frac{1}{3}} = \frac{2}{3}AH$ .

After passing the lowest point *G*, the body will ascend in the circle to *B* where the force of gravity in direction *BO* is equal to the centrifugal force; after which it will come within the circle and ascend in a parabola like any other projectile. To determine the point *B*, make



$IB = z$ ,  $IK = PH = c$ , and  $2d = 32\frac{1}{2}$ ; then as  $OB : BK :: 2d : 2d \times (c - z) \div a =$  the force of gravity in direction  $no$ ; but  $2\sqrt{dz}$  being the velocity at  $B$ , the centrifugal force there will be  $4dz \div a$ ; hence  $4dz \div a = 2d \times (c - z) \div a$ , and  $z = \frac{1}{2}c$ , or  $BK = c - z = \frac{1}{2}c = \frac{1}{2}AH = \frac{1}{2}a\sqrt{\frac{1}{6}} =$  the sine of  $10^\circ 27' 14''$ . Therefore the whole arc described  $cb = 166^\circ 21' 32''$ . Moreover the  $\perp$  ascent in the parabola will be  $IB \times OK \div OB = 470a \div 2187\sqrt{6}$ , a very little less than  $BI$ .

*The same answered by C. Bumpkin.*

Let  $s$  be put for  $32\frac{1}{2}$ ,  $r$  for the length of the string, and  $x$  for  $AH$  the height from which the body falls above the horizontal line  $noK$  in which the tack  $o$  is fixed. Then will  $2\sqrt{sx}$  be the velocity after descending through  $2x$ , and  $2r^{-1}\sqrt{(r^2sx - sx^2)}$  the initial velocity in the circle; to acquire which a body must fall from the height  $(2r^2x - 2x^2) \div r^2$ ; to which height adding  $r - x$  (the descent in the circle) we have  $r + x - 2x^2r^{-2}$  for the height from which a body must descend to acquire the velocity the body will have at the lowest point of the circle. Which being a maximum,  $x$  will be  $= r \div \sqrt{6}$ , and  $r + 2r \div 3\sqrt{6} =$  the last-mentioned height. Moreover,  $\sqrt{2sy}$  will be the velocity of the body after ascending the height  $2r \div 3\sqrt{6} - y$  above  $noK$  in the circle, when its centrifugal force will be  $2syr^{-1}$ : Which being put  $= 2s \div 3\sqrt{6} - sy \div r$ , the force with which gravity then urges it towards the tack, we find  $y = 2r \div 9\sqrt{6}$ ; and consequently  $4r \div 9\sqrt{6} =$  the height above  $noK$  at which the body will quit the circle and begin to describe a parabola. In which its horizontal velocity will be  $8\sqrt{rs} \div (27 \times 6^{\frac{1}{4}})$ , and its vertical velocity at first  $= 2\sqrt{(235rs)} \div 3^{\frac{1}{2}} \times 6^{\frac{1}{4}}$ ; its absolute velocity at quitting the circle being  $2\sqrt{(rs)} \div (3 \times 6^{\frac{1}{4}})$ . Whence it follows, that  $470r \div 3^7\sqrt{6}$  will be the ascent in the parabola,  $2\sqrt{(235r)} \div \sqrt{(3^7s\sqrt{6})}$  the time of that ascent, and  $8r\sqrt{235} \div \sqrt{3^{15}}$  the space the body will be carried horizontally during that time.

*The same answered by Mr. Wm. Sewell.*

Besides the lines in the figure as before, he produces  $ac$  to any point  $L$ , and draws  $LM \perp oc$  produced.

Then, if unity or 1 denote the force of gravity,  $2\sqrt{AC}$  will measure the perpendicular velocity acquired by the free descent through  $AC$ ; let this  $2\sqrt{AC}$  or its equal  $2\sqrt{2AH}$  be represented by  $CL$ , which being resolved into the two  $CM$ ,  $ML$ , the latter  $ML$  will measure the velocity with which the body enters the arc at  $c$ . Now, by similar triangles,  $OA^2 : OA^4 - AH^2 (OH^2) :: 8AH (CL^2) : ML^2$  the square of the velocity at  $c$ , which is also  $= 4CF$ , therefore  $CF = (OA^4 - AH^2) \times 2AH \div OA^4$ ; from this take away  $CH = AH$ , and there is  $HF = (OA^4 \times AH - 2AH^3) \div OA^4$  a maximum, or  $OA^4 \times AH - 2AH^3$  a maximum, and hence  $AH$

$= OA \div \sqrt{6} = \text{sine of } 24^{\circ} 5' 41''$  half the arc AC. To determine the point B, we have  $2IB \div OB$  or its equal  $(2HF - 2KB) \div OB =$  the centrifugal force in B; but  $OB : BK :: 1$  (gravity) :  $BK \div OB =$  force of gravity at B in the direction BO, which must there be = the centrifugal force,  $2HF - 2KB = KB$ , or  $BK = \frac{2}{3}HF = 4OA \div 9\sqrt{6} =$  the sine of  $10^{\circ} 27' 14''$ . Hence the whole arc described is  $166^{\circ} 21' 33''$ . He then finds the height in the parabola as in the other solutions.



*Questions proposed in 1779, and answered in 1780.*

I. QUESTION 742, by Mr. John Penberthy.

I'm in love with a damsel, the pride of the plain,  
Have courted and talk'd in Ovidian strain;  
But vain is the rhetoric us'd by my tongue,  
She says I'm too old and that she is too young:  
From the following equations, dear ladies, unfold;  
If she be too young, or if I be too old.

$$\begin{aligned} x^2 + xy^2 &= 4640y & \text{Where } x \text{ represents my age,} \\ x^2y - y^3 &= 537.6x & \text{and } y \text{ the damsel's.} \end{aligned}$$

*Answered by Mr. Tho. Truswell.*

Put the given numbers in the question  $= a$  and  $b$ , multiply the two equations together, and divide by  $xy$ , so shall  $x^2 - y^2 = ab$ ; hence  $x = \sqrt{(ab + y^2)}$ ; this substituted for  $x$  in the second equation, and reduced, we obtain  $4y^4 - (a^2 - 4ab - b^2) \cdot y^2 = -ab^2$ ; then by completing the square, &c. is found  $y = 16$ ; and then  $x = 40$ ; the two ages.

*The same by Mr. Wm. Reynolds.*

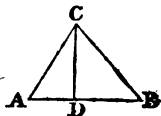
Put  $xy = x$ , which substitute in the original equations, and divide by  $y$ , so shall  $x^2y^2 + zy^2 = a$ , and  $x^2y^2 - y^2 = bz$ ; hence  $y^2 = a \div (x^2 + z) = bz \div (x^2 - 1)$ ; this gives  $bz^2 - (a - b)x^2 = -a$ ; dividing by  $b$ , completing the square, &c. gives  $z = 2.5$ . Then  $y = 16$ , and  $x = 40$ , the ages.

II. QUESTION 743, by Mr. Wm. Spicer.

In a plane triangle, given the sum of each segment added to its adjacent side, equal to 26 and 37; required the sides of the triangle when the area is a maximum?

*Answered by Mr. John Eaden, Jun.*

Put  $AC + AD = 26 = a$ ,  $BC + BD = 37 = b$ , and  $CD = x$ . Then  $a : x :: x : x^2 \div a = AC - AD$ , therefore  $\frac{1}{2}a - \frac{1}{2}x^2 \div a = (a^2 - x^2) \div 2a = AD$ ; in like manner  $(b^2 - x^2) \div 2b = BD$ . Consequently the area  $= (a^2 - x^2) \cdot x \div 4a + (b^2 - x^2) \cdot x \div 4b = (a + b) \times (abx - x^3) \div 4ab$  a maximum, or  $abx - x^3$  a maximum. Its fluxion put  $= 0$ , gives  $ab - 3x^2 = 0$ , and  $x = \sqrt{\frac{1}{3}ab}$ . Then  $AC = \frac{1}{2}a + \frac{1}{6}b = 19\frac{1}{6}$ , and  $BC = \frac{1}{2}b + \frac{1}{6}a = 22\frac{5}{6}$ , and  $AB = \frac{1}{3}a + \frac{1}{3}b = 21$  the base.



*By Mr. Kelly, Schoolmaster, Little Russel Street, Bloomsbury.*

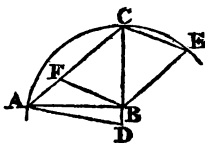
Let  $a = 26$ ,  $b = 37$ ,  $x = AD$ , and  $y = BD$ . Then  $a - x = AC$ ,  $b - y = BC$ , and (Euc. 1, 47)  $a^2 - 2ax = b^2 - 2by = CD^2$ , hence  $y = (b^2 - a^2 + 2ax) \div 2b$ ; therefore  $AB = (b^2 - a^2 + 2ax + 2bx) \div 2b = (b + a) \times (b - a + 2x) \div 2b$ ; and  $(b + a) \times (b - a + 2x) \times \sqrt{(a^2 - 2ax) \div 4b} =$  the area a maximum, or  $(b - a + 2x)^2 \times (a - 2x) =$  a maximum. Its fluxion  $= 0$ , gives  $x = \frac{1}{2}a - \frac{1}{6}b$ , and all the rest as above.

### III. QUESTION 744, by Mr. Nathan Parnel.

To determine a point in the hypotenuse of a given right-angled triangle, standing with one leg perpendicular to the horizon, such, that if a right line be drawn from that point to the right angle, a heavy body may descend down the said line in the same time as down the hypotenuse.

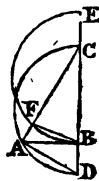
*Answered by Mr. Nathan Parnel.*

Let  $ABC$  be the triangle, having  $BC$  perpendicular to the horizon. Draw  $AD$  making the  $\angle CAD = \angle ACD$ , and meeting  $CB$  (produced if necessary) in  $D$ . With centre  $D$  and radius  $DA = DC$  describe a circle meeting  $BE \parallel AC$  in  $E$ ; draw  $CE$  and  $BF \parallel CE$  meeting  $AC$  in  $F$  the point required. For  $FB$  being both  $\parallel$  and  $= CE$ , the time of descending down  $FB =$  the time in  $CE =$  the time in  $CA$ .



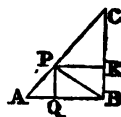
*Otherwise by Mr. Wm. Francis, Master of Shinfield School.*

$ABC$  being the triangle, draw  $AD \perp AC$  meeting  $CB$  produced in  $D$ . In  $BC$  produced take  $BE = DC$ , on which diameter describe a circle cutting  $AC$  in  $F$  the point required.—For, draw  $FB$ , and the semicircle  $DAC$ ; then the time in  $FB =$  time in  $EB =$  time in  $CD =$  time in  $CA$ .



*Another Solution by Mr. Alex. Rowe.*

Let ABC be the given triangle, and P the required point: Join PB, and draw PQ and PR  $\perp$  AB and BC. Now the squares of the times being directly as the spaces and inversely as the accelerating forces, and the forces on CA and PB being as  $CB \div CA$  and  $PQ \div PB$ , the squares of the times will be as  $CA^2 \div CB$  and  $PB^2 \div PQ$ ; and since the times are equal, therefore  $CA^2 \times PQ = PB^2 \times BC$ . But AC : CB :: AP : PQ = AP  $\times$  BC  $\div$  CA, this being substituted for it, the equation becomes  $CA \times AP = PB^2$ .—Put now AC = a, AB = b, BC = c, and AP = x. Then  $a : c :: x : cx \div a = PQ$ , and  $a : b :: a - x : (a - x)b \div a = PR = BQ$ , therefore  $PB^2 = c^2 x^2 \div a^2 + (a - x)^2 \cdot b^2 \div a^2 = CA \times AP = ax$ ; this equation gives  $x^2 - (a^2 + 2b^2)x \div a = -b^2$ , and  $x = (a^2 + 2b^2 - \sqrt{(a^2 + 4b^2)}) \div 2a$ .

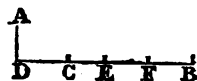


## IV. QUESTION 745, by Mr. Tho. Moss.

To produce, geometrically, a given right line DC, so that the rectangle under the whole compounded line BD and a given right line AD, shall be equal to the square of half the produced part BC.

*Answered by Mr. Wm. Cole.*

In DC produced take CE = AD, and EF = a mean proportional between AD and DE, as also FB = FC, and it is done.—For  $FB^2$  or  $CF^2 = CE^2 + EF^2 + CE \times 2EF = CE^2 + CE \times DE + CE \times 2EF = CE \times (DE + CE + 2EF) = CE \times (DC + 2CE + 2EF) = CE \times (DC + 2CF) = AD \times DB$ .



*The same answered by Mr. Thomas Moss.*

*Geometrical Analysis.* Conceive BC, bisected in E, to be the produced part required: Then, by the problem and hypothesis,  $BD : CE :: CF (BE) : AD$ , or by division,  $BD - CE : CE :: CE - AD : AD$ , or (if  $CF = AD$ )  $BD - CE : CE :: CE - CF : CF$ , that is,  $DE : CE :: FE : FC$ ; or, again by division  $DE - CE : CE :: FE - FC : FC$ , that is (if  $FM = CF$  or  $AD$ )  $DC : CE :: ME : CF$ , or  $DC \times CF = CE \times ME$ . But, DC, CF, and CM are given; and therefore the problem is evidently reduced to this, to produce CM (= 2CF or 2AD) so that the rectangle under CE and ME shall be = the given rect.  $DC \times CF$  or  $CD \times AD$ . Whence the following

*Construction.* In AC produced take CF and FM each = AD; then,

by problem 18, page 106, Simpson's Geometry, produce  $cm$ , so that  $ce \times me$  may be  $= dc \times cf$ ; lastly, in  $ae$  produced take  $eb = ce$ , and it is done.

*Demonstration.* By construction  $dc : ce :: me : cf$  or  $mf$ ; then, by composition,  $cd + ce : ce :: me + mf : mf$ , or  $de : ce (be) :: fe : fc$ ; and again by composition,  $de + be : be (ce) :: ef + cf (ce) : cf = ad$ , that is  $bd : ce :: ce : ad$ , or  $bd \times ad = ce^2$ .

V. QUESTION 746, by the Rev. Mr. Lawson, from an old Greek Epigram.

Two sorts of wine a vint'ner did contrive  
To mix :—of eight-pence, some ; and some at five.  
The price, in pence, of all the pints, he made  
A square ; and if a number, given, you add  
Thereto, a perfect square 'twill yet remain,  
Whose root does all the pints, thus mix'd contain.  
Tell, ye who into such like questions dive,  
What pints he mix'd at eight-pence ; what at five.

\* \* This question is re-proposed, because it is thought to admit of a more general as well as a more elegant answer than has yet been given to it.

*Answered by the Rev. Mr. Lawson.*

This is the 33d question of Diophantus's 5th book. And the 14th of Vieta's 5th book of Zetetics. It is also inserted by Dr. Saunderson among his Diophantine questions. They all exemplify by the addition of the number 60. But the epigram implies that any number at pleasure may be added. Alexander Anderson, in his 4th Mathematical Exercitation, has revised Vieta's solution, and shewn that the limits are not there pointed out with sufficient accuracy. To trace this problem through its limits, when fractions are allowed, would be much too long for the Diary : I therefore send only a short solution, which I was favored with by my worthy and most ingenious friend the Rev. Mr. Wildbore.

Let  $x$  = the number of pints at 8 pence, and  $y$  = those at 5. Then  $8x + 5y$  = price of the whole, which must be a square ; let it  $= z^2 - 2nz + n^2$  to this an integer being added must make a square, whose root is  $x + y$  ; let that integer be  $4nz$ , then  $z + n = x + y$ , and  $8x + 5y = 3x + 5z + 5n = z^2 - 2nz + n^2$  ; whence  $x = \frac{1}{3}z^2 - (2n + 5) \cdot \frac{1}{3}z + \frac{1}{3}n^2 - \frac{1}{3}n$ . Let  $z = 3u$ , and  $n^2 - 5n = 3mn$  ; then  $n = 3m + 5$ , and  $x = 3u^2 - (2m + 5) \cdot 3u + 3m^2 + 5m$ , also  $y = 3u \cdot (2m + 6) - 3u^2 - 3m^2 - 2m + 5$ , and  $4nz = 12u \cdot (3m + 5)$  which must be so given as that  $u$  may be an integer, and  $m$  an integer or  $= 0$ , so always an answer in integers can be obtained. And hence all the possible limits both for whole numbers and fractions,

and for all values of  $4nz$ , are readily found. For that  $x$  may be positive, it is necessary that  $3u^2 + 3m^2 + 5n$  be greater than  $3u \times (2m + 5)$ , and hence  $u$  greater than  $m + 2\frac{1}{2} + \sqrt{(3\frac{1}{2}m + 6\frac{1}{4})}$ ; and that  $y$  may be positive,  $u$  must be less than  $m + 3 + \sqrt{(5\frac{1}{2}m + 10\frac{1}{2})}$ . *Ex.gr.* if  $m = 0$ ,  $u$  must be greater than 5 and less than  $3 + \sqrt{10\frac{1}{2}}$ ; in which case therefore the only integral value of  $u$  is 6; and hence we find  $x = 18$ , and  $y = 5$ . Now 18 pints at 8 pence come to 144, and 5 at 5 pence to 25; then  $144 + 25 = 169 = 13^2$ , to which adding  $4nz = 360$ , it becomes  $529 = 23^2 = (x + y)^2$  as required.

*The same by Mr. Thomas White.*

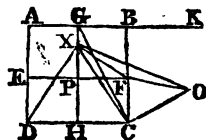
Let  $x$  and  $y =$  the number of pints at 8 pence and 5 pence, and  $n =$  the given number. Then  $8x + 5y =$  a square  $=$  suppose  $(\frac{1}{2}z - \frac{1}{2}v)^2$ , and  $8x + 5y + n = (x + y)^2 =$  suppose  $(\frac{1}{2}z + \frac{1}{2}v)^2$ ; by subtracting,  $n = vz$ , and  $z = n \div v$ ; this substituted in the 2nd equation gives  $x + y = (n + v^2) \div 2v$ , and  $y = (n + v^2) \div 2v - x$ ; which values of  $y$  and  $z$  written in the first equation, &c. give  $x = ((n - v^2)^2 - 10v \cdot (n + v^2)) \div 12v^2$ , and  $y = (16v \cdot (n + v^2) - (n - v^2)^2) \div 12v^2$ , where  $v$  may be taken at pleasure provided  $(n - v^2)^2$  is greater than  $10v \cdot (n + v^2)$  and less than  $16v \cdot (n + v^2)$ .

VI. QUESTION 747, by Mr. George Sanderson.

Mr. Hobbes begins his Geometrical Roses with a new method of cutting a line in extreme-and-mean-proportion, which is this; Let ABCD be a square, and let the sides be bisected in the points E, F, G, H, and the opposite points of bisection joined by the lines EF, GH; then with the centre D, and radius DA, one side of the square, describe a circle cutting GH in x; lastly, if x, F be joined, he says that  $xf$  will be equal to the greater segment of a side of the square divided in extreme-and-mean-proportion. Now it is required to shew by a direct demonstration the falsity of this proposition?

*Answered by Mr. Geo. Sanderson, the Proposer.*

The figure being constructed as in the question, join  $xc$  and  $gc$ , and in  $GB$  produced take  $CK = GC$ . Then  $BK$  is the greater segment of  $AB$  divided in extreme-and-mean-proportion by Eucl. 2, 11, and  $xf =$  the same greater segment by Hobbes; But it is to be proved that  $xf$  is greater than  $BK$ . In order to which make  $co \perp xc$  and  $CF = \frac{1}{2}cx$ , and join  $fo$  and  $xo$ : then the right-angled triangles  $xco$  and  $gch$  having  $xc$  and  $co$  in the one equal to  $gh$



and  $HC$  in the other, have  $xo = gc$  and  $\angle cox = \angle hcg$  (Eucl. 1, 4): again, from the equal angles  $dcr$  and  $xco$  take the common angle  $xcr$ , then  $rco = xcd$ , and therefore the  $\triangle rco$  is equilateral, being similar to the  $\triangle xcd$ , whence the  $\angle cor = fco = xch$ ; but  $xch$  is less than  $gch$  or  $cox$ , therefore  $cor$  is less than  $cox$ , and  $xf + fo$  or  $xf + gc$  is greater than  $xo$  (Eucl. 1, 20); but  $xo = gc = gk = gb + bk$ , therefore  $gb + xf$  is greater than  $gb + bk$ , and  $xf$  greater than  $bk$ .

*The same by Mr. John Fatherley.*

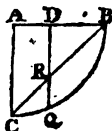
First  $gc = \sqrt{(gh^2 + hc^2)} = \sqrt{(gh^2 + \frac{1}{4}gh^2)} = \sqrt{\frac{5}{4}gh^2} = \frac{1}{2}gh \sqrt{5}$ , and therefore  $bk = gk - gb = gc - gb = \frac{1}{2}gh \sqrt{5} - \frac{1}{2}gh = \frac{1}{2}gh \times (\sqrt{5} - 1) =$  Euclid's greater segment. Again,  $xh = \sqrt{(xc^2 - ch^2)} = \sqrt{(gh^2 - \frac{1}{4}gh^2)} = \sqrt{\frac{3}{4}gh^2} = \frac{1}{2}gh \sqrt{3}$ , and  $xf = xh - ph = \frac{1}{2}gh \sqrt{3} - \frac{1}{2}gh = \frac{1}{2}gh \times (\sqrt{3} - 1)$ , then  $xf = \sqrt{(xf^2 + pf^2)} = \sqrt{(\frac{1}{4}gh^2 \times (\sqrt{3} - 1)^2 + \frac{1}{4}gh^2)} = \frac{1}{2}gh \sqrt{(5 - 2\sqrt{3})} =$  Hobbes' greater segment, different from the former or Euclid's, and therefore is wrong: for if it were right then would  $\sqrt{(5 - 2\sqrt{3})} = \sqrt{5} - 1$ , and by squaring, &c. the equation reduces at last to 49 equal 48, an absurdity.

VII. QUESTION 748, *by Mr. Simon Woolcott.*

Let the ring  $q$  begin to descend freely from  $B$  down the given quadrantal arc  $BQC$ , when another ring  $R$  begins to descend freely down its chord  $BC$ . It is required to find the places of the rings  $q$  and  $R$ , and the time when they are in the same vertical line, or in a parallel to  $AC$  perpendicular to the horizon, the radius of the circle being 60 feet?

*Answered by Mr. Rob. Phillips.*

To find when  $R$  and  $q$  are in the sine  $BQ$  of the arc  $BQ$ ; first put  $AB = AC = 60 = \frac{1}{2}d$ ,  $BD = x$ ,  $BQ = z$ ,  $16\frac{1}{2} = s$ ,  $t =$  the time of the bodies in motion, and  $v =$  the velocity of  $q$ . Then  $BQ = \sqrt{(dx - x^2)}$ , and  $v = 2\sqrt{(s \times BQ)} = 2\sqrt{s} \times \sqrt{(dx - x^2)}$ ; also  $z = \frac{1}{2}dx \div BQ = \frac{1}{2}dx \div \sqrt{(dx - x^2)}$ ; therefore  $t = \frac{z}{v} = \frac{\frac{1}{2}dx}{4s^{\frac{1}{2}}(dx - x^2)^{\frac{3}{2}}}$



$= \frac{d^{\frac{1}{2}}x}{4s^{\frac{1}{2}}} \times 1 + \frac{3x^{\frac{1}{2}}}{4d} + \frac{3 \cdot 7x^{\frac{5}{2}}}{4 \cdot 8d^2} + \frac{3 \cdot 7 \cdot 11x^{\frac{7}{2}}}{4 \cdot 8 \cdot 12d^3}$  &c. and  $t = \frac{d^{\frac{1}{2}}x^{\frac{1}{2}}}{s^{\frac{1}{2}}}$   
 $\times 1 + \frac{3x}{4 \cdot 5d} + \frac{3 \cdot 7x^2}{4 \cdot 8 \cdot 9d^2} + \frac{3 \cdot 7 \cdot 11x^3}{4 \cdot 8 \cdot 12 \cdot 13d^3}$  &c. the time down the arc  $BQ$ .

Again  $\sqrt{s} : \sqrt{x} :: 1'' : \sqrt{x} \div \sqrt{s} =$  the time of falling down BR, and DR : BR :: time in DR :  $\sqrt{2x} \div \sqrt{s} = t$  the time in BR.

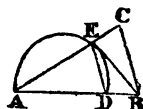
Therefore  $\sqrt{2x} = \sqrt{dx} \times 1 + \frac{3x}{4 \cdot 5d} + \frac{3 \cdot 7x^2}{4 \cdot 8 \cdot 9d^2} + \frac{3 \cdot 7 \cdot 11x^3}{4 \cdot 8 \cdot 12 \cdot 13d^3}$  &c. from which equation is found  $x = 37 \cdot 219$ . Consequently  $t = \sqrt{(2x \div s)} = 2 \cdot 151''$  the time of the bodies in motion.

VIII. QUESTION 749, *by the Rev. Mr. Crakelt.*

Having the vertical angle and base of a triangle given, and also a line drawn from either of the angles at the base to cut the opposite side in a given ratio; to construct the triangle.

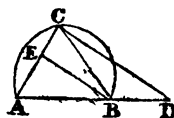
*Answered by Mr. Wm. Walton.*

Make AB = the given base, which divide in D in the given ratio; on AD describe the segment of a circle capable of containing the given vertical angle, and from B apply to it BE = the given line; draw AE and DE, and lastly EC  $\parallel$  DE meeting AE produced in C. Then is ABC the triangle required.—For AB = the given base and BE = the given line by construction; also, because of the parallels, AE : EC :: AD : DB the given ratio, and  $\angle C = \angle AED$  the given angle.



*The same by Mr. Robert Hartley.*

Upon the given base AB describe a segment capable of containing the given vertical angle; and produce AB till AB be to BD in the given ratio; apply DC a 4th proportional to AB, the given line, and AD; draw AC and BC, and ABC will be the triangle.—For, drawing BE  $\parallel$  DC, BE : BA :: DC : DA :: (by construction) the given line : BA; therefore BE = the given line. Again AE : EC :: AB : BD the given ratio by construction.



IX. QUESTION 750, *by Mr. Wm. Sewell.*

What particular expression is that which, being constituted of the variable quantity  $x$  only, its value, when the said expression is a minimum, is obtained from the equation  $x = \pm \sqrt{(x-4) \div 4x} - \frac{1}{2}$ , where  $x$  denotes the hyperbolic logarithm of  $x$ ?

*Answered by the Proposer Mr. Wm. Swell.*

By reducing the given equation we have  $xx + x + x^{-1} = 0$ ; and in order that this be an equation derived from a minimum, the fluxion of the minimum will be found by multiplying this equation by  $\dot{x}$ , whence  $xxx + x\dot{x} + x^{-1}\dot{x} = 0$ ; but  $x^{-1}\dot{x} = \dot{x}$ , and  $x\dot{x} = x\ddot{x}$ , therefore by substitution  $xxx + x\ddot{x} + \dot{x} = 0$ , or  $x\dot{x} + x\ddot{x} + x^{-1}\dot{x} = 0$ , the fluent of which gives  $xx + \text{hyperbolic logarithm of } x$ , an expression whose value, when a minimum, is evidently obtained as per question. But to obtain an expression constituted of the variable quantity  $x$  only, let that above exhibited be considered as a logarithm, then it is plain that its natural value  $x^x$  will be a minimum when the said logarithm is so. By the same way of reasoning, if the last found expression be also considered as a logarithm its natural value  $x^{x^x}$  will evidently be the expression required.

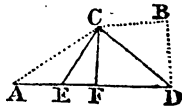
X. QUESTION 751, *by the Rev. Mr. Wildbore.*

In a plane triangle are given the base and the greater of the other two sides, to construct it when the rectangle under the third side and the difference of the segments of the base is a maximum.

*Answered by the Rev. Mr. Wildbore, the Proposer.*

On the given side  $CD$  as hypotenuse make a right-angled  $\triangle BCD$ , whose legs may be in the ratio of the side of a square to its diagonal, and make the less leg  $CB$  the third side  $CE$  of the  $\triangle CDE$ , whose base  $DE$  and side  $DC$  are the given ones; and the thing is done.

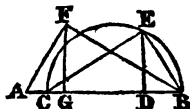
*Demonstration.* Take  $FA = FD$ ; then when  $EA \times EC$  is a maximum,  $ED \times EA \times EC$  is a maximum because  $ED$  is given; but  $ED \times EA$  is  $= (CD + CE) \times (CD - CE) = CD^2 - CE^2 = BD^2$ , therefore  $EC \times BD^2$  or  $BC \times BD^2$  is a maximum; wherefore by theorem 18 of Mr. Simpson on the Maxima and Minima of geometrical quantities,  $BD^2 = 2BC^2$  as per construction.



*Corollary.* If  $AD$  be given instead of  $ED$ , and the solid  $ED \times EA \times EC$  be required to be a maximum; then since the solid  $BC \times BD^2$  is still a maximum,  $BD^2 = 2BC^2$  as before; which is the 439th Diary question. It is moreover evident that the same solid  $ED \times EA \times EC$  being to be a maximum, if instead of  $ED$ , we have given  $EA$ , or the  $\angle ECD$ , or  $CED$ , or a line from  $C$  to  $ED$  specified how drawn, or any similar datum along with  $CD$ , still  $BD^2 = 2CE^2$ , and the solution is effected with nearly the same facility as before.

*The same by Mr. Nathan Parnel.*

Make  $AB$  = the given base,  $BC$  = the given side, on which describe the semicircle  $CEB$ ; divide  $BC$  in  $D$  so that  $CD = 2DB$ , and draw  $DE \perp BC$ ; join  $B, E$  and  $C, E$ ; then with the sides  $AF = BE$ , and  $BF = BC$ , constitute on  $AB$  the  $\triangle AFB$ , which will be that required.



For, let fall the  $\perp FG$ , then  $(FB^2 - FA^2) \div AB = GB - GA$ , therefore per question  $FA \times (FB^2 - FA^2) \div AB$  is a maximum and since  $AB$  is given,  $FA \times (FB^2 - FA^2) = BE \times (BC^2 - BE^2) = BE \times CE^2$  is likewise a maximum, which will be when  $BC \times CD$ , or  $CE^2$  is  $= 2BE^2$  or  $BC \times 2BD$  by Simpson's Geometry, page 208, theorem 18, or when  $CD = 2DB$  as by construction.

*An Algebraical Solution by Mr. Joel Lean.*

Let  $AB = a$ ,  $BF = b$ , and  $AF = x$ . (See the last figure). Then  $a : b + x :: b - x : (b^2 - x^2) \div a = GB - GA$ ; hence  $(b^2x - x^3) \div a$  or  $b^2x - x^3$  is a maximum whose fluxion equated to 0, gives  $x = b\sqrt{\frac{1}{3}} = AF$ .

#### XI. QUESTION 752, by C. Bumpkin.

Parallel to a right line, given in position, to draw another line which shall intersect in three points a line of the third order, whose equation is given, so that, of those three points of intersection, the middle one shall be equidistant from the other two.

*Answered by Plus Minus.*

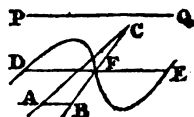
If neither the abscissa  $x$  nor ordinate  $y$  be parallel to the line given in position, change the inclination of one of them (suppose the ordinate) that it may be parallel to it, in the same manner as in my solution of the prize question 1774 I changed it to make it parallel to the asymptote; and let the equation resulting be  $v^3 + (a + bz) \cdot v^2 + \&c. = 0$ ; now construct the locus of the equation  $3v = -a - bz$ , and it will cut the curve in the point or points sought.

Note, the line given in position must not be parallel to an asymptote, nor to the diameters of the diverging parabola, nor to the general diameter of the cubic parabola; for if it be, it cannot cut the curve in 3 points. If either  $y$  or  $x$  be parallel to the line given in position, there is no occasion to transform the equation, but only to make  $y$  or  $3x$  equal to minus the coefficient of  $y^2$  or  $x^2$  respectively, the cube  $y^3$  or  $x^3$  having unity for its coefficient. If  $y^3$  or  $x^3$  be not found in the given equation, the abscissa itself, or the first ordinate itself, cuts the curve in the point sought respectively.

*The same by Mr. Roberts, teacher of Mathematics.*

Let  $y^3 + (ax + b) \times y^2 + \&c. = 0$  be the equation of the curve

to the abscissa AC and ordinates  $\parallel$  PQ the line given in position. Take the abscissa AC =  $b \div a$ , and the ordinate AB =  $\frac{1}{2}b$ ; draw CB cutting the curve in F, thro' which draw DFE  $\parallel$  PQ, and it will be the line sought making FD = FE; as is evident from page 78, Emerson on curves.

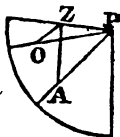


### XII. QUESTION 753, by Nauticus.

It is required to find at what time, on May 21st, 1779, the difference of azimuth between Aldebaran and  $\alpha$  Orion, will be a maximum or minimum at London?

*Answered by Mr. Taylor.*

Let ZP be the meridian of London, P the pole, z the zenith, A Aldebaran, and o  $\alpha$  Orion when the  $\angle oZA$ , the difference of their azimuths, is a maximum or a minimum. It is obvious that when this is the case, its fluxion being then nothing, the fluxions of the angles AZP and OZP must be equal. Now, by Article 273, Crakelt's Translation of Mauduit's Trigonometry  $P : AZP :: \sin^2 ZA : \cos. ZP - \cos. PA \times \cos. ZA$ , and  $P : OZP :: \sin^2 ZO :$



cosine ZP — cosine PO  $\times$  cosine ZO; consequently, P being constant and common,  $(\cosine ZP - \cosine PO \times \cosine ZA) \div \sin^2 ZA = (\cosine ZP - \cosine PO \times \cosine ZO) \div \sin^2 ZO$ .—Put c and s for the sine and cosine of ZP, a and b sine and cosine PA, d and e sine and cosine  $\angle APO$ , p and q sine and cosine PO, and z the cosine  $\angle OPZ$ , or what  $\alpha$  Orion wants of the meridian. Then  $cx - d\sqrt{(1-z^2)} = \cosine \angle APZ$ ; and, by spherics,  $cpz + qs = \cosine zo$ , and  $caez - cad\sqrt{(1-z^2)} + bs = \cosine ZA$ : then  $1 - (cpz + qs)^2 = \sin^2 zo$ , and  $1 - (acez - axd\sqrt{(1-z^2)} + bs)^2 = \sin^2 ZA$ ; make these substitutions in the above equation, so shall

$$\frac{s - abcez + abcd\sqrt{(1-z^2)} - sb^2}{1 - (acez - axd\sqrt{(1-z^2)} + bs)^2} = \frac{s - cpqz - sq^2}{1 - (cpz + qs)^2} \quad \left\{ \begin{array}{l} \text{When } z \\ \text{may be} \\ \text{found.} \end{array} \right.$$

### XIII. QUESTION 754, by Mr. Henry Clarke.

Required the sum of the infinite series

$$\frac{1}{1.7.11.15} - \frac{2}{5.9.7.19} + \frac{2}{3.11.17.23} - \frac{1}{7.13.5.27} + \frac{1}{2.15.23.31} - \frac{2}{9.17.13.35} + \frac{2}{5.19.29.39} - \frac{1}{11.21.8.43} \text{ \&c.}$$

N. B.—This series is of that kind which has been hitherto thought incapable of summation; the factors in the denominators of the terms (however transformed by multiplying, dividing, &c.) not being in the same arithmetical progression. The sum of the whole series may however be exhibited by the means of ratios and angles in a finite expression; and which is hereby required to be done.

*Answered by the Rev. Mr. Wildbore.*

The given series is  $= \frac{4}{4.7.11.15} - \frac{4}{5.9.14.19} + \frac{4}{6.11.17.23} - \frac{4}{7.13.20.27} \&c.$  Now

$$\frac{1}{4.7.11.15} = \frac{6}{11.12.14.15} = 6 \times \left( \frac{1}{11} - \frac{1}{12} \right) \cdot \left( \frac{1}{14} - \frac{1}{15} \right) = 6 \times : \frac{1}{11.14} - \frac{1}{11.15} - \frac{1}{12.14} + \frac{1}{12.15} ;$$

$$\frac{1}{5.9.14.19} = \frac{6}{14.18.18.19} = 6 \times \left( \frac{1}{14} - \frac{1}{18} \right) \cdot \left( \frac{1}{18} - \frac{1}{19} \right) = 6 \times : \frac{1}{14.18} - \frac{1}{14.19} - \frac{1}{18.18} + \frac{1}{18.19} ;$$

$$\frac{1}{6.11.17.23} = \frac{6}{17.18.22.23} = \dots \dots \dots 6 \times : \frac{1}{17.22} - \frac{1}{17.23} - \frac{1}{18.22} + \frac{1}{18.23} ;$$

$$\frac{1}{7.13.20.27} = \frac{6}{20.21.26.27} = \dots \dots \dots 6 \times : \frac{1}{20.26} - \frac{1}{20.27} - \frac{1}{21.26} + \frac{1}{21.27} ;$$

$$\text{Moreover } \frac{1}{11.14} - \frac{1}{14.18} + \frac{1}{17.22} - \frac{1}{20.26} \&c. = 3 \times : \frac{1}{21.22} - \frac{1}{27.28} + \frac{1}{33.34} - \frac{1}{39.40} \&c.$$

$$= 3 \times : \frac{1}{21} - \frac{1}{22} - \frac{1}{27} + \frac{1}{28} + \frac{1}{33} - \frac{1}{34} - \frac{1}{39} \&c. = \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} \&c. - \frac{3}{2} \times : \frac{1}{11} - \frac{1}{14} + \frac{1}{17} \&c.$$

$$\text{and } \frac{1}{11.15} - \frac{1}{14.19} + \frac{1}{17.23} \&c. = 12 \times : \frac{1}{44} - \frac{1}{45} - \frac{1}{56} \&c. = 3 \times : \frac{1}{11} - \frac{1}{14} + \frac{1}{17} \&c. - 4 \times : \frac{1}{15} - \frac{1}{19} \&c.$$

$$\text{and } \frac{1}{4} - \frac{1}{15.18} + \frac{1}{18.22} \&c. = \frac{1}{3} \times : \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \frac{1}{10} + \frac{1}{11} - \frac{1}{12} - \frac{1}{13} \&c.$$

$$\text{and } \frac{1}{12.15} - \frac{1}{15.19} + \frac{1}{18.23} \&c. = \frac{4}{5} \times : \frac{1}{15} - \frac{1}{16} - \frac{1}{19} \&c. = \frac{4}{3} \times : \frac{1}{15} - \frac{1}{19} \&c. - \frac{1}{3} \times : \frac{1}{4} - \frac{1}{6} + \frac{1}{6} \&c.$$

Which several series collected give

$$\frac{8}{9} \times : \frac{1}{7} - \frac{1}{9} + \frac{1}{11} \&c. - \frac{1}{6} \times : \frac{1}{4} - \frac{1}{5} + \frac{1}{6} \&c. - \frac{9}{2} \times : \frac{1}{11} - \frac{1}{14} + \frac{1}{17} \&c. + \frac{16}{5} \times : \frac{1}{15} - \frac{1}{19} + \frac{1}{23} \&c.$$

Now the series  $\frac{x^3}{2} - \frac{x^5}{5} + \frac{x^8}{8} - \frac{x^{11}}{11} + \frac{x^{14}}{14} \&c.$  = the fluent of  $\frac{x^{\frac{1}{2}}}{1+x^3}$ , which when  $x = 1$  is  $= \frac{1}{2}$  arc to sine  $\frac{1}{2}\sqrt{3}$  (rad. 1.)  $- \frac{1}{2}$  hyp. log. 2 = A suppose.

Again,  $\frac{x^3}{3} - \frac{x^7}{7} + \frac{x^{11}}{11} - \frac{x^{14}}{14} \&c.$  = the fluent of  $\frac{x^{\frac{1}{2}}}{1+x^4}$ , which when  $x = 1$  is  $= \frac{1}{2\sqrt{2}} \times$  arc to sine  $\frac{1}{2}\sqrt{(2+\sqrt{2})} + \frac{1}{2\sqrt{2}} \times$  arc to sine  $\frac{1}{2}\sqrt{(2-\sqrt{2})} - \frac{1}{4\sqrt{2}} \times$  hyp. log.  $\frac{2+\sqrt{2}}{2-\sqrt{2}}$  = B suppose. — Then  $\frac{1}{x^{\frac{1}{2}}}$  of the given series is  $= \frac{2}{3} \times (1 - \frac{1}{3} + \frac{1}{3} - \text{arc sine } \sqrt{\frac{1}{2}}) - \frac{1}{3} \times (1 - \frac{1}{2} + \frac{1}{2} - \text{h. l. 2}) - \frac{2}{3} \times (\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - A) + \frac{1}{3} \times (\frac{1}{3} - \frac{1}{4} + \frac{1}{11} - B)$  the expression required.

*Scholium.* The sum of any series of the form  $\frac{1}{n} \pm \frac{1}{m+n} + \frac{1}{m+2n} \pm \frac{1}{m+3n} \&c.$  will be had from the fluent of  $\frac{x^{m-1}}{1 \mp x^n}$ , which is the general Cotesian form. And thus may the sum of any other series compounded like that above be found.

*The same answered by Mr. Wm. Sewell.*

The given series reduces to  $\frac{4}{4 \cdot 7 \cdot 11 \cdot 15} - \frac{4}{5 \cdot 9 \cdot 14 \cdot 19} + \frac{4}{6 \cdot 11 \cdot 17 \cdot 23} - \&c.$  Now  $\frac{1}{4}w^4 - \frac{1}{5}w^5 + \frac{1}{6}w^6 - \&c. = w - \frac{1}{2}w^2 + \frac{1}{3}w^3 - \text{hyp. log. } (1+w)$ , for each side of the equation is the fluent of  $w^{\frac{1}{2}} \div (1+w)$ . Put  $x^2$  for  $w$ , and A for the said log. multiply each side of the equation by  $x^{-2}$ , take the fluents, and there will arise  $\frac{x^7}{4 \cdot 7} - \frac{x^9}{5 \cdot 9} + \frac{x^{11}}{6 \cdot 11} - \frac{x^{13}}{7 \cdot 13} + \&c. = x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{3 \cdot 5} - B.$

Again write  $y^3$  for  $x$ , multiply by  $y$ , and take the fluents, so will  $\frac{1}{2} \times : \frac{y^{22}}{4 \cdot 7 \cdot 11} - \frac{y^{28}}{5 \cdot 9 \cdot 14} + \frac{y^{34}}{6 \cdot 11 \cdot 17} - \&c. = \frac{y^4}{4} - \frac{y^{10}}{2 \cdot 3 \cdot 16} + \frac{y^{16}}{3 \cdot 5 \cdot 16} - C.$

Again write  $z^3$  for  $y$ , multiply by  $z$ , and the fluents give  $\frac{1}{6} \times :$

$$\frac{z^{45}}{4 \cdot 7 \cdot 11 \cdot 15} - \frac{z^{37}}{5 \cdot 9 \cdot 14 \cdot 19} + \frac{z^{29}}{6 \cdot 11 \cdot 17 \cdot 23} - \&c. = \frac{z^9}{4 \cdot 9} \\ - \frac{z^{21}}{2 \cdot 3 \cdot 10 \cdot 21} + \frac{z^{13}}{3 \cdot 5 \cdot 16 \cdot 33} - D.$$

Lastly, write 1 for  $z$ , multiply by 24, &c. and there will arise

$$\frac{4}{4 \cdot 7 \cdot 11 \cdot 15} - \frac{4}{5 \cdot 9 \cdot 14 \cdot 19} + \frac{4}{6 \cdot 11 \cdot 17 \cdot 23} \&c. = \frac{501}{770} - 24 D \text{ the sum required.}$$

Now to deduce the value of  $D$ , we have

$$A = w \cdot (1 + w)^{-1},$$

therefore  $A = \text{hyp. log. of } 2$ ;

$$B = Ax^{-2}x,$$

therefore  $B = -Ax^{-1} + E$ ;

$$C = Ey = -Ax^{-1}y + Ey,$$

therefore  $C = \frac{1}{2}Ay^{-2} + Ey - F - G$ ;

$$D = Cz = \frac{1}{2}Ay^{-2}z + Eyz - Fz - Gz,$$

$$\text{therefore } D = -\frac{1}{8}Az^{-3} + \frac{1}{8}Ez^3 - Fz - Gz + H - I + K + L$$

$$= -\frac{1}{8}A + \frac{1}{8}E - F - G + H - I + K + L \text{ when } z = 1;$$

$$E = Ax^{-1} = 2x \cdot (1 + x^2)^{-1},$$

therefore  $E = \frac{1}{2}c$ , where  $c = 2 \times 3.14159$ , &c.

$$F = \frac{1}{2}Ay^{-2} = 3y^3 \cdot (1 + y^6)^{-1},$$

therefore  $F = \frac{1}{2}\sqrt{3} \times \text{arc sin. } \frac{1}{2}\sqrt{3} - \frac{1}{2}h.l. 2 = \frac{1}{12}c\sqrt{3} - \frac{1}{2}h.l. 2$ ;

$$G = yE = y^{-2}A = 2F,$$

therefore  $G = 2F$ ;

$$H = \frac{1}{8}z^{-3}A = 2z^6 \cdot (1 + z^{12})^{-1},$$

therefore  $H = \frac{1}{8}\sqrt{2} \times \text{arc sin. } \frac{1}{2}\sqrt{(2 + \sqrt{2})} + \text{arc sin.}$

$$\frac{1}{2}\sqrt{(2 - \sqrt{2})} - \frac{1}{2}h.l. \frac{2 + \sqrt{2}}{2 - \sqrt{2}} = \frac{\sqrt{2}}{12} \times \left( \frac{1}{2}c - h.l. \frac{2 + \sqrt{2}}{2 - \sqrt{2}} \right);$$

$$I = \frac{1}{3}z^2E = 4z^5 \cdot (1 + z^{12})^{-1} = 2H,$$

therefore  $I = 2H$ ;

$$K = zF = 6z^8 \cdot (1 + z^{12})^{-1} = 3H,$$

therefore  $K = 3H$ ;

$$L = zG = 2zF = 2K, \text{ therefore } L = 2K = 6H. \text{ Finally we obtain}$$

$$D = \frac{1 + 4\sqrt{2} - 3\sqrt{3}}{12} \times c + \frac{1}{3}h.l. 2 - \frac{1}{3}\sqrt{2}h.l. 1 + \sqrt{2}.$$

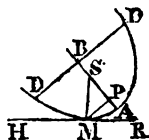
#### XIV. QUESTION 755, by Plus Minus.

My brother being lately returned from abroad, brought with him a solid piece of gold cast in the form of a prolate spheroid; I told him that since he was grown so rich, he might afford his poor brother a little of it; on which he generously gave me leave to cut off as much as would tumble (or rest on a horizontal plane in no position but with the axis perpendicular to it); but added that he would be paid double

the value for the overplus, if I should cut off any more ; the section to be perpendicular to the longer axis, which is double the shorter. What point of the axis would you cut through were it your own case?

*Answered by Plus Minus.*

Let the annexed figure represent the frustum sought, placed with its axis  $AB$  oblique to the horizontal plane  $HR$ . At the point of contact  $M$  erect  $MS \perp HR$ , cutting the axis in  $s$ : now if the centre of gravity of the frustum be in  $s$ , it will rest in this position ; if not, it will fall towards that side of  $MS$  on which the centre of gravity lies, as is plain from mechanics ; therefore that the solid may tumble, the centre of gravity must be nearer to the vertex  $A$  than is the point  $s$ , or any other point where a  $\perp$  to a tangent cuts the axis. If  $AP$  be called  $x$  ;  $PM, y$  ; and  $c$  and  $t$  be the conjugate and transverse semi-axes ; then  $As = x + (cct - ccx) \div tt$  ; and when the point of contact falls in  $A$ , or  $x$  is infinitely small,  $As = cc \div t$ , which being the radius of curvature at the vertex, or the nearest approach of  $s$  towards  $A$ , is the limit beyond which the centre of gravity must not pass ; but because the greatest frustum of the kind is required, I place the centre as far from  $A$  as the nature of the thing will admit, viz.  $= cc \div t$ . So that the problem is now reduced to this. To find the point  $B$  in the axis through which I must cut, that the part cut off may have the distance of its centre of gravity from the vertex equal to half the parameter of the axis. Call  $AB, z$  ; then the distance of the centre of gravity from  $A$  will be  $= (8tz - 3zz) \div (12t - 4z)$  which must  $= cc \div t$  ; hence  $z = (2 \div 3t) \times (cc + 2tt - \sqrt{(c^4 - 5c^2t^2 + 4t^4)}) = (\text{when } t = 2c) c \times (3 - \sqrt{5})$ . Had  $DAD$  been a parabolic conoid,  $AB$  had been  $= \frac{3}{4}$  of the parameter of the axis.

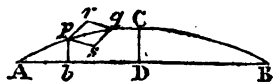


THE PRIZE QUESTION, *by* Peter Puzzlem.

The length, tension, and weight of a musical string being given ; it is required to find how many vibrations it will make in a given time when a small given weight is fastened to its middle and vibrates with it.

*Answered by the Proposer, Peter Puzzlem.*

Let  $ADB$  be the string when straight,  $ACB$  when curved ;  $D$  being the middle point in the former position,  $C$  in the latter. Let  $DC = a$ ,  $AB = b$ ,  $Ab = x$ ,  $bp = y$ ,  $Ap = z$ . Then  $pq$  being a particle of the curve, and  $pr, qr$  tangents ; if  $ps, qs$  be parallel to those tangents the tension of the string will be to the force urging the parti-



cle  $pq$  towards  $AB$ , as the sine of  $psr$  to the sine of  $rps$ ; i. e. as  $\div z$  to  $z \div R$ , or as  $Rz$  to  $z^2$ ,  $R$  being the radius of curvature at  $p$ . Let the tension be  $Ang$ ,  $n$  being a perpendicular section of the string,  $g$  ( $= 32\frac{1}{2}$  feet) the accelerating force of gravity, and  $bng$  the weight of the string: then  $Rz : z^2 :: Ang : Angz \div Rz$ , the motive force urging the particle  $pq$  towards  $AB$ . Therefore  $Angz \div Rz$  will be the accelerative force of  $pq$  in that direction; which by the nature of the motion of a musical string must be  $= dy$ ,  $d$  being some invariable quantity yet unknown. From which equation we have  $Angz \div dy = R = -z^2 \div xy$ ,  $z$  being considered as invariable: whence  $Ag^2y \div z^2 (1 + y^2x^{-2}) = -dy$ . But, the value of  $yx^{-1}$  being indef. small in comparison with  $z$ ,  $1 + y^2x^{-2}$  may be considered as  $= 1$ : Therefore  $Ag^2y \div z^2 = -dy$ ; and the fluents give  $Ag^2y^2d^{-1}x^{-2} = Ag^2d^{-1} + a^2 - y^2$ ,  $z = \sqrt{(Ag \div d) \times y \div \sqrt{(Ag^2d^{-1} + a^2 - y^2)}}$ , and  $x = \sqrt{(Agd^{-1})} \times \text{circ. arc, radius 1, sine } y \div \sqrt{(Ag^2d^{-1} + a^2)}$ ,  $e$  being  $= yx^{-1}$  when  $y = a$ . Now the motive force urging the weight at  $c$  towards  $D$  will be to  $(2Ang)$  twice the tension, as  $e$  to  $\sqrt{(1 + e^2)}$ , or ( $e$  being indefinitely small) as  $e$  to 1: therefore if that weight be denoted by  $bng$ , its accelerative force will be  $2Agce^{-1} = da$ . Hence  $e = dab \div 2Ag$ ; and, by substitution,  $x = \sqrt{(Agd^{-1})} \times \text{circular arc, radius 1, sine } y \div a \sqrt{(1 + \frac{1}{4}db^2a^{-1}g^{-1})}$ . Where, taking  $x = \frac{1}{2}b$ , and  $y = a$ , we have  $\frac{1}{2}b = \sqrt{(Agd^{-1})} \times \text{circular arc, radius 1, sine } (1 + \frac{1}{4}db^2a^{-1}g^{-1})^{-\frac{1}{2}}$ ; from whence we have  $d = 4Agc^2b^{-2}$ ,  $c$  denoting the circular arc, radius 1, sine  $b \div \sqrt{(b^2 + b^2c^2)}$ . The accelerative force  $dy$  will therefore be  $= 4Agc^2yb^{-2}$ , and  $-4Agc^2b^{-2}yy = \dot{v}\dot{v}$ ,  $v$  denoting the velocity of the point  $p$  towards  $AB$ . The fluents give  $4Agc^2b^{-2} \times (h^2 - y^2) = v^2$ ,  $h$  being the value of  $y$  when  $v = 0$ : therefore  $t$  will be  $= by \div 2c\sqrt{(Ag)} \times \sqrt{(b^2 - y^2)}$ , and the time  $t = b \div 2c\sqrt{(Ag)} \times \text{circular arc, radius 1, cosine } y \div h$ . Consequently the time of a whole vibration will be  $= bq \div c\sqrt{(Ag)}$ ,  $q$  being the quadrantal arc of a circle whose radius is 1.

If  $b$  be  $= 0$ ,  $c$  will be  $= q$ ; and the time of vibration, when no weight is fastened to the string,  $= b \div \sqrt{(Ag)}$ .

If  $r$  be the cotangent of  $c$ , and  $n$  be to  $b$  as  $r$  to  $c$ ; the times of vibration of the string without the weight ( $ang$ ) and of the string with the weight, will be to each other as  $c$  to  $q$  respectively.

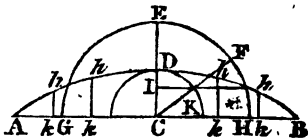
If  $r$  be  $= 1$ ,  $c$  will be  $= \frac{1}{2}q$ : and, if  $n : b :: 2 : q$ , the string without the weight will make two vibrations in the time wherein the string with the weight will make one vibration.

If  $n$  be very great in comparison with  $b$ ,  $c$  will be equal  $b \div \sqrt{(b^2$



*The same answered by Mr. Wm. Herschel.*

Let  $ACB = GEH = a$  length of the string,  $m =$  its weight,  $w =$  weight that stretches it,  $s = 16\frac{1}{2}$  feet,  $DI = x$ ,  $IL = y$ ,  $DK = z$ ,  $EF = v$ ,  $EC = R$ , and  $DC = r$ . Then  $\sqrt{(2sw \div am)} =$  number of vibrations in one second. To find the effect of a small given weight  $n$  fastened to the middle, we must consider, that its power of retarding the vibrations of the string in that place, will be to its power of retarding them if it were diffused all over the string equally, as the rectangle  $ra$  to the area  $ADBCA = A$ . To illustrate this, suppose the string to be a line without weight, and let  $m$  be divided into several equal parts placed at equal distances at  $kkk$ , &c. Now since the momentum of a body is as velocity  $\times$  weight, and that when the string vibrates, the velocity of each part is  $kh$ , therefore if the number of parts be infinitely increased, they will form the string  $a$ , and all the  $kh$ 's  $\times$  all the particles  $= ma$ . But if, instead of the equal velocities, all the particles are supposed to move with the velocity  $r$ , that is, if all the  $kh$ 's  $=$  so many  $r$ 's, then all the particles,  $\times$  all the  $r$ 's, or  $mra$ , will express the momentum of  $m$  when it is all in  $c$ . But  $ma : mra :: A : ra$ .



Now by the nature of the harmonic curve (granting it to be as defined, page 251, Dr. Smith's Harmonics)  $y$  is always  $= v$ , and by similar sectors,  $r : R :: z : v = Rz \div r$ , therefore the fluxion of the area  $DIL = y\dot{x} = vx = Rz \div r$ , the fluent of which is  $Rr^{-1} \times : xz + \sqrt{(2rx - x^2)} - 2r \times \text{arc to radius } r \text{ and sine } \sqrt{(\frac{1}{2}rx)} = \text{area } DIL$ ; and when  $x = r$ , then  $xz = 2r \times \text{arc to sine } \sqrt{(\frac{1}{2}rx)}$ , and the fluent becomes barely  $Rr =$  the area  $DCB$ . But  $c(3.14159, \&c.) : 1 :: a : R = a \div c$ , therefore  $A = 2DCB = 2ar \div c$ .

Having obtained  $A$ , we shall be able to reduce  $n$  to the same condition under which  $m$  acts; for  $2arc^{-1} : ar :: n : \frac{1}{2}cn$ . And this being put into the former expression, gives  $\sqrt{(2sw \div a(m + \frac{1}{2}cn))} =$  the number of vibrations in one second.

*Scholium.* When  $a$  and  $w$  are given, the times of vibrations are in the subduplicate ratio of the weight of the string; therefore to find what  $n$  must be to cause a given interval, for instance a semi-tone, or a tone, we have  $15 : 16 :: \sqrt{m} : \sqrt{(m + \frac{1}{2}cn)}$ , and hence  $n = 62m \div 225c = .08771m$  for a semi-tone; also  $n = .16888m$  for a tone. The experiment is easily made by weighing a little globe of wax, and squeezing it on the middle of the string; when the sound will be found a semi-tone or tone lower as  $n$  is put on  $= .08771m$  or  $= .16888m$ . As nothing is here allowed for the resistance of the air and the disturbance in the vibrations, we cannot expect the experiment to answer with larger intervals, as the sound must soon become too flat and un-

intelligible if  $n$  be enlarged. It must also be observed that the above solution is not to be considered as mathematically true, but as a practical solution approaching near the truth : since the chord loaded with a single weight cannot vibrate exactly in the same curve as when unloaded, nor as when uniformly loaded throughout.



*Questions proposed in 1780, and answered in 1781.*

I. QUESTION 757, *by Mr. Stephen Roberts.*

Fair Phillida sighs for—and sighs for in vain.  
Ye fair-ones, oh pity her woes ;  
And from those equations the object explain.  
That so greatly disturbs her repose.

$$\left. \begin{aligned} x + y &= z \\ z^3 - x^3 + y^3 &= 470 \\ z^3 - x^3 - y^3 &= 468 \end{aligned} \right\} \begin{array}{l} \text{Where the values of } x, y, z, \text{ shew the letters} \\ \text{in the alphabet that compose the name.} \end{array}$$

*Answered by Mr. Wm. Terril.*

The difference between the second and third equations gives  $y = 1$ , answering to the letter A ; and their sum gives  $z^3 - x^3 = 469$  ; but by the first,  $x = z - y = z - 1$  ; which substitute in the last ; it becomes  $3z^2 - 3z + 1 = 469$ , or  $z^2 - z = 156$  ; hence  $z = 13$ , answering to the letter N. Lastly  $x = z - 1 = 12$ , denoting the letter M. So that MAN is what the fair one sighs for.

*The same answered by Mr. Robert Dowden.*

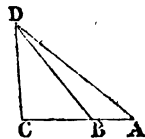
The third equation taken from the second, there results  $2y^3 = 2$ , or  $y = 1$  ; hence the first gives  $z = x + 1$ , which written in either the second or third, it becomes  $x^3 + x = 156$ , consequently  $x = 12$ , and  $z = 13$  ; and poor Phillida sighs for a MAN.

II. QUESTION 758, *by Mr. Wm. King of Lofthouse.*

Given the difference of the distances of the hour-lines of 8 and 9 in the forenoon, from the meridian, on a horizontal dial, equal to 15 degrees 25 minutes. Required the latitude of the place the dial was made for ?

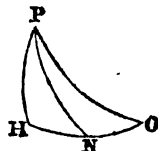
*Answered by Mr. Thomas Bosworth.*

Let DA, DB, DC be the hour-lines of 8, 9, 12 respectively. Then, by Gnomonics, CB, CA will be in the ratio of the tangents of  $45^\circ$  and  $60^\circ$  respectively; and, by the question, the  $\angle ADB = 15^\circ 25'$ . Then, by prop. 4, Simpson's Trigonometry, as  $CA - CB : CA + CB :: \sin \angle ADB : \sin (\angle ADB + 2\angle BDC) = 82^\circ 48'$ ; hence the  $\angle BDC = 33^\circ 41\frac{1}{2}'$ . Then, as tangent of  $45^\circ$  (the angle on the sphere) : tangent  $33^\circ 41\frac{1}{2}'$  (the  $\angle BDC$ , its representation on the dial) :: radius : sine  $41^\circ 48' 48''$ , the latitude required.



*The same answered by Mr. J. Nicholson.*

Let P represent the north pole, PO and PN arcs of 8 and 9 o'clock hour-circles, HO an arc of the horizon, and PH the latitude of the place. Then, by spherics, 1 (radius) :  $x$  (sine PH) ::  $\tau$  (tangent  $\angle HPO = 60^\circ$ ) :  $\tau x$  = tangent HO, and 1 (radius) :  $x :: 1$  (tangent  $\angle HPN = 45^\circ$ ) :  $x$  = the tangent HN; hence, putting  $t$  = tangent NO =  $15^\circ 25'$ , by article 43, Crakell's Trigonometry, we have  $\tau x$  (= tangent HO or tangent HN + NO) =  $(x + t) \div (1 - tx)$ ; consequently  $\tau tx = (\tau - 1) \cdot x = -t$ , and  $x = (\tau - 1) \div 2\tau t \pm \sqrt{((\tau - 1) \div 2\tau t)^2 - 1 \div \tau} = .8659670$  or  $.6667110$  the sines of  $59^\circ 59' 34''$  and  $41^\circ 48' 47''$ , the two latitudes.



### III. QUESTION 759, by Mr. Rich. Rawle, of Redruth.

There are two northern latitudes, both under the same meridian, the one being as far from the equator as the other from the pole: now on the 21st of June the sun rises 1 hour 18 minutes sooner at the place in the greater latitude than at the other. Required both latitudes?

*Answered by Mr. Joseph James.*

Put  $t$  = the cotangent of  $23^\circ 28'$ , and  $s$  and  $c$  for the sine and cosine of  $19^\circ 30' = 1$  hour 18 minutes. Then, by Thacker's theorem,  $8 \div (2 + 2c + s^2 t^2) = 1.7870983 =$  versed sine of  $141^\circ 54' 54''$ , the one-fourth of which gives  $35^\circ 28' 43\frac{1}{2}''$  for the less latitude, and consequently  $54^\circ 31' 16\frac{1}{2}''$  is the greater.

### IV. QUESTION 760, by Mr. Jonathan Mabbot, Excise Officer.

Required, without substituting for the ratio of the fluxions, the relation of  $x$  and  $y$  from  $a^2 \dot{x}^4 + y^2 \dot{y}^4 = b^2 \dot{y}^2 \dot{x}^2 + 2ay\dot{y}\dot{x}^2$ ; supposing  $x$  and  $y$  to begin together.

*Answered by Mr. Tho. White.*

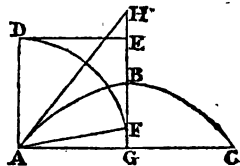
By transposing the term  $2ay\dot{y}^2x^2$  we have  $a^2\dot{x}^4 - 2ay\dot{y}^2x^2 + y^2\dot{y}^4 = b^2\dot{y}^2x^2$ , and extracting the root on both sides it is  $a\dot{x}^2 \pm y\dot{y}^2 = b\dot{y}x$ ; then, by compleating the square, &c. is found  $2ax \pm b\dot{y} = \dot{y}\sqrt{(b^2 + 4ay)}$ , the correct fluents of which give  $b^3 + 12a^2x \pm 6aby = (b^2 + 4ay)^{\frac{3}{2}}$  for the relation sought, which is an equation of the parabolic kind, and when  $b = 0$ , it becomes barely  $9ax^3 = 4y^3$ .

v. QUESTION 761, *by Mr. John Willis, of Marsk.*

Being on the sands at Marsk, which are truly horizontal, I saw a sea-mew flying, at which I threw a stone with a velocity of 60 feet per second, and observed that it just touched the bird in the vertex of its path. It is required to find the angle of projection, the time the stone was in motion, and the height of the bird; the sum of the said height and the horizontal distance of the stone's projection being a maximum.

*Answered by Mr. Nathan Parnel.*

Let ABC be the parabolic curve described by the stone, F the focus, and DE the directrix of the parabola; and GAH the angle of projection. Then put  $a = AD = AF = GE = FH = 60 \times 60 \div 16 \cdot \frac{1}{2} \times 4 = 55 \frac{1}{3} \frac{2}{3} = 55 \cdot 9585337$  feet, the space a body must fall by its gravity to acquire the velocity of 60 feet per second, and  $x = FG$ ; then  $\frac{1}{2}a + \frac{1}{2}x = BG$ , and  $\sqrt{(a^2 - x^2)} = AG$ , and  $\frac{1}{2}a + \frac{1}{2}x + 2\sqrt{(a^2 - x^2)}$  is a maximum, by the question; the fluxion of which made  $= 0$ , gives  $x = a \div \sqrt{17} = 13 \cdot 5719120 = FG$ ; whence  $BG = 34 \cdot 7652379$  the height of the bird; and by trigonometry as  $AF : \text{radius} :: FG : 1 \div \sqrt{17} = \cdot 2425357 = \text{sine of } 14^\circ 2' 10'' = \angle GAF$ , consequently  $45^\circ + \frac{1}{2}GAF = GAF + FAH = GAH = 52^\circ 1' 5''$  the angle of projection; also, by the laws of projectiles,  $\text{radius} : \text{sine GAH} :: 2\sqrt{(a \div 16 \cdot \frac{1}{2})} : 2 \cdot 94045$  seconds, the time the stone was in motion.



*The same answered by Mr. Robert Hartley.*

Put  $a = 60 \times 60 \div 16 \cdot \frac{1}{2} \times 4$  the impetus, and  $x =$  the sine of the angle of projection. Then  $ax^2 =$  the vertical height of the stone, and  $4ax\sqrt{(1-x^2)} =$  the amplitude; therefore  $x^3 + 4x\sqrt{(1-x^2)} =$  a maximum, which put into fluxions, &c. we have  $x = \sqrt{(\frac{1}{2} + \sqrt{\frac{1}{3}})} = \cdot 783205$  the sine of  $52^\circ 1' 5''$ . Therefore  $ax^2 = 34 \cdot 7647$  the

height of the bird, and  $60x \div 16\frac{1}{2} = 2'' 56''' 25''$  the time the stone was in motion.

*The same answered by Mr. John Fletcher.*

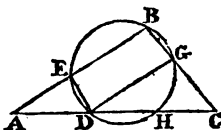
Put  $x$  and  $v$  for the sine and versed sine of double the elevation,  $k = 55.959$  the perpendicular projection. Then  $\frac{1}{2}hv$  is the altitude, and  $2hx$  the horizontal projection; therefore  $4hx + hv$  is a maximum, or  $4x + 1 + \sqrt{(1 - x^2)}$  is a maximum; hence  $x = \sqrt{\frac{1}{4}} = .9701428$  the sine of double  $52^\circ 1'$  the angle of projection; also the altitude is  $34.765$ , and the time  $= 2.948''$ .

# VI. QUESTION 762, by Mr. Nathan Parnel.

Given the four sides of a trapezium, which has two of its adjacent angles equal to each other, to construct it.

*Answered by Mr. Henry Clarke.*

*Analysis.* Let BCDE be a trapezium of which the sides are those given, and the  $\angle CBE = \angle DEB$ . Produce CD, BE till they meet in A, and through D, E, B describe a circle, and join D, G. Then, because the  $\angle B = \angle E$ , and the points B, G, D, E in the circumference of a circle, DG is  $=$  ED, and  $DG \parallel BE$ ; and therefore GC is given, as also DA from the similar triangles ABC, DGC. And because  $BC \cdot CG$  is  $=$   $DC \cdot HC$ , CH is given, and consequently HA; but  $HA \cdot AD$  is  $=$   $(BE + EA) \cdot EA$ , from whence EA is given. Hence this



*Construction.* On the indefinite line AC take  $CD =$  any one of the given sides of the trapezium, then take any two of the remaining sides as BC, ED, and make  $CG (BC - ED) : CD :: CB : CA$ , and  $CD : CB :: CG : CH$ , and find AE such that  $HA \cdot AD = (BE + AE) \cdot AE$ . With the lines AE, DE, on the base AD constitute the  $\triangle AED$ , and on AE produced take  $EB =$  the remaining side of the trapezium, and join B, C; and the thing is done. The demonstration is evident from the analysis.

*Scholium.* It appears from the above construction that there may be 12 different trapezia formed from the data, which will equally answer the conditions of the question.

# VII. QUESTION 763, by Mr. Mic. Taylor.

To find the sum of the infinite series  $\frac{2}{1.3} + \frac{1}{3.5.2} + \frac{1.3}{4.5.7.2} + \frac{1.3.5}{4.6.7.9.2^2} + \frac{1.3.5.7}{4.6.8.9.11.2^3}$  &c. in finite terms.

*Answered by Mr. Mic. Taylor.*

First, since  $\frac{1}{\sqrt{(1-\frac{1}{2}x^2)}} = 1 + \frac{x^2}{2 \cdot 2} + \frac{1 \cdot 3x^4}{2 \cdot 4 \cdot 2^2} + \frac{1 \cdot 3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 2^3} \&c.$

Mult. by  $\dot{x}$ , then  $\frac{\dot{x}}{\sqrt{(1-\frac{1}{2}x^2)}} = \dot{A} = \dot{x} + \frac{x^2 \dot{x}}{2 \cdot 2} + \frac{1 \cdot 3x^4 \dot{x}}{2 \cdot 4 \cdot 2^2} \&c.$

And double the fluent is  $2A = 2x + \frac{x^3}{3 \cdot 2} + \frac{1 \cdot 3x^5}{4 \cdot 5 \cdot 2^2} + \frac{1 \cdot 3 \cdot 5x^7}{4 \cdot 6 \cdot 7 \cdot 2^3} \&c.$

Mult. by  $x\dot{x}$ , so shall  $\dot{B} = 2A x \dot{x} = 2x^2 \dot{x} + \frac{x^4 \dot{x}}{3 \cdot 2} + \frac{1 \cdot 3x^6 \dot{x}}{4 \cdot 5 \cdot 2^2} \&c.$

Take the fluents, so shall  $B = \frac{2x^3}{3} + \frac{x^5}{3 \cdot 5 \cdot 2} + \frac{1 \cdot 3x^7}{4 \cdot 5 \cdot 7 \cdot 2^2} \&c.$

which is the given series when  $x = 1$ .—Now to find  $A$  and  $B$ , since  $\dot{A} = \dot{x} \div \sqrt{(1-\frac{1}{2}x^2)}$ , the fluent is  $A = \sqrt{2} \times \text{arc, radius 1, sine } x \div \sqrt{2}.$

Then,

$\dot{B} = 2A x \dot{x}$ , hence  $B = Ax^2 - c$ . Therefore

$c = x^2 A = x^2 \dot{x} \div \sqrt{(1-\frac{1}{2}x^2)}$ , hence  $c = -x \sqrt{(1-\frac{1}{2}x^2)} + \sqrt{2} \times \text{arc, sine } \frac{1}{2}x = -x \sqrt{(1-\frac{1}{2}x^2)} + A.$

Therefore  $B = Ax^2 - c = Ax^2 + x \sqrt{(1-\frac{1}{2}x^2)} - A = x \sqrt{(1-\frac{1}{2}x^2)} - A \cdot (1-x^2)$ . That is  $x \sqrt{(1-\frac{1}{2}x^2)} - A \cdot (1-x^2) = \frac{2x^3}{3}$

$+ \frac{x^5}{3 \cdot 5 \cdot 2} + \frac{1 \cdot 3x^7}{4 \cdot 5 \cdot 7 \cdot 2^2} \&c.$  in general. { And when  $x = 1$ , these become

$\sqrt{\frac{1}{2}} = \frac{2}{3} + \frac{1}{3 \cdot 5 \cdot 2} + \frac{1 \cdot 3}{4 \cdot 5 \cdot 7 \cdot 2^2} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 7 \cdot 9 \cdot 2^3} \&c.$   
the sum required.

*The same answered by Mr. Henry Clarke.*

The sum of this series may be had from several theorems in my summation of series, but peculiarly belongs to No. 232. For multiply (F) and ( $\Sigma$ ) by 3, (reducing the latter expression) and the result is

$\frac{3}{2 \cdot 4 \cdot 5 \cdot 7 \cdot 2} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 2^2} + \&c. = \frac{1}{6x} \times \text{fluent of}$

$\left( \frac{\dot{x}}{(1-x)^{\frac{3}{2}}} - x \right) - \frac{1}{4x^{\frac{3}{2}}} \times \text{fluent} \left( \frac{x^{\frac{1}{2}} \dot{x}}{(1-x)^{\frac{3}{2}}} - x^{\frac{1}{2}} \dot{x} \right) + \frac{1}{12x^{\frac{5}{2}}}$

$\times \text{fluent} \left( \frac{x^{\frac{3}{2}} \dot{x}}{(1-x)^{\frac{3}{2}}} - x^{\frac{3}{2}} \dot{x} \right)$ . The correct fluent of the first

term is  $\frac{1}{6x} \times \left( \frac{2}{\sqrt{(1-x)}} - x - 2 \right)$ , and the fluents of the 2d and

third terms are respectively  $-\frac{1}{4x^{\frac{3}{2}}} \times \left( \frac{2x^{\frac{1}{2}}}{\sqrt{1-x}} - \frac{2}{3}x^{\frac{3}{2}} - 2q \right)$ ,  
 and  $\frac{1}{12x^{\frac{5}{2}}} \times \left( \frac{3x^{\frac{1}{2}} - x^{\frac{3}{2}}}{\sqrt{1-x}} - \frac{2}{3}x^{\frac{5}{2}} - 3q \right)$ , (which need no correc-  
 tion)  $q$  being the circular arc, radius 1, sine  $\sqrt{x}$ . Now take  $x = \frac{1}{2}$ ,  
 and the expression becomes  $\frac{1}{3} \times (2\sqrt{2} - \frac{4}{3}) - \frac{\sqrt{2}}{2} \times \left( \frac{2}{3\sqrt{2}} - \right.$   
 $2q) + \frac{\sqrt{2}}{3} \times \left( \frac{7}{5\sqrt{2}} - \frac{1}{3} - 3q \right)$ , which, by reduction, and adding  
 the first two terms of the proposed series ( $\frac{2}{3}\sqrt{2}$ ), produces  $\frac{1}{2}\sqrt{2}$  for the  
 required sum.

*The same answered by Mr. Robert Phillips.*

Putting  $z$  = the circular arc to radius 1 and sine  $x$ , we have  $2x +$   
 $\frac{x^3}{3} + \frac{1 \cdot 3x^5}{4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{4 \cdot 6 \cdot 7}$  &c. =  $2z$ ; multiply by  $xz$ , and the fluents  
 give  $\frac{2x^3}{3} + \frac{x^5}{3 \cdot 5} + \frac{1 \cdot 3x^7}{4 \cdot 5 \cdot 7}$ , &c. =  $x^2z - \frac{1}{2}z + \frac{1}{2}x\sqrt{1-x^2}$ ;  
 where taking  $x = \sqrt{\frac{1}{2}}$ , and dividing by  $\frac{1}{2}\sqrt{\frac{1}{2}}$ , we have  $\frac{2}{3} + \frac{1}{3 \cdot 5 \cdot 2}$   
 $+ \frac{1 \cdot 3}{4 \cdot 5 \cdot 7 \cdot 2}$ , &c. =  $\sqrt{2} \times (c - \frac{1}{2}c \times 2 + \frac{1}{2}) = \frac{1}{2}\sqrt{2} = \sqrt{\frac{1}{2}}$ ,  
 the sum of the series required.

*Corollary.* Hence, in general, if  $x$  be taken =  $\sqrt{\frac{1}{n}}$ , and  $c$  = the  
 arc, whose sine is  $\sqrt{\frac{1}{n}}$ , then shall  $\frac{2}{3} + \frac{1}{3 \cdot 5n} + \frac{1 \cdot 3}{4 \cdot 5 \cdot 7n^2} +$   
 $\frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 7 \cdot 9n^3}$  &c. be =  $\sqrt{n} \times (c - \frac{1}{2}nc + \frac{1}{2}\sqrt{(n-1)})$ . And, in par-  
 ticular, if  $n = 1$ , then  $\frac{2}{3} + \frac{1}{3 \cdot 5} + \frac{1 \cdot 3}{4 \cdot 5 \cdot 7} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 7 \cdot 9}$  &c.  
 =  $\frac{1}{2}c$  = one-eighth of the circle whose radius is 1.

#### VIII. QUESTION 764, by the Rev. Mr. Lawson.

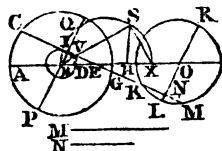
Having two circles given in magnitude and position, and a point in  
 the right line joining their centres; to draw through that point a right  
 line to cut the circles in such a manner, that the chords shall have a  
 given ratio?

*Note.* This question, relating to circles, is analogous to the 15th

problem annexed to a dissertation on the Geometrical Analysis of the Ancients, relative to right lines, which is there proposed in the case of the rhombus, but is applicable to all parallelograms in general.

*Answered by Mr. Henry Clarke.*

**Construction.** Let the given ratio of  $CG$  to  $KM$  be that of  $M$  to  $N$ ; and  $E$  the given point through which  $CM$  is to pass. Make  $BX : BE :: EO : M$ , and describe the semicircle  $BSX$ , in which apply  $XS = N$ , and join  $S, B$ . Take  $Q^2 = M \times OR$ , and  $R^2 = N \times BP$ , also take  $BH, BV$  such that  $BH^2 = Q^2 + R^2$ , and  $BV^2 = Q^2 - R^2$ . Join  $S, H$ , and drawn  $VD \parallel SH$ , then with centre  $B$  describe the circle  $DI$ , to which through the given point  $E$  draw the tangent  $CEM$ , and it is done.



**Demonstration.** For, by similar triangles,  $BS : BH :: BV : BD$ , that is, (from the construction, and Euc. 1, 47),  $BX^2 - N^2 : Q^2 + R^2 :: Q^2 - R^2 : BD^2$ , or  $BX^2 - N^2 : OR \cdot M + BP \cdot N :: OR \cdot M - BP \cdot N : BD^2$ ; hence by composition, &c.  $M^2 \cdot OR^2 - BX^2 \cdot BD^2 : M^2 \cdot (BP^2 - BD^2) :: N^2 : M^2$ . Now, by construction,  $BE : EO :: M : BX$ , and by exposit.  $BE : EO :: BD : ON$ , *ex æquali*  $BD : ON :: M : BX$ ; hence  $M^2 \cdot (OR^2 - ON^2) : M^2 \cdot (BP^2 - BD^2) :: N^2 : M^2$ , or  $OR^2 - ON^2 : BP^2 - BD^2 :: N^2 : M^2$ . But  $OR^2 - ON^2 = RNL = KN^2$  and  $BP^2 - BD^2 = PIQ = GI^2$ , by Euc. 3, 35, and expos. therefore  $KN^2 : GI^2 :: N^2 : M^2$ , or  $KM : CG :: N : M$ .

#### IX. QUESTION 765, by Mr. J. Dymond.

If one end of an indefinite line move round the circumference of a given circle and continually pass through a given point; it is required to find the nature of the curve described by a given point in the revolving line?

*Answered by Mr. Henry Clarke.*

Let  $DI$  be any position of the indefinite line passing through the given point  $O$ , and  $FI$  a part of the curve described by the motion of the given point  $I$ , as per question. Produce  $ID$  to  $d$ , and through the centre  $B$  of the given circle draw  $AF$ ; make  $IG \perp OF$ , and  $BV \perp dd$ . Put  $AB = a$ ,  $OC = b$ ,  $ID = d$ ,  $OG = x$ , and  $GI = y$ ; then will  $OI = \sqrt{(x^2 + y^2)}$ , and  $OD = d - \sqrt{(x^2 + y^2)}$ ; but, per property of the circle,  $OD : OC :: OA : od = (2ab + b^2) \div (d - \sqrt{(x^2 + y^2)})$ , and since  $BV$  is  $\perp$  to  $od$ , we have  $\frac{1}{2}(OD + od) = OV = (ab + \frac{1}{2}b^2) \div (d - \sqrt{(x^2 + y^2)}) + \frac{1}{2}d - \frac{1}{2}\sqrt{(x^2 + y^2)}$ . And, by similar triangles,  $OB : OV :: OI : OG = \frac{\sqrt{(x^2 + y^2)}}{2 \cdot (a + b)} \times \frac{(2a + b) \cdot b}{d \mp \sqrt{(x^2 + y^2)}} + d \mp \sqrt{(x^2 + y^2)} = x$ , the equation of the curve.



If  $DI$  be less than  $AO$ , as  $Ap$ , the curve will have a *nodus* between  $O$

and  $p$ , the *punctum duplex* being at  $o$ ; the affirmative sign taking place in the equation if the value of  $x$  be taken downwards from  $o$  towards  $p$ . If the above expression be reduced, we shall have an equation of the eighth power for the locus of  $x$ , six of whose roots will be found impossible in the case of  $DI$  being greater than  $AO$ , and four when  $DI$  is less than  $AO$ . In the former case therefore a right line will cut the curve in two points only, and in the latter in four.

X. QUESTION 766, by Mr. Thomas Moss.

To draw a right line through a given point  $E$  within a given triangle  $ABC$ , cutting two of its sides so, that the sum of the perpendiculars  $GM$ ,  $HN$ , drawn from the points of intersection  $G$  and  $H$  to the other side  $AB$ , shall be the greatest possible.

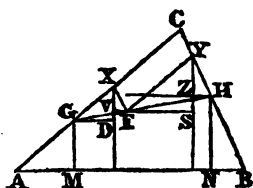
*Answered by Amicus.*

*Construction.* Through the given point  $E$  draw  $EX$ ,  $EY$ ,  $VS$ , parallel to the given sides, let fall  $XV$ ,  $YS$ , on which take  $YZ : YS :: XV : YZ$ , draw  $ZH \parallel AB$ , and draw  $HEG$  the line required.

*Demonstration.* Draw  $GD \parallel AB$ . Now when  $GM + HN$  is a maximum, 'tis evident that  $XD + YZ$  is a minimum; but, by similar triangles,  $YS : XD :: YE : XG :: YH : XE :: YZ : XV$ , therefore the rectangle under  $XD$ ,  $YZ$  being given  $= YS \times XV$ , the perimeter and consequently the sum of its sides will be a minimum when it is a square as per construction.

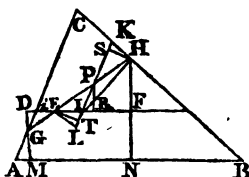
*Corol.* 1. If the sum of the perpendiculars be a given quantity, the sum of the sides of the given rectangle is given, and consequently the sides.

*Corol.* 2. Since  $XG$  is in a given ratio to  $XD$ ,  $YH$ , to  $YZ$ , and  $CX$ ,  $CY$ , given, the problem to draw a line through  $E$ , so that the sum of  $CG$  and a line to which  $CH$  has a certain given ratio, may be a minimum or given, is included in the above.



*The same answered by Mr. Parnel.*

*Analysis.* Suppose that  $GH$  is drawn through the given point  $E$  as required; and draw  $DEF \parallel AB$  cutting the perpendiculars  $MG$  and  $NH$  in  $D$  and  $F$ , and the side  $AC$  in  $i$ ; and in  $DF$  take  $Ei = Ei$ , also draw  $EL \parallel BC$ , and  $LIK \parallel AC$  cutting  $GH$  and  $BC$  in  $P$  and  $K$ ; likewise draw  $ET$  and  $HS \perp LK$ , and  $PR \perp DF$ . Then since  $MG + NH = 2FN + FH - PR$  ( $DG$ ) is a maximum,  $HF - PR$  must be so likewise, because  $FN$  is constant; and since the triangles  $EHI$  and  $EPI$  have a common base  $Ei$ , their areas are, as their



perpendiculars HF and PE, and therefore the  $\triangle EHI - EPI = PHI$  or  $IP \times SH = PI \times ET \times PK \div PL$  (because  $SH = ET \times PK \div PL$ , by similar triangles)  $= ET \times (LP - LI) \times (LK - LP) \div LP = ET \times (LK + LI) - ET \times (LP^2 + LI \times LK) \div LP$  is a maximum. But  $ET, LK$ , and  $LI$  being constant,  $(LP^2 + LI \times LK) \div LP$  must evidently be a minimum; which may be considered as the hypothenuse of a right-angled triangle, of which  $LI \times LK =$  the square of the perpendicular let fall from the right angle, and  $LP$  one of the segments; which hypothenuse must evidently be a minimum when the segments are equal; and then  $LP^2 = LI \times LK$ : hence this

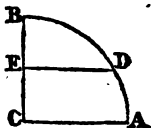
*Construction.* Draw  $DF, EL$ , and  $LK$  as in the analysis, and take  $LP$  a mean proportional between  $LI$  and  $LK$ , then draw  $HPEG$  the line required.

XI. QUESTION 767, by Mr. Rob. Phillips.

Suppose the quadrant ABC of a circle, whose radius CA or CB is 100 feet, to revolve about the perpendicular radius AC with a uniform velocity, making each revolution in five seconds of time, whilst a ring at the upper end at B is left to descend along the curve BA by its own gravity: it is required to determine in what time it will arrive at the lowest point A?

*Answered by Mr. Rob. Phillips.*

This question was proposed last year with only one certain position of the quadrant, but this gentleman has given two solutions, one for each position. And, first, for the quadrant placed as in the annexed figure.—Put the radius CA or CB = 100 feet =  $a$ , the force of gravity  $32\frac{1}{2}$  feet =  $s$ , the velocity of the point A = 125.66 feet per second =  $b$ , BE =  $x$ , ED =  $y$ , the arc BD =  $z$ , the velocity of the ring D along the curve



=  $v$ , and the time of describing  $BD = t$ . Then  $b^2 a^{-1} =$  the centrifugal force of the point A, and  $a : y :: b^2 a^{-1} : b^2 y a^{-2} =$  the centrifugal force of the ring at D in the direction ED, also  $\dot{z} : \dot{y} :: b^2 y a^{-2} : b^2 y \dot{y} a^{-2} \dot{z}^{-1} =$  the effect of the centrifugal force to urge the ring down the curve BDA; but  $\dot{z} : \dot{x} :: s : s \dot{x} \dot{z}^{-1} =$  the effect of gravity in the same direction; consequently  $s \dot{x} \dot{z}^{-1} + b^2 y \dot{y} a^{-2} \dot{z}^{-1}$  is the whole force which accelerates the velocity of the ring down the curve, which by the principles of motion is  $= v \dot{v} \dot{z}^{-1}$ , therefore  $v \dot{v} = s \dot{x} + b^2 a^{-2} \dot{y} \dot{y}$ , and the fluents give  $v^2 = 2s x + b^2 a^{-2} y^2 = 2s x + b^2 a^{-2} \cdot (2ax - x^2)$  by the property of the

circle, and putting  $2s + 2b^2a^{-1} = m$ , and  $b^2a^{-2} = n$ , we have  
 $v = \sqrt{(mx - nx^2)}$ , hence the fluxion of the time  $t = \frac{dx}{v} =$

$$\frac{dx}{\sqrt{(mx - nx^2)} \times \sqrt{(2ax - xx)}} = \frac{dx}{x\sqrt{(2am - (m + 2an) \cdot x + nx^2)}}$$

$$= \sqrt{\frac{a}{2m}} \times \frac{x^{-1} dx}{\sqrt{(1 - cx + rx^2)}}, \text{ where } r \text{ is } = \frac{n}{2am} \text{ and } c =$$

$$\frac{m + 2an}{2am}; \text{ and the fluent of this expression is } t = -\sqrt{\frac{a}{2m}} \times$$

hyp. log. of  $\frac{2 - cx + 2\sqrt{(1 - cx + rx^2)}}{x\sqrt{(c^2 - 4r)}}$ . But when  $x = 0$ , this

fluent and consequently the time  $t$ , is infinite, which shews that the ring must be put at some small distance from the upper end B, otherwise it will not descend; wherefore when  $t = 0$ , let  $x = a$  very small

quantity  $d$ , then the fluent corrected will be  $t = \sqrt{\frac{a}{2m}} \times \text{hyp. log.}$

$$\frac{2x - cdx + 2x\sqrt{(1 - cd + rd^2)}}{2d - cdx + 2d\sqrt{(1 - cx + rx^2)}}, \text{ in which if } d \text{ be taken } = \frac{1}{10000}$$

part of a foot, the whole time when  $x = 100$  feet  $= a$ , will come out 5.2258'', the time of descent through nearly the whole quadrant.

*Scholium.* I give this only as an approximate answer. For  $v$  should have a small correction; and then the time  $t$  will, by Landen's tables, be obtained by elliptic arcs.—This problem is treated generally in Landen's 8th Memoir, vol. 1.

In the other case, when the quadrant is inverted, the method of solution is the very same, except that the effect of the centrifugal force in the direction of the arc is to be subtracted from that of gravity, instead of added, because it then tends to retard the descent of the body.

## XII. QUESTION 768, by Mr. George Sanderson.

In a given circle to inscribe the greatest triangle that can have two of its sides in a given ratio?

*Answered by Mr. Henry Clarke.*

*Analysis.* Suppose ABC to be the required greatest triangle inscribed in the given circle ABDC, and having its two sides AB, AC in the given ratio. Draw the diameter AD, its centre being E, and CF  $\perp$  AD; then  $\triangle ACF$  is evidently similar to  $\triangle ABC$ , and having the side AC common, ACF will be a maximum when ABC is a maximum, but when a triangle, as ACF is a maximum, the base and perpendicular are to each other directly as their increments or decrements, namely, AG : CF :: AG' : CF'. With centre H in AD, suppose a semicircle AFI drawn

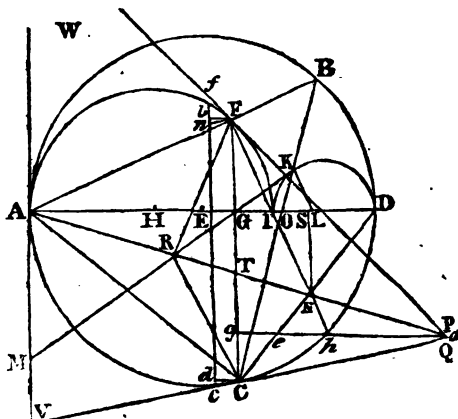
through  $A$  and  $F$ , (the diameter  $AI$  of which will be given, since  $AI : AD :: AF : AB :: AC' : AB'$  in the duplicate of the given ratio) and suppose  $cf$  to be indefinitely near to  $cf$ ,  $cd$  and  $fb \parallel AG$  and  $fn \parallel cc$ ; then  $cd$  or  $fb$  is the decrement of  $AG$ , and  $fn = fb + cd$  is the increment of  $CF$ ; therefore

$$AG : CF :: FB : fn ::$$

(by drawing the tangents  $ca$ ,  $fa$ , forming the  $\Delta caf$  similar  $\Delta ncf$ , and drawing  $ag \parallel AG$  or  $\perp CF$ )  $ag : CF$ ; and as the consequents are the same  $CF$ , the antecedents  $AG$ ,  $ag$ , are equal. So that the problem is now reduced to this. Dividing  $AD$  in  $r$  in the duplicate of the given ratio, and describing the semicircle  $AFI$ , to find the point  $g$  so, that drawing  $cgf \perp AD$ , and the tangents  $ca$ ,  $fa$ , and  $ag \parallel AG$ , then  $ag$  may be  $= AG$ .

Now the point  $g$  will be found so as to answer this condition, by describing the semicircle  $IKD$  on the diameter  $ID$  and centre  $I$ , then taking  $AM \perp AD$  and  $= 3IL$ , and drawing  $mek$  to touch  $IKD$  in  $k$ .—For, draw  $finh$  and  $cend$ , also  $AN$ , producing it to meet the tangent  $fa$  in  $p$ , and tangent  $ca$  in  $q$ ; bisect  $AN$  in  $R$ , and join  $RF$ ,  $RC$ ; also on  $GD$  take  $GS = GK$ , and join  $NS$ . Because  $ACN$  and  $AFN$  are right angles,  $R$  is the centre of a circle passing through the points  $A$ ,  $F$ ,  $N$ ,  $C$ ; but the same circle passes through  $s$ , (because  $GA \cdot GI = GF^2$  (Euc. 8, 6) and  $GD \cdot GI = GK^2$  (37.3)  $= GS^2$ , therefore  $GF^2 : GS^2 :: GA : GD :: GA^2 : GA \cdot GD = GC^2$ , therefore  $GF : GS :: GA : GC$ , and  $GF \times GC = GA \cdot GS$ ), and as  $AN$  is the diameter of it,  $s$  is a right angle, and  $AGT$ ,  $ASN$  similar triangles; but, by construction,  $AM = 3KL$ , therefore  $AG = 3GK = 3GS$ , and  $AT = 3TN$ ,  $AN = 4TN$ ,  $RN = 2TN$ , and  $TR = TN$ . But  $\angle IFP = IAF$  (32.3)  $= IFT$  (32.1) and  $BFT + IFT$  (NFT)  $= RNF$  (5.1)  $= NPF$  (NFG)  $+ NPF$  (32.1), therefore  $NPF = RFT$ ; and the  $\angle R$  is common, therefore the triangles  $BFT$ ,  $RFP$  are equiangular; but  $RF$  ( $RN$ )  $= 2RT$ , therefore  $RP = 2RF$  or  $2RN$ . In the very same manner, because  $CQ$  is a tangent to the circle  $ACD$ , it is proved that  $RQ = 2RN$ . Consequently  $P$  and  $Q$  both coincide with  $a$ ; therefore  $ra = AN$  ( $2RA$ ), and  $TA = Ta$ ; and hence, by similar triangles,  $GA = ga$ .

*Scholium.* Many curious properties are easily drawn from what has been done above; such as that  $AO = 3DO$ , or the base  $CB$  of the greatest  $\Delta ABC$  always passes through the same point  $o$  in the diameter



AD as the inscribed equilateral triangle having the same vertex A; whatever be the given ratio of AB to AC; that the parts *ge*, *eh*, *ha* are respectively = *GE*, *EH*, *HA*; that the greatest  $\triangle AFC$ , having its vertex in A, and base *FC*  $\perp$  AD and terminating in the two given semicircles AFI, ACD, will be, when the two tangents *arw*, *acv*, terminated by *VMW*  $\perp$  AD, are bisected in the points of contact *r*, *c*, &c.

*A Fluxionary Solution to the same.*

Put *AG* = *x*, *AI* = *a*, and *AD* = *d*, then *GF* =  $\sqrt{(ax - x^2)}$ , and *ac* =  $\sqrt{(dx - x^2)}$ ; consequently  $x\sqrt{(ax - x^2)} + x\sqrt{(dx - x^2)}$  = *a* maximum, the fluxion of which made = 0 gives  $8x^2 = 9x \cdot (a + d) - 9ad$ ; subtract each side from  $9x^2$ , so shall  $x^2 = 9 \times (x^2 - x \cdot (a + d) + ad) = 9 \cdot (a - x) \cdot (d - x) = 9GI \cdot GD = 9GK^2$ , hence *x* = *AG* = 3*GK*, and consequently *AM* = 3*KL*, as before.

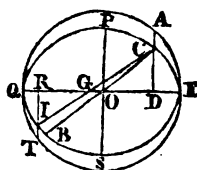
XIII. QUESTION 769, by Plus Minus.

Had I been at the antipodes with Captain Cook, I might have won a wager of him; for he says, that he then was as far removed from his friends in London as possible. Now I desire to know what course he ought to steer, and how far from the place where he then was, to be as far from London as possible; and how much farther he would then be than when at the antipodes; supposing the polar diameter of the earth to be to the equatorial diameter as 229 to 230, and that the latter is 8000 miles?

*Answered by Nauticus.*

Let *epqs* be the elliptical meridian of London, *eq* the equator, *rs* the earth's axis, *r* the north, and *s* the south pole. On *eq* describe a circle, and take the arc *EA* =  $51^\circ 23' 42''$ , the tang. of which is to the tangent of the latitude of London ( $51^\circ 31'$ ) as *or* to *oe*: draw *AD*  $\parallel$  *ro*, cutting the ellipse in *c*, which will be the situation of London on the spheroid: draw the diameter *cb*, and *b* will be antipodes to it. Now if a circle be described from *c* as a centre, to touch the ellipse, as in *i*, it is plain that *i* will be the farthest possible distance that the ship can be from London; and this point will manifestly be between *b* and the equator at *q*: the course must therefore be due north, and the distance will be the difference of the latitudes of the points *b* and *i*. Also the radius *ci* will be perpendicular to a tangent drawn to the ellipse and circle at the point *i*. Draw the ordinate *ia*, which produce to meet the circle in *t*; and let *o* be the point where *ic* cuts the transverse axis.

If *oe* be = 1, then will *ro* =  $\frac{229}{230} = c$ ,  $c^2 (= \frac{52241}{52900})$  will be half the parameter, and  $1 - cc (= \frac{459}{52900})$  the difference between the semi-transverse and half the parameter. Diminish *AD*, the sine of  $51^\circ$

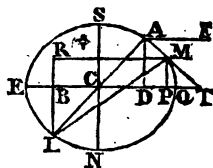


23' 42'', in the ratio 230 to 229, and we gain  $cd$ , which put  $= a$  and let  $b$  = do the cosine of  $51^\circ 23' 42''$ ; also let  $x$  = or the cosine of the arc  $qr$ . Then, by the properties of the ellipse,  $1 : cc :: (1 + x) \cdot (1 - x)cc : (1 - xx)cc = Rr^2$ ;  $1 : cc :: x : ccr = Rg$ ; and  $1 : 1 - cc :: x : x \cdot (1 - cc) = go$ ; hence  $b + x \cdot (1 - cc) = cd$ ; and, by similar triangles,  $b + x \cdot (1 - cc) : a :: ccr : c\sqrt{(1 - xx)}$ ; consequently  $ac^2x = (b + x \cdot (1 - cc)) \times c\sqrt{(1 - xx)}$ , or  $b \div ac = x \div \sqrt{(1 - xx)} - x(1 - cc) \div ac$ . Here it is plain that  $x \div \sqrt{(1 - xx)}$  is the cotangent of the arc  $rq$ , its cosine being  $x$ ; from which consideration we have the following easy and expeditious method of obtaining the value of  $x$ . Compute  $b \div ac = A$ , and  $(1 - cc) \div ac = B$ , and find the logarithm of the latter. Now, because the ellipse differs but little from a circle, the arc  $qr$  will be very little less than the arc  $EA$ : assume it  $50^\circ 55'$ ; take out the natural cotangent and the logarithm cosine; to the latter add the logarithm of  $B$ , find the number answering to the sum, and take it from the natural cotangent of  $50^\circ 55'$ ; if the remainder be  $= A$ ,  $50^\circ 55'$  is rightly assumed; but it will be found  $\cdot 0002869$  too little. Assume the arc  $qr = 50^\circ 54'$ ; repeat the operation, and the result will be found  $\cdot 0001935$  too great. Then  $4804 (2869 + 1935) : 60'' :: 1935 : 24''$ ; which being added to  $50^\circ 54'$  gives  $50^\circ 54' 24''$  for the arc  $qr$ . And as  $229 : 230 :: \text{tangent } qr : \text{tangent } 51^\circ 1' 43\frac{1}{2}'' S$ . the latitude sought: consequently the distance to be sailed will be  $29' 16\frac{1}{4}''$ .

Again, compute  $Rr = \cdot 7727454$ . Then  $CI = \sqrt{(CD + Rr)^2 + Rr^2} = 1\cdot 9947122$ ,  $CB = 2\sqrt{(OD^2 + DC^2)} = 1\cdot 9946939$ , and  $(CI - CB) \times 4000 = \cdot 0732 = \frac{3}{41}$  of a mile, which the ship will be farther from London at the point  $i$  than at  $s$  the antipodes.

*The same answered by Plus Minus.*

Let the ellipsis  $ENQS$  represent the meridian of London,  $c$  the centre,  $N$  and  $s$  the north and south poles,  $EQ$  the equatorial diameter,  $L$  London,  $A$  its antipodes, and  $M$  the point sought, the course being evidently due north. Draw  $MP$ ,  $AD$ ,  $LB \perp$  and  $MR \parallel EQ$  also draw the tangent  $MT$ .



Then if you call  $CD$  or  $CB$ ,  $a$ ;  $AD$  or  $LB$ ,  $b$ ;  $CE$ ,  $t$ ;  $CS$ ,  $c$ ;  $CP$ ,  $x$ ; and  $PM$ ,  $y$ ; you will have  $ty = c\sqrt{(t^2 - x^2)}$ . And because  $LM$  is  $\perp$   $MT$ , the triangles  $MRL$ ,  $MPT$ , will be similar; and consequently  $LR (b + y) : RM (a + x) :: PT ((t^2 - x^2) \div x) : PM (y \text{ or } (c \div t) \sqrt{(t^2 - xx)})$ , that is, as  $\sqrt{(t^2 - xx)} \div x$  to  $c \div t$ , or as  $(c \div t) \sqrt{(t^2 - xx)}$  to  $ccx \div tt$ , or as  $y$  to  $ccx \div tt$  or as  $tty$  to  $ccx$ ; therefore  $bccx + ccxy = utty + ttty$ . Now if you exterminate  $y$  from this equation by means of the equation  $ty = c\sqrt{(t^2 - xx)}$ , you will have a biquadratic equation, one of whose roots is  $= -a$ ; it being therefore divided by  $x + a$ , the following cubic equation will arise

having only one possible root, namely  $x^3 + \frac{tt + cc}{tt - cc} ax^2 + \frac{2cc - tt}{(tt - cc)^2} x$

$t'x - \frac{at^6}{(tt - cc)^3} = 0$ . But to save the labour of solving such an

equation as this, let us try more simple means.—The equation  $bccx + ccxy = atty + ttxy$  is that of an hyperbola, of which that small portion which falls within the earth, is almost a right line; for it passes through the points  $t$  and  $c$ , and by reason our ellipsis is almost a circle, can pass by  $A$  but a very little way from it; of course that still smaller portion of it which falls between  $AD$  and  $AF$ , parallel to  $CD$ , may be considered as a right line without sensible error. Now because when  $x = a$ ,  $y = bcc \div (2tt - cc)$ ; and when  $y = b$ ,  $x = att \div (2cc - tt)$ , the equation of the chord of that portion of the hyperbola will be

$$y = \frac{2cc - tt}{2att - acc} bx + \frac{tt - cc}{2tt - cc} b:$$

instead of the equation of the hyperbola, so we may also use the equation of the tangent of the ellipsis at the point  $A$ , instead of the equation for the ellipsis itself; but the equation of that tangent is  $(tt - aa) \cdot y = (tt - ax) \cdot b$ ; hence  $x$  and  $y$  may be found by simple equations, viz.

$$x = a + \frac{(2at^6 - 2a^3) \cdot (tt - cc)}{tt \cdot (2cc - tt) + 3a^3 \cdot (tt - cc)}, \text{ and } y = b -$$

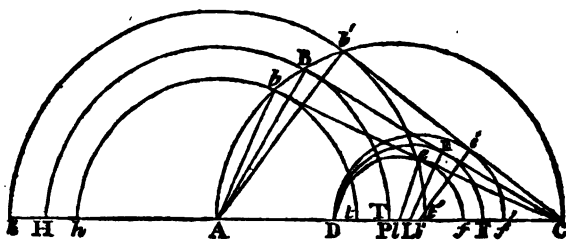
$\frac{2a^3b \cdot (tt - cc)}{tt \cdot (2cc - tt) + 3a^3 \cdot (tt - cc)}$  which give  $x = 2522.4325$ , and  $y = 3091.0941$ ; and  $\sqrt{((x - a)^2 + (b - y)^2)} \div CA$  or  $\sqrt{(a^2 + b^2)} =$  tangent of  $29^\circ 19''$  the difference of latitude; also  $LM - LA = \sqrt{((x + a)^2 + (y + b)^2)} - 2\sqrt{(a^2 + b^2)} =$  the difference of distance sought very nearly.

#### XIV. QUESTION 770, by the Rev. Mr. Wildbore.

A semicircle and a point in its diameter or base being given; to draw another semicircle through the given point, having its centre between it and the angular point of the given semicircle that is farthest from it so that a line being drawn to touch it, and tend to the said angular point, the part of that line intercepted by the semicircles may be a maximum: and to give the geometrical demonstration. This being wanted to complete the determinations in the restitution of the second book of Apollonius on Inclinations.

*Answered by the Rev. Mr. Wildbore.*

*Construction.* From  $P$ , the centre of the given semicircle, take  $PM$  a mean proportional between  $PC$  and  $PC + 2DC$ , take  $AB = AM$ , join  $AC$ , to touch which through the given point  $D$  draw the semicircle  $DEF$ , and it will be that required.



**Demonstration.** Since by construction  $PH^2 - PC^2$  or  $HC \cdot HA = 2CF \cdot CD = CA \cdot CD$ , therefore  $CH : CA :: CD : AH = AB$ , but by sim. triangles,  $CH : CA :: CD : CL$  ( $L$  being the centre of  $DEF$ ), consequently  $AB = CL$ ; then  $BE$  is a maximum and  $BE^2 = CB^2 - CE^2 = 2CB \cdot CE = CA^2 - AB^2 + CD \cdot CF = 2CD \cdot CT$  (because by similar triangles,  $CE : CD :: CT : CB$ )  $= CA^2 - AB^2 + CD \cdot (2CL - CB) = 2CD \cdot (CA - AB)$  is a maximum or, because  $CA$  and  $CD$  are given,  $2CD \cdot (CL + AB) - AB^2$  is a maximum. For suppose the contrary, and that  $Ab$  is less than  $AB$ , and consequently  $cl$  greater than  $CL$  when the intercepted part of the tangent is a maximum then  $2CD \cdot (cl + Ab) - Ab^2$  is greater than  $2CD \cdot (CL + AB) - AB^2$ , or, taking away  $2CD \cdot AB - Ab^2$  from both,  $2CD \cdot cl$  is greater than  $2CD \cdot (CL + Hh) - 2Ah \cdot Hh - Hh^2$ ,  $2CD \cdot cl$  than  $2CD \cdot Hh - 2Ah \cdot Hh - Hh^2$ ,  $2CD \cdot CA$  than  $CH \cdot ch \cdot (2CD - 2Ah - Hh)$  (because  $Ll \cdot CH \cdot ch = Hh \cdot CA \cdot CD$ ) or than  $CH \cdot ch \cdot (2DL + Hh)$  (because  $2AH - Hh = 2Ah + Hh$ ,  $AH = CL$ , and  $CD - CL = DL$ ),  $2CD \cdot cl$  than  $2DL \cdot CH \cdot ch + CH \cdot Hh \cdot ch$  (because  $CH \cdot cl = CA \cdot CD = CH \cdot CL$ ),  $2CD \cdot (cl - CL)$  than  $CH \cdot Hh$  (because  $DL \cdot CH = CD \cdot AH = CD \cdot CL$ ),  $2CD^2 \cdot CA$  than  $CH^2 \cdot ch^2$ ,  $2CH^2 \cdot CL^2$  than  $CH^2 \cdot ch^2$ ,  $CA$ , or  $2CL^2$  than  $ch \cdot CA$ ,  $2CL \cdot cl$  than  $CH \cdot ch \cdot CA$  (because  $CH \cdot CL = ch \cdot cl$ ), or  $2CL \cdot cl$  than  $CH \cdot CA$ , which is absurd, because  $2CL = TH$  is less than  $CH$ , and  $cl$  than  $CA$ . Therefore, &c.

Again, if  $Ab' = Ah'$  be greater than  $AB$  when  $b'e'$  is a maximum then, reasoning as before,  $2CD \cdot (cl' + Ah') - Ah'^2$  is greater than  $2CD \cdot (CL + AH) - AH^2$ ,  $2CD \cdot (cl' + Hh') - 2AH \cdot Hh' - Hh'^2$  than  $2CD \cdot CL$ ,  $2CD \cdot Hh' - 2AH \cdot Hh' - Hh'^2$  than  $2CD \cdot Ll'$ ,  $2CD \cdot CH \cdot ch'$  than  $2CD^2 \cdot CA + 2AH \cdot CH \cdot ch' + CH \cdot ch' \cdot Hh'$ ,  $2CD \cdot CH$  than  $2CD \cdot cl' + 2AH \cdot CH + CH \cdot Hh'$ ,  $2DL \cdot CH$  or  $2CD \cdot AH$  or  $2CD \cdot CL$  than  $2CD \cdot cl' + CH \cdot Hh'$ ,  $2CD \cdot Ll'$  than  $CH \cdot Hh'$ , or  $2CD^2 \cdot CA$  than  $CH^2 \cdot ch'^2$ , which is impossible, for it has been proved above \*, not to be greater than  $CH^2 \cdot ch^2$ . Therefore, &c.

**Scholium.** This, according to Dr. Simson, is what the Ancients called a final determination, or one that followed the composition, and whose use was to distinguish at first sight from the data, whether the problem could be constructed at all, or not. Thus, if it be required to apply a line, verging to  $c$ , between the concavity of the semicircle  $ABC$ , and the convexity of  $DEF$ ,  $DE'f'$ , or any other that passes through  $D$ , and that line be given greater than  $BE$ , it is manifest immediately that the problem cannot be constructed.



of projection in E, D, and M: draw also EM and DM; which, because all great circles are represented by right lines in this projection, will be the representations of the circles KIQ and AIB, and consequently the former of them will be the line of measures to the latter, and be  $\perp$  to DE, which is the representation of the circle APBQ. Hence therefore the  $\angle EDM$  is the angle made by the representations of the circles APBQ and AIB on the plane of projection. It is also farther manifest that the plane angle ECM, being measured by the arch KI, is = the spherical  $\angle KAI$ , which is measured by the same arch. Take EF = EC, and draw FM. Then the  $\angle s$  FEM, CEM being both right, FE = CE, and EM common, the  $\angle EFM$  is =  $\angle ECM$ , that is, = the spherical  $\angle KAI$ , formed by the circles on the sphere: consequently the  $\angle DMF = \angle EFM - \angle EDM$  must be a maximum. But it is manifest that when the  $\angle DMF$  is a maximum a circle described through the points D, F, M, will touch EM; and consequently (Eucl. 3, 37, and 6, 17)  $EM' = DE \cdot EF = DE \cdot EC$ ; and hence comes the following

*Construction.* Draw CA, meeting DE the plane of projection in D, and CKE  $\perp$  CD; take ED = EC, describe the semicircle DMD erect EM  $\perp$  DE; draw MD, and it will be the representation required.

*Scholium.* If the difference of the angles be required equal to a given angle: take EF = EC, describe the circle DFM to contain that angle and draw the representation from D to the point where the circle cuts EM. But if, in this case, the circle neither cut nor touch EM, the problem is impossible.



*Questions proposed in 1781, and answered in 1782.*

I. QUESTION 772, by Mr. John Penberthy.

For Susan tho' long I have sigh'd, her coldness still causes me smart,  
Yet every effort's been try'd, to warm with affection her heart.  
But my wooing, alas! is in vain, and my doom irrevocable stands,  
Unless you the mystery \* explain, what 'tis that my fair one demands.

$$\left. \begin{aligned} *v^2 + vx + vy + vz &= 252 \\ x^2 + xv + xy + xz &= 504 \\ y^2 + yv + yx + yz &= 396 \\ z^2 + zv + zx + zy &= 144 \end{aligned} \right\} \begin{array}{l} \text{where } v, x, y, z \text{ denote the letters} \\ \text{in the alphabet composing the word.} \end{array}$$

*Answered by Mr. Thomas Serjeant.*

Since each equation contains the square of a letter, together with the product of that letter by the sum of all the other letters, it is evident that the sum of all the equations will be the square of the sum of all the letters, and therefore the square root of the sum of the equations is  $v + x + y + z = \sqrt{1296} = 36$ ; by which dividing each of the equa-

tions, we have  $v = 252 \div 36 = 7$ ,  $x = 504 \div 36 = 14$ ,  $y = 396 \div 36 = 11$ , and  $z = 144 \div 36 = 4$ .

But the proposer, Mr. *J. Penberthy*, dividing each equation respectively by  $v, x, y, z$ , the four quotients give each an expression for  $v + x + y + z$ ; which therefore being equated to each other, the values of three of the letters are obtained in terms of the fourth, and which being substituted in one of the original equations the whole becomes known—And Mr. *Richard Dening* and Mr. *John Mole* solve it thus: Since in the four equations, each letter is multiplied by the same quantity ( $v + x + y + z$ ), these letters are respectively proportional to the given numbers: and as these are to be whole numbers, the given numbers divided by 36 their common measure give the same values as before.

## II. QUESTION 773, by Philaethes.

A Gentleman has a right-angled triangular garden whose hypothenuse is one hundred yards, and in one of the sides of it stands a tree at the distance of twenty yards from the right angle: moreover, standing at the opposite angle, I observed the tree appeared to stand in the middle of the side. Hence I would know the area of the garden.

*Answered by the Rev. Mr. Hellins.*

If the  $\triangle ABC$  represent the garden, and  $D$  the place of the tree; and if  $C, D$  be joined, and there be drawn  $DE \perp AC$ ; it is evident from the principles of optics and geometry, that  $\angle ACD = \angle DCB$ , and  $DE = DB$ . Now as  $\frac{1}{2}AC \times DE + \frac{1}{2}BC \times DB = \text{area of } \triangle ABC$ , and  $AC$  and  $DE (= DB)$  are given, we have only to find  $BC$ , and the area will be known.

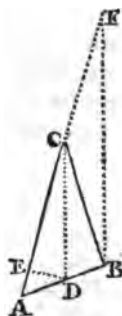
Put  $AC = 100 = a$ ,  $BD = 20 = b$ , and  $BC = x$ : then  $AB = \sqrt{(aa - xx)}$ , and the area  $= \frac{1}{2}x \sqrt{(aa - xx)} = \frac{1}{2}b \times (a + x)$  (by what hath been premised). This equation being cleared of the surd and divided by  $a + x$ , we get  $xx \times (a - x) = bb \times (a + x)$ , or  $x^3 - ax^2 + bbx + abb = 0$ , or in numbers,  $x^3 - 100x^2 + 400x + 40000 = 0$ , where the two positive values of  $x$  are 90.7326 and 26.1355; and the corresponding areas 1907.326 and 1261.355 square yards, either of which answers the question.—The third root is  $-16.8681$ .

*Scholium.* It is evident from what is done above, that, if there be drawn  $BF \parallel DC$ , meeting  $AC$  produced in  $F$ , it will be  $AF : AE :: BC^2 : BD^2$ .

*Note.* This question was inserted by particular desire, on account of a dispute which it occasioned.

## III. QUESTION 774, by Mr. Mark Elstob, of Shotton, Durham.

In surveying a circular field, which resembled the surface of a cone,



I took a station on the top of the hill in the very middle of it, and found the greatest sum of the angles formed at that point, quite around, to amount to 342 degrees. I also took the angle of depression of an object, which I knew to be 6 chains distant on a level from the hedge, in an adjacent inclosure, equal to 13 degrees one minute. Required the area of the field?

*Answered by Mr. Wm. Richardson, of Backworth.*

Let  $nco$  be a vertical section of half the hill,  $o$  the vertex,  $n$  the object, and  $c$  the centre of the circular base.

Then  $360 : 342$  or  $20 : 19 :: BO : BC$

$:: 1 : .95$ , the sine of  $\angle BOC$ , which therefore is  $= 71^\circ 48' 18''$ , and its complement

or  $\angle OBC = 18^\circ 11' 42''$ ; hence  $\angle B - \angle D = \angle BOD = 5^\circ 10' 42''$ ; then  $s. \angle O$

$: s. \angle D :: BD = 6 : BO = 14.97305$ ;

lastly, by Rule 2, page 98, of Hutton's Mensuration, the area of the field is  $BO^2 \times 3.14159 \&c. \times 19 \div 20 = 669.1045$  square chains  $= 66$  ac.  $3$  r  $25.672$  p.

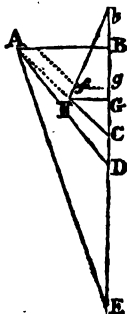


#### IV. QUESTION 775, by Mr. Joel Lean.

Standing at an unknown distance from an octagonal house, in the line perpendicular to one of its sides, I observed the angle subtended by the extreme visible corners; then advancing one hundred feet nearer in the same line, I found the same angle to be triple of what it was at the first station; and advancing twenty feet still nearer, found I was in a line with two of the sides. Required the area of the floor within; the walls being two feet thick?

*Answered by the Proposer, Mr. Joel Lean.*

Let  $E, D, C$  be the three stations,  $AF$  one side of the outer octagon, and  $FG, \perp BGCE$ , half of another,  $AB \perp BE$ . Put  $CE = 120 = a$ ,  $CD = 20 = b$ ,  $AB = BC = z$ , and  $x = \text{tangent } \angle z$ : then (by Emerson's Trigonometry b. 1, prop. 33, cor. 1.)  $(3x - x^3) \div (1 - 3x^3) = \text{tangent } \angle D$ . But, by trigonometry,  $1 : x :: a + z : z$ , and  $1 : \text{tangent } \angle D :: b + z : z$ ; from the first equation  $z = ax \div (1 - x) = (\text{from the 2nd}) (3bx - bx^3) \div (1 - 3x - 3x^3 + x^3)$ , which equation in numbers gives  $x^3 - 3.4x^3 - 3x + .6 = 0$ , the root of which is  $x = .16917$ ; and hence  $z = 24.4339 = BC = CB$ , if  $b$  is the centre of the octagon.—Then, in the  $\triangle bfg$  are given  $bg = 22.4339$ , and all the angles, viz.  $\angle f = 45^\circ$ , and  $\angle b = 22^\circ 30'$ , hence  $16 \times \triangle bfg = 8$



$\times bg^2 \times (\sqrt{2} + 1) \times (2 - \sqrt{2}) \div (2 + \sqrt{2}) = 8bg^2 \times (\sqrt{2} + 1) \times (\sqrt{2} - 1)^2 = 8 \times bg^2 \times (\sqrt{2} - 1) = 3.313708 bg^2 = 1667.724$ ,  
the area of the floor within the walls.

V. QUESTION 776, by Mr. John Turner.

If through the focus of a conic hyperbola any right line be drawn, terminated by the curve on both sides ; the rectangle of the two parts intercepted by the focus and the curve, applied to the whole line, will be a constant quantity. Required the demonstration ?

*Answered by Horticultura.*

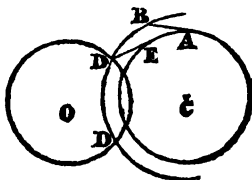
It is demonstrated, (Simson's Conics, cor. to prop. 29, lib. v. or Emerson's Conics, cor. 3, to prop. 65,) that the rectangle under the segments of the line drawn through, and made at the focus of an (ellipsis or) hyperbola, is equal to the rectangle under the whole line and  $\frac{1}{4}$  the latus rectum of the axis. Consequently the constant quantity in question is  $= \frac{1}{4}$  the said latus rectum.

VI. QUESTION 777, by the Rev. Mr. Crakelt.

Two circles being given both in position and magnitude, it is required to find a point in the circumference of one of them, from whence if a tangent be drawn to cut that of the other, the part of it intercepted between the two circumferences may be equal to a given line, and to determine the limits of possibility.

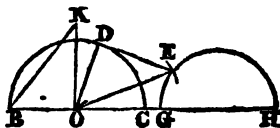
*Answered by Amicus.*

To any point A of the circle, let a tangent AB of the given length be drawn, and with CB radius describe a circle cutting the other given one in D, then drawing the tangent DE to the former given one, E will be the point required. For DE is evidently equal BA ; and when the circle BD neither cuts nor touches DD, it is manifest that the problem is impossible.



*The same by Mr. John Fletcher, of Chester.*

BH being the line passing through the centres of the two given circles, on O the centre of one of them erect OK  $\perp$  BH and  $=$  the given tangent ; and with the centre O and radius BK cut the other circle in E, the point required. For, drawing the tangent ED, and OD, then BO  $=$  OD, BK  $=$  OE, and  $\angle O = \angle D$  a right angle, therefore DE  $=$  OK, the given line.



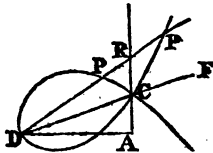
*Limit.* It is evident, that the given line cannot be greater than the tangent drawn from H, nor less than the tangent from G, which are the greatest and least tangents; that is, it must not be greater than a mean proportional between HC and HB, nor less than a mean proportional between GC and GB.

VII. QUESTION 778, by Mr. Wm. Cole.

If from the acute angle at the base of a given right-angled triangle, a right line be drawn to intersect the perpendicular, produced ad libitum; and if the said line be supposed to revolve about the angular point from which it is drawn; it is required to find the nature of the curve described by a point in that line whose distance from the intersection is, always, equal to the distance of the intersection from the vertex of the given triangle?

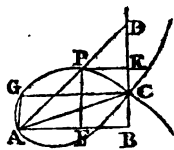
*Answered by Amicus.*

Suppose DAC to be the given triangle, and DEP a position of the revolving line; then the figure corresponds with figure 38, table 5, of Maclaurin's Geometrica Organica, where the curve is proved at page 36, to be Sir Isaac Newton's 34th species; and when CR is taken = CD, its asymptote passes through F and || AC.



*The same by Mr. Nathan Parnel.*

Let ABC be the triangle, AD the revolving line, and P the point. Then put  $a = AB$ ,  $b = BC$ ,  $x = PE$ , and  $y = CE$ ; and we shall have  $a - x = AE$ ,  $b + y = FP = BE$ ; also, by similar  $\Delta$ s,  $a - x : b + y :: x : (b + y) \times x \div (a - x) = DE$ ; and, by 47 Eucl. 1.  $x^2 + (b + y)^2 \times x^2 \div (a - x)^2 = PD^2 = DC^2 = (y + (b + y) \times x \div (a - x))^2$  by the question; this equation, reduced, becomes  $ax^2 - x^3 = ay^2 + xy^2 + 2bxy$ , the equation expressing the nature of the curve.—When  $y = 0$ ,  $x$  is  $= 0$  or  $= a$ ; which shews that the curve passes through the angular point C of the rectangle AOCB. And when  $y = -b$ ,  $x$  is  $= a$  or  $= b$ : which also shews the curve passes through the angular point A, and likewise decussates the base AB at the distance of BC from the right angle.

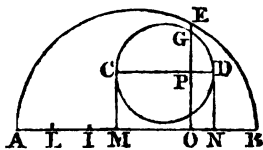


VIII. QUESTION 779, by Mr. Nathan Parnel.

Given the magnitude and position, of two circles, to draw a right line parallel to a right line given in position, to cut both circles, so that the chords of the segments cut off shall be in a given ratio.

*Answered by the Proposer, Mr. Nathan Parnel.*

**Construction.** Draw the diameters  $AB$  and  $CD$ , of the given circles  $AEB$  and  $CGD$ , in a direction perpendicular to the line given in position, also  $CM$  and  $DN$  parallel to the same line or  $\perp AB$ , and in  $AB$  take  $AI : IB :: R : s$  in the given ratio, also take  $AL : AI :: AI : IB$  the same ratio, or  $AL \times IB = AI^2$ ; lastly,  $\perp$  to  $AB$  draw  $EGPO$  (by problem 7, book 2, Wales's Determ. Section) cutting  $AB$  in  $O$ , so that  $AO \times OB :: MO \times ON :: IB : AL$ ; so shall  $EO : GP :: s : R$ , as required.

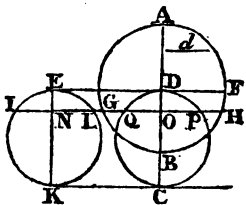


**Demonstration.** By construction  $MO = CP$ , and  $ON = PD$ , therefore by prop. of circle  $AO \times OB = EO^2$ , and  $MO \times ON = PG^2$ ; whence by construction  $EO^2 : GP^2 :: IB : AL :: IB^2 : AL \times IB = AI^2$ , or  $EO : GP :: IB : AI :: s : R$ .

**Note.** The construction would be performed nearly in the same manner, let the positions of the circles be what they may.

*The same by Mr. John Hampshire.*

**Construction.** Let  $d$  be the line given by position,  $R$  to  $s$  the given ratio, and  $AGBH$ ,  $EIKL$  the two given circles; to the lesser of which parallel to  $d$  draw the indefinite tangents  $EF$  and  $KC$  and,  $\perp$  to  $EF$  or  $KC$  the diameters  $ADB$  and  $ENK$  to cut the tangents in  $D$ ,  $C$  and  $E$ ,  $K$ . Then (by problem 4 of Lawson's, or problem 7 of Wales's Determ. Section) cut  $AB$  (produced if necessary) in  $O$ , so that  $OA \times OB :: OC \times OD :: R^2 : s^2$ ; through  $O$  and  $\parallel$  to  $d$  draw  $ILGOH$ , the line required.



**Demonstration.** On the diameter  $nc$  describe the circle  $npcq$  to touch the tangents  $EF$  and  $KC$  in  $D$  and  $C$ . Then, by parallel lines,  $CD = KE$ ,  $CO = KN$ , and  $DO = EN$ , consequently  $pQ = IL$ . But, by construction and the property of the circle,  $OA \times OB = OH^2 : OC \times OD = OP^2 = IN^2 :: R^2 : s^2$ , or  $GH : IL :: R : s$ .

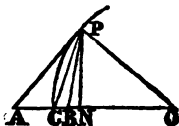
#### IX. QUESTION 780, by Nauticus:

Two ships sail at the same time from a port in latitude 48 degrees 16 minutes, north, and arrive at the same time also at two others lying in the same parallel of latitude; one sails at the rate of 7, and the other at the rate of 9 knots; and the angle included by their two tracks is bisected by the N. N. E. rhumb, it is also known that the

sum of the distances run by the two ships, and the distance between the two ports, is 656 miles. Required the distance between the ports, the latitude they are in, and the course and distance run by each ship?

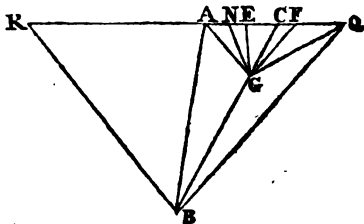
*Answered by Amicus.*

If  $P$  be the port sailed from,  $PN$  its meridian,  $PC$  a small part of the N. N. E. rhumb,  $NO \perp PN$ ,  $O$  the centre of a circle passing through  $P$  and  $C$ , and  $PN = \text{unity}$ ; then  $PO = CO = \sqrt{2}$ , and  $CN = \sqrt{2} - 1$ ; and, by the lemma at page 336, Simpson's Algebra, if there be taken  $AC : CO :: AC - BC : BC :: AP - PB : PB ::$  (by the question)  $2 : 7$ , then  $PA$  and  $PB$  will be the courses of the ships,  $AC = \frac{2}{7}\sqrt{2} = .4040610$ ,  $CB = \frac{5}{7}\sqrt{2} = .3142697$ ,  $AB = .7183307$ ,  $AN = .8182746 = \text{tangent of } 39^\circ 17'$  one course,  $AP = 1.292118$ ,  $BN = .0999439 = \text{the tangent of } 5^\circ 42'$  the other course,  $PB = \frac{1}{7}AP = 1.0049806$ . And since the whole rhumbs are proportional to their parts  $PA$ ,  $PB$ , and the distance of the ports may without sensible error be taken along the parallel of latitude that passes through them, we have  $3.0154293 = AP + PB + AB : 656$  (per question)  $:: AP : 281.0977 = \text{the distance run on the course } PA :: PB : 218.63 = \text{its distance run} :: AB : 156.2708 = \text{the distance of the ports arrived at} :: PN : \text{the difference of latitude} = 217.5478 \text{ miles.}$



*The same by Mr. John Brinkley, of Harleston.*

Upon an east-and-west line  $RQ$  take  $AQ = 656$ , the sum of the three sides of the triangle formed by the three ports which divide in  $c$  in the given ratio, namely  $7 : 9 :: AC : cq$ , and draw  $CB$  the N. N. E. rhumb; also take  $CR : cq :: CA : cq - CA$ , and with centre  $R$  and radius  $RC$  cut  $CB$ ; draw  $gq$  to bisect the  $\angle Q$ ; then draw  $GF \parallel BQ$ , and  $GE \parallel BA$ ; so shall  $C$ ,  $E$ ,  $F$  be the three ports.



For since  $AB : BQ :: AC : cq :: 7 : 9$ , by lemma, page 336, Simpson's Algebra, therefore  $\angle B$  is bisected by  $BC$ , and  $ABQ$  is similar to the triangle formed by the three ports. Because  $GF \parallel BQ$ , therefore  $\angle FgQ = \angle gQB = \angle FgQ$ , therefore  $GF = FQ$ . And because  $gq$  bisects  $\angle Q$  therefore  $BG : GC :: BQ : QC ::$  (by construction)  $BA : AC$ , therefore  $AC$  bisects the  $\angle A$ ; and for the like reason that  $gQ = FQ$ , is  $EA = EG$ . Consequently  $EG + EF + GF = AQ$  the given sum, and  $EGF$  is similar to  $ABQ$ .

*Calculation.* In the  $\triangle BQR$  are known  $RB$ ,  $RQ$ , and the  $\angle R (= 2 \text{ compl. } \angle c)$ , whence is found  $\angle Q = 50^\circ 42'$ , and from thence the  $\angle QAB = 95^\circ 42'$ , and  $QBA = 33^\circ 36'$ . Also in the  $\triangle CQB$  are known  $cq$

and the angles, hence  $CG$  will become known, and from thence  $FG = 281.068$ ,  $EG = 218.608$ , and  $EF = 156.324$ ; also the difference of latitude  $GN = 217.502$  miles; therefore the latitude sailed to was  $51^\circ 53' N.$  and the course of both ships was from the  $N.$  towards the  $E.$  that of the one  $5^\circ 42'$ , and the other  $39^\circ 18'$ .

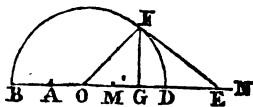
X. QUESTION 781, by Mr. Geo. Sanderson.

To produce the diameter  $BD$  of a given circle to  $E$ , so that, drawing the tangent  $EF$ , and from the point of contact  $F$  letting fall the perpendicular  $FG$  on the diameter  $BD$ , it may divide the line  $AE$  in a given ratio at  $G$ ;  $A$  being a given point in the diameter  $BD$ .

*Answered by Amicus.*

Produce the given diameter  $BD$  to  $N$  till  $DN : DO$  in the given ratio, and from the given point  $A$  take  $AM : AO :: ON : OD$ , then by Simpson's Geometry 5, 18, take  $OE : OD :: ON : ME$ , and  $E$  is the point required.

For since  $OE : OD :: ON : ME ::$  (by similar  $\Delta$ s)  $OF : OD :: OG$ , therefore  $ON : OD :: ME : AE - AM : OG = AG - AO$ ; but, by construction,  $AM : AO :: ON : OD$ , consequently  $ON : OD :: AE : AG$ , or by division,  $DN : DO :: GE : GA$ .



*The same by Mr. Thomas White, of Anwick.*

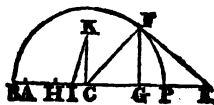
$c$  being the centre, and  $m : n$  the given ratio, bisect  $AC$  in  $u$ , and take  $HI : HC :: m : m + n$  and  $CK^2 : CD^2$ , erecting  $CK \perp CD$ ; make  $IG = IK$ , erect the  $\perp GF$ , and draw the tangent  $FE$  meeting  $BD$  in  $E$ , the point required.

**Demonstration.** For  $m + n : m :: CD^2 : CK^2$  by construction. But  $CD^2 = CF^2 = CG \times CE$  (Eucl. 6, 8), and  $CK^2 = IK^2 - IC^2$  (1, 47)  $= IG^2 - IC^2$  (construction)  $= CG^2 + CG \times 2CI$  (2, 4). Therefore  $m + n : m :: (CG \times CE : CG^2 + CG \times 2CI ::) CE : CG + 2CI$  (6, 1).

Again by construction  $m + n : m :: HC : HI :: (2HC \text{ or } AC : 2HI$  (5, 15).

Therefore (5, 12)  $m + n : m :: (CE + CA \text{ or } AE : (CG + 2CI + 2HI = CG + 2CH = CG + CA =) AG$ , and  $n : m :: GE : GA$  (5, 17.)

**Scholium.** Hence the least triangle and cone may be geometrically described about a circle and a sphere, or any segment of them;  $A$  being the centre of the base of the segment,  $F$  the point of contact, and  $E$  the vertex of the triangle or cone; also in the triangle and circle  $AG = GE$ , but in the cone and sphere  $AG = \frac{1}{2}GE$ . See the Scholia, page 201 and 209, Simpson's Geometry.



## XI. QUESTION 782, by Mr. Henry Clarke.

It is required to exhibit (without circular arcs or logarithms) a finite value of the expression  $\frac{1}{2}x^{-\frac{1}{2}} \int \left\{ x \times \frac{2}{3} \int (x^{-\frac{1}{2}} \times \frac{1}{4} \int ((1-x)^{-\frac{1}{2}} \dot{x} + (1-x)^{-\frac{1}{2}} x \dot{x} - \dot{x})) \right\}$ , when  $x$ , (which is supposed to begin from 0) is ultimately expounded by  $\frac{1}{2}$ ;  $\int$  denoting the fluent of the whole quantity under its respective vinculum.

*Answered by Mr. Rob. Hartley, of Daresbury.*

It is evident that  $\frac{1}{4} \text{ fl. } ((1-x)^{-\frac{1}{2}} \dot{x} + (1-x)^{-\frac{1}{2}} x \dot{x} - \dot{x})$  is  $= \frac{1}{4} \text{ fl. } (\dot{x}(1-x)^{-\frac{1}{2}} - \dot{x})$  (See page 116 of Clarke's Infinite Series)  $= \frac{1}{4} \times (2(1-x)^{-\frac{1}{2}} - x - 2)$ ; therefore the whole given expression becomes  $(1 \div x^{\frac{1}{2}} \sqrt{x}) \text{ fl. } (\dot{x} \times \text{fl. } (x^{-\frac{1}{2}} \dot{x} (1-x)^{-\frac{1}{2}} - \frac{1}{2} x^{\frac{1}{2}} \dot{x} - x^{-\frac{1}{2}} \dot{x}))$   $=$  (by page 16 of the same book)  $\frac{1}{2} x^{-\frac{1}{2}} \text{ fl. } (x^{-\frac{1}{2}} \dot{x} (1-x)^{-\frac{1}{2}} - x^{\frac{1}{2}} \dot{x} - x^{-\frac{1}{2}} \dot{x}) - \frac{1}{2} x^{-\frac{1}{2}} \text{ fl. } (x^{\frac{1}{2}} \dot{x} (1-x)^{-\frac{1}{2}} - \frac{1}{2} x^{\frac{1}{2}} \dot{x} - x^{\frac{1}{2}} \dot{x}) = ax^{-\frac{1}{2}} - \frac{1}{6} - x^{-1} - \frac{1}{2} ax^{-\frac{1}{2}} + \frac{1}{2} x^{-2} \sqrt{1-x} + \frac{1}{10} + \frac{1}{3} x^{-1} = \frac{2x-1}{2x^{\frac{1}{2}} \sqrt{x}} a + \frac{\sqrt{1-x}}{2x^{\frac{1}{2}}} - \frac{2}{3x} - \frac{1}{15}$ , where  $a$  is the circular arc, radius 1, sine  $\sqrt{x}$ . And when  $x = \frac{1}{2}$ , the first term is  $= 0$ , and the others become  $\sqrt{2} - \frac{1}{3}$ , the quantity sought.

*The same by Amicus.*

The last correct fluent of the given expression, as above, is  $= \frac{1}{4} \times 2(1-x)^{-\frac{1}{2}} - x - 2$ , hence that of  $\frac{2}{3} x^{-\frac{1}{2}} \dot{x} \times \frac{1}{4} \times \left\{ 2(1-x)^{-\frac{1}{2}} - x - 2 \right\}$  or of  $\dot{x} (x - xx)^{-\frac{1}{2}} - \frac{1}{2} \dot{x} \sqrt{x} - x^{-\frac{1}{2}} \dot{x}$  is  $v - \frac{1}{3} x \sqrt{x} - 2 \sqrt{x}$  ( $xv$  being  $= \dot{x} x^{\frac{1}{2}} (1-x)^{-\frac{1}{2}}$ ; hence the fluent of  $v \dot{x} - \frac{1}{3} x \dot{x} \sqrt{x} - 2 \dot{x} \sqrt{x} = vx - \text{fl. } xv - \frac{1}{3} x \sqrt{x} - \frac{2}{15} x^{\frac{3}{2}} \sqrt{x} = vx - \text{fl. } x^{-\frac{1}{2} + \frac{1}{2}} \dot{x} (x - xx)^{-\frac{1}{2}} - \frac{1}{3} x \sqrt{x} - \frac{2}{15} x^{\frac{3}{2}} \sqrt{x} = (x - \frac{1}{2}) \cdot v + \sqrt{(x - xx)} - \frac{2}{15} x \sqrt{x} \times (10 + x) = (\text{when } x = \frac{1}{2}) \frac{1}{2} - \frac{1}{15} \sqrt{\frac{1}{2}} \times 10\frac{1}{2}$ ; which being multiplied by  $\frac{1}{2} x^{-\frac{1}{2}}$  or  $2\sqrt{2}$ , it becomes  $\sqrt{2} - \frac{1}{3}$ .

## XII. QUESTION 783, by Mr. Thomas Moss.

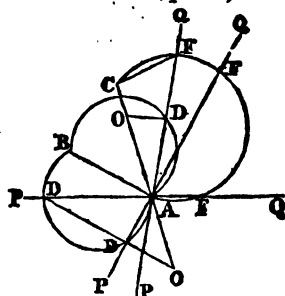
Through a given point A, so to draw an indefinite right line  $pq$ , to which if lines  $bd$ ,  $cf$  be drawn from two other given points  $b$ ,  $c$ , and forming given angles with the said indefinite line  $pq$ , the rectangle contained under the parts  $ad$ ,  $af$ , intercepted by the given point A and the two lines so drawn, shall be equal to the square of a given line  $mn$ .

*Answered by Mr. Geo. Sanderson.*

Draw  $ba$ ,  $ca$ , on which describe the segments  $abd$ ,  $acf$  to contain

the given angles; on CA, or CA produced as the case requires, take AO such, that  $AO \times AC = MN^2$ ; draw OD to make the  $\angle AOD =$  the angle which CF is to make with FQ, and meeting the circle BDA in D; through A, D draw FQ cutting the other circle in F, and the thing is done.

For the  $\Delta s$  ACF, ADO, having  $\angle O = \angle F$ , and angles at A equal or common, are similar; whence  $AD : AO :: AC : AF$ , and  $AD \times AF = AO \times AC = MN^2$ , as required.

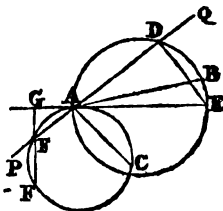


*The same by Mr. Robert Bownas, of Bramley near Leeds.*

**Construction.** Draw AB, AC, on which describe the segments ABD, ACF to contain the given angles. Draw either of the diameters AE, on which continued take AG such, that  $EA \times AG = MN^2$ ; make  $GF \perp GE$ , and through its intersection F with the other circle draw the required line FAQ.

**Demonstration.** When DE is drawn, the  $\Delta s$  ADE, AGF will be similar; therefore  $DA \times AF = EA \times AG = MN^2$ .

**Note.** It is evident that when GF touches the circle, the rectangle is the greatest. And that when it meets it not, the problem is impossible.

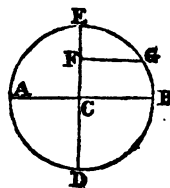


### XIII. QUESTION 784, by Terricola.

If a ball be let fall from the surface, down a perforation made diametrically through the earth; it is required to find its velocity and time of falling to the centre, and to any given point, with the other circumstances of its motion; abstracted from the effect of the earth's rotation; and on the supposition that the earth is a homogeneous sphere of 8000 miles diameter.

*Answered by Mr. Rob. Phillips, of St. Agnes.*

Put the earth's radius CE or CB = 4000 miles = 21120000 feet =  $a$ ,  $32\frac{1}{2}$  feet = the force of gravity =  $s$ , the distance CF of the body from the centre at the end of any time ( $t$ ) =  $x$ , and the velocity at F =  $v$ . Then, by the laws of attraction,  $a : s :: x : sxa^{-1}$  = the force at F, and by the principles of motion  $v\dot{v} = -sxa^{-1}$ , and taking the fluents  $v^2 = -sx^2 a^{-1}$ ; but when  $v = 0$ ,  $x = a$ , therefore the correct fluent



is  $v^2 = sa^{-1} \times (a^2 - x^2)$ , and hence  $v = \sqrt{(s \div a)} \times \sqrt{(a^2 - x^2)} = \sqrt{(s \div a)} \times ra$ , the velocity of the ball at  $r$ . And when  $x = 0$ , this becomes  $\sqrt{sa} = \sqrt{(s \times Ac)}$  the greatest velocity, or that at the centre  $c$ , which is = 26064½ feet per second.

Again, the Fluxion of the time, or  $\dot{t}$  is  $-\dot{x}v^{-1} = \sqrt{(a \div s)} \times -\dot{x} \times (a^2 - x^2)^{-\frac{1}{2}}$ , and the correct fluent gives  $t = \sqrt{(a \div s)} \times \text{circ. arc whose radius is 1 and cosine } xa^{-1} = (as)^{-1} \times \text{arc, radius } a$  and cosine  $x = EG \div \sqrt{(s \times Ac)} =$  the time of describing  $EF$ . And when the ball arrives at  $c$ , this becomes  $t = EB \div \sqrt{(s \times CB)} = (EB \div CB) \sqrt{(CB \div s)} = (EB \div CB) \sqrt{(a \div s)} = 2 \times .78539$ , &c.  $\sqrt{(a \div s)} = 1272.82$  seconds =  $21^m 12.82^s$  the time of falling to the centre.

*Corol.* Since any active force, acting in contrary directions, always generates or destroys an equal quantity of motion in the same time, it is evident that after the body passes the centre, its velocity at all equal distances on either side will be equal; and when it arrives at the opposite surface, its velocity will be quite destroyed, and the body will again fall towards the centre, and proceed till it arrives at the surface again; and thus it will oscillate forward and backward continually; and the whole time of a double oscillation, or of quitting  $z$  till it arrive at  $z$  again, will be quadruple the time above found for passing over the radius  $ec$ , and will therefore be  $2 \times 3.14159$ , &c.  $\sqrt{(a \div s)} = 1^h 24^m 51.28^s$ .

#### XIV. QUESTION 761, by Amicus.

Given the perimeter, and the two differences of the sides and segments of the base in one sum, to construct the triangle a maximum.

*Answered by Mr. George Sanderson.*

*Lemma.* If the point  $A$  in the right line  $GE$  be given, and it be required to find the points  $O$  and  $I$ , in  $GA$  and  $GE$ , respectively, such that  $GA : AE :: GO : EI$ ; I say the solid  $OI \times GO^2$  will be the greatest possible, when  $GO = 2AO$ . For  $GE : GA :: OI : OA :: OI \times GO^2 : OA \times GO^2$ , therefore when  $OI \times GO^2$  is a maximum,  $OA \times GO^2$  is so too, which by theorem 17, Simpson on the Maxima and Minima, is when  $GO = 2OA$ .



*Construction.* On  $GA$  the given sum of the differences, and  $AE =$  the given perimeter, take  $GK$  and  $AK$  each =  $\frac{1}{2}GA$ , also take  $EF = \frac{1}{2}AE$ ; on the diameter  $KF$  describe a semicircle, to which apply  $AD =$

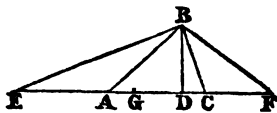
$ak$  ( $AG - AK$ ) which continue to  $B$ , and from the centre  $c$  draw  $CB$ ; so is  $ACB$  the triangle required.

*Demonstration.* Take  $GO = 2Gk$  ( $2AK$ ), and  $EI = 2EF$ , draw  $CD$ , and the  $\perp$   $CP$ . Then  $AD$  ( $AG - AK$ ) is manifestly  $= AP - PB$  ( $DP$ ), and  $AF = AC + CB$ ,  $KA = AC - CB$ , and (by Euc. cor. 3, 36, and 6, 14.)  $AK (Gk) : AB :: AG : AE :: Gk : FE$ , by construction, therefore  $AB = EF$ , and  $AC + CB$  ( $AF$ )  $+ AB = AE$  the given perimeter. Also the  $\triangle ABC$  is a maximum, for  $2AB (EI)$  is to  $2AB (GO)$  in the given ratio of  $AE$  to  $AG$ , therefore when  $\frac{1}{2}AB \times CP$  or the area of the  $\triangle ACB$  is a maximum,  $AK \times PC$  must be so likewise, and consequently  $2AK (GO) \times 2CP$  and  $GO^2 \times 4CP^2$  are so; but  $4CP^2 = 4CD^2 - 4DP^2 = FK^2 - DB^2 = (FK + DB) \times (FK - DB)$ , therefore  $GO^2 \times (FK - DB)$  is a maximum, because  $FK + DB = AE - AG$ , a given quantity, but  $FK - DB = AF - AB + AD - AK = OI$ , and by the lemma  $GO^2 \times OI$  is a maximum when  $GO = \frac{2}{3}AO$ , therefore the  $\triangle ACB$  is a maximum.

In the very same manner is the solution given by the proposer, *Amicus*, who farther remarks, that in this triangle are given the sums of each side, and its adjacent segment of the base, which is the 713d Diary question.

*The same answered by Mr. Nathan Parnel, of Nuneaton.*

If  $ABC$  be the triangle required, and  $BD$  the  $\perp$ ; then if  $AB + BC + AC = AB + BC + AD + DC$ , and  $AB + BC - AD - DC$  be given, the half sum and half difference of these two quantities are also given, that is,  $AB + BC$ , and  $AC$  are given, in which case (Simpson's Geometry, page 198, theorem 5), the triangle will be an isosceles one when a maximum, and in this case I have omitted the construction, it being so well known. But if  $AB + AD - BC - DC$  be given, instead of  $AB + BC - AD - DC$ , the construction will be as follows. Make  $EG = GF = \frac{1}{2}$  the given perimeter;  $GD = \frac{1}{2}$  the given sum of the two differences of the sides, and of the segments of the base; and  $BD \perp EF$ , and of such a length that  $BD^2 = \frac{1}{3}ED \times DF$ ; then draw  $EB$ ,  $BF$ ; as also  $BA$ ,  $BC$  making the  $\angle ABE = \angle E$ , and  $\angle CBF = \angle F$ ; and  $ABC$  will be the triangle required. — For by reason of the equal angles, we have  $AB = AE$ , and  $CB = CF$ ; consequently  $AB + AD = ED$ , and  $BC + CD = DF$ ; therefore  $AB + BC + AC = AB + BC + AD + DC = ED + DF$ , the given perimeter; and  $AB + AD - BC - DC = ED - DF = 2DG$ , the given sum; and (47, Euc. I. &c.)  $AB - AD = BD^2 \div DE$ , and  $BC - DC = BD^2 \div DF$ ; and consequently  $2AD = ED - BD^2 \div DE$ , and  $2DC = DF - BD^2 \div DF$ ; and because the  $\triangle ABC$  is a maximum,  $2AC \times BD = (2AD + 2DC) \times BD = (ED + DF - BD^2 \div ED - BD^2 \div DF) \times BD = (ED \times DF - BD^2) \times BD \times (ED + DF) \div ED \times DF$  is a maximum, and since  $ED$  and  $DF$  are given,  $(ED \times DF - BD^2) \times BD$  is evidently a maximum, therefore (by Simpson's Geometry, page 208, theor. 18),  $ED \times DF - BD^2 = 2BD^2$ , or  $BD^2 = \frac{1}{3}ED \times DF$ , as by construction.



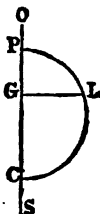
*Note.* The above contains a solution to the 743d Diary question.

XV. QUESTION 786, *by Plus Minus.*

It is required to find the length of a pendulum whose vibrations are isochronous to those of a given cylinder, when a given point in its axis is made the point of suspension, and also what point in its axis is the point of suspension when the time of vibration is the shortest?

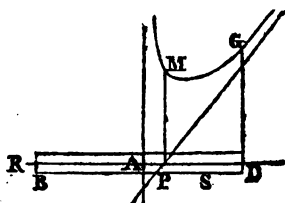
*Answered by Amicus.*

If  $os$  be the axis of a cylinder, the radius of whose base is  $= r$ , it is proved by writers on this subject, that when it is suspended at one end  $o$ , the distances  $co$  and  $cg$  from the centre of oscillation  $c$  to the point of suspension  $o$ , and to the centre of gravity  $g$ , will be respectively  $(4os^2 + 3r^2) \div 6os$  and  $(os^2 + 3r^2) \div 6os$ . And when it is suspended at any other point  $p$ , we have  $pg : og = \frac{1}{2}os :: (os^2 + 3r^2) \div 6os : (os^2 + 3r^2) \div 12pg = gc$ , the distance of the centres of gravity and oscillation when it is suspended at  $p$ ; and erecting  $gl \perp os$  so that  $12gl^2 = os^2 + 3r^2$ , and with the centre in  $os$  through the points  $p$  and  $l$  describing a semicircle, its diameter  $pc = pg + gl \div pg$  will be the length of an isochronous pendulum, and will evidently be a minimum when  $gl = gp = gc =$  the radius of the circle.



*The same answered by Plus Minus, the Proposer.*

Let  $bd$  be the given cylinder,  $p$  the given point of suspension in the axis  $bd$ . Bisect  $bd$  in  $a$ ; produce it also to  $r$ , which suppose to be the centre of oscillation when  $p$  is the point of suspension; then will  $r$  be the centre of oscillation if  $r$  be the point of suspension. And if you call  $bd$ ,  $a$ ;  $ap$ ,  $x$ ;  $br$ ,  $z$ ; and the radius of the cylinder's base,  $r$ ; you will have, by page 238, Emerson's Fluxions,  $pr = (12x + 12ax + 4a^2 + 3r^2) \div (12z + 6a) = x + \frac{1}{2}a + z (= ap + ab + br)$ , consequently  $z = (a^2 + 3r^2 - 6ax) \div 12x$ ; and  $z + x + \frac{1}{6}a (pr) = x + \frac{1}{2}a + (a^2 + 3r^2 - 6ax) \div 12x = (12x^2 + a^2 + 3r^2) \div 12x = y (pm)$ . Or  $12xy - 12x^2 - a^2 - 3r^2 = 0$ .



The hyperbola, which is the locus of this equation, is constructed by drawing two indefinite right lines through  $a$ , one  $\perp ad$ , the other at an angle of  $45^\circ$ , for asymptotes; and then drawing  $de \perp ad$  cutting

one asymptote, and  $= \frac{2}{3}a + \frac{1}{3}r^2a^{-1}$ ; then through  $a$  describe the opposite sections.

Every ordinate of these hyperbolas is the length of a pendulum, whose vibrations are isochronous to the vibrations of the cylinder, when the point, which is the foot of that ordinate, is its point of suspension. The above equation of the hyperbola being reduced, gives  $x = \frac{1}{2}y \pm \frac{1}{2}\sqrt{(y^2 - r^2 - \frac{1}{3}a^2)}$ , and the ordinate ( $y$ ) being a minimum when the two values of the abscissa become equal, in that case we shall have  $x = \frac{1}{2}y$ , and  $\sqrt{(y^2 - r^2 - \frac{1}{3}a^2)} = 0$ , or  $y = \pm \sqrt{(r^2 + \frac{1}{3}a^2)}$ , and consequently  $x = \pm \frac{1}{2}\sqrt{(r^2 + \frac{1}{3}a^2)}$ . If then you take  $A$  as this value of  $x$ ,  $s$  will be the point of suspension when the vibrations of the cylinder are made in the shortest time. And in this case the point of suspension and centre of oscillation are equidistant from  $A$ , the middle of the axis.

THE PRIZE QUESTION, by Peter Puzzlem.

To find the force and its direction requisite (at every instant) to cause a projectile to describe such a trajectory, that the body shall always be found in the arc of a given conic parabola, revolving with an invariable angular velocity about its axis; the direction of the required force being always in a plane passing through the focus at right angles to the plane of the revolving parabola?

*Answered by Peter Puzzlem, the Proposer.*

Let  $b$  be the angular velocity of the parabola about its axis, measured at the distance  $r$  therefrom;  $v$  the velocity of the projectile from, and  $v$  the velocity parallel to the axis;  $f$  the force towards, and  $h$  the force parallel to the same axis; and  $g$  the force at right angles to the plane of the parabola.

Then, by Landen's IXth Mathematical Memoir,  $f$  will be  $= \frac{b^2 y \dot{y} - r^2 v \dot{v}}{r^2 \dot{y}}$ ,  $g = \frac{2bv}{r}$ , and  $h = \frac{v \dot{v}}{x}$ ; where  $v : v :: \dot{y} : \dot{x}$ . Now,

$4rx = y^2$  being the equation of the parabola,  $f \cdot (x - r) + hy$ , the force at right angles to the ray from the focus, by the question must be  $= 0$ ; and therefore the force towards the focus, or  $(fy + h \cdot (r - x)) \div (r + x)$ , will be  $= ((r - x)^2 + y^2) \div (r^2 - x^2) + h$ . Moreover, by substituting properly in the equation  $f \cdot (x - r) + hy = 0$ , we get  $v \dot{v} + \frac{2v^2 y \dot{y}}{4r^2 + y^2} = \frac{b^2 y \dot{y}}{r^2} \times \frac{4r^2 - y^2}{4r^2 + y^2}$ : whence  $v = \frac{b}{r}$

$\times \frac{\sqrt{(k + 16r^4 y^2 - \frac{1}{3}y^6)}}{4r^2 + y^2}$ , where  $a$  and  $c$  are contemporary values of  $y$  and  $v$ , and  $k = c^2 r^2 b^{-2} \cdot (4r^4 + a^2)^2 - 16r^4 a^4 + \frac{1}{3}a^6$ . The value of  $v$  being thus found, the value of  $v$  ( $= \frac{1}{2}vyr^{-1}$ ) will be

known; and consequently the force  $g$  and the force towards the focus, of which the required force will be compounded.

Putting  $w$  for  $(4r^2 + y^2) \div 4r$ , the focal distance;  $t$  the fluxion of the time, will be  $= \frac{\sqrt{3} \times r w \dot{w}}{2b\sqrt{(w-r)} \times \sqrt{(k + 3rw^2 - w^3)}}$ ,  $k$  being  $= 3k \div 64r^3 - 2r^2$ : the fluent of which fluxion will, in some cases, be assigned by the arcs of the conic sections. See the *Append.* to LANDEN's *Mathematical Memoirs*.

*Remark 1.* If  $k$  be  $= 0$ , the body, either immediately, or some time after it is put in motion, will continually approach nearer and nearer to the vertex of the parabola, yet never can arrive at it!

*Remark 2.* If  $k$  be positive, the body will pass and repass through the vertical point, and its recess therefrom will be limited by a circle, to whose plane the axis of the parabola will be perpendicular, and whose radius will be the positive root ( $y$ ) of the equation  $y^6 - 48r^4y - 3k = 0$ ; which circle will touch the trajectory at each apsis.

*Remark 3.* If  $k$  be negative, the trajectory will be situated between, and limited by, two circles, to each of whose planes the axis of the parabola will be perpendicular, and the radius of each will be a positive root of the equation  $y^6 - 48r^4y - 3k = 0$ ; which circles will touch the trajectory, the one at the apsides nearest to the vertex of the parabola, and the other at the apsides most remote from the said vertex.



### Questions proposed in 1782, and answered in 1783.

#### I. QUESTION 788, by Mr. Wm. Swift, of Stow near Lincoln.

Required the value of  $x$  as below \*,  
And you will oblige your most humble at Stow.

\* By a quadratic, supposing  $x^4 - 2x^2 + x = a$ .

*Answered by Mr. Charles Trelease.*

Compare the given equation,  $x^4 - 2x^2 + x = a$ , with the general one  $x^4 + ax^3 + bx^2 + cx + d = 0$ , and we shall have  $a = -2$ ,  $b = 0$ ,  $c = 1$ , and  $d = -a$ . Assume  $f = b - \frac{1}{4}aa = -1$ . Now it is a known property of biquadratic equations (see *Simpson's Algebra*, page 154 and 155), that if  $c = \frac{1}{2}af$  or  $d$  either  $= c^2 \div 4f$  or  $= c^2 \div a^2$ , the root may be obtained by the resolution of a quadratic only; the first case of which succeeding in the equation before us, we have

$$x = -\frac{1}{2}a \pm \sqrt{(\frac{1}{16}aa - \frac{1}{2}f \pm \sqrt{(\frac{1}{4}ff - d))} = \frac{1}{2} \pm \sqrt{(\frac{1}{4} \pm \sqrt{(\frac{1}{4} + a))}}.$$

*The same answered by the Rev. Mr. Hellins.*

The given equation,  $x^4 - 2x^2 + x = a$ , is evidently  $x^4 - 2x^2 + x^2 - x^2 - x = (x^2 - x)^2 - 1 \times (x^2 - x) = a$ , a quadratic; from which, completing the square, &c. we get  $x^2 - x = \frac{1}{2} \pm \sqrt{(a + \frac{1}{4})}$ , another quadratic, from which we obtain  $x = \frac{1}{2} \pm \sqrt{(\frac{1}{4} \pm \sqrt{(a + \frac{1}{4})})}$ , which expression comprehends the four values of  $x$ .

*Corol. 1.* If  $a$  be less than  $-\frac{1}{4}$ , the term  $\sqrt{(a + \frac{1}{4})}$  becoming an imaginary or impossible quantity, all the four roots will be impossible.

*Corol. 2.* If  $a$  be  $= -\frac{1}{4}$ , the four values of  $x$  will be  $\frac{1}{2} + \frac{1}{2}\sqrt{3}$ ,  $\frac{1}{2} + \frac{1}{2}\sqrt{3}$ ,  $\frac{1}{2} - \frac{1}{2}\sqrt{3}$ , and  $\frac{1}{2} - \frac{1}{2}\sqrt{3}$ ; i. e. the equation will have two pair of equal roots.

*Corol. 3.* If  $a$  be greater than  $\frac{5}{16}$ , then  $\sqrt{(\frac{1}{4} - \sqrt{(a + \frac{1}{4})})}$  will be an impossible quantity, and consequently two roots will be imaginary.

*Corol. 4.* If  $a$  be  $= \frac{5}{16}$ ,  $\sqrt{(a + \frac{1}{4})}$  being  $= \frac{1}{2}$ ,  $\sqrt{(\frac{1}{4} - \sqrt{(a + \frac{1}{4})})}$  becomes  $= 0$ , and the four roots are  $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{1}{2} + \frac{1}{2}\sqrt{6}$ , and  $\frac{1}{2} - \frac{1}{2}\sqrt{6}$ .

*Corol. 5.* Therefore, when the value of  $a$  is between  $-\frac{1}{4}$  and  $+\frac{5}{16}$  (except when it is  $= 0$ ), all the four roots will be real.

*Corol. 6.* Where  $a = 0$ , the roots are  $\frac{1}{2} + \frac{1}{2}\sqrt{5}$ ,  $\frac{1}{2} - \frac{1}{2}\sqrt{5}$ ,  $\frac{1}{2} + \frac{1}{2}\sqrt{1} = 1$ , and  $\frac{1}{2} - \frac{1}{2}\sqrt{1} = 0$ .

*Scholium.* And hence it is obvious that, if there be a biquadratic equation of this form,  $y^4 - 2py^2 + p^2y = q$ , and you put  $a = q \div p^2$ , the four values of  $y$  will be obtained from this expression,  $y = p \times \left\{ \frac{1}{2} \pm \sqrt{(\frac{1}{4} \pm \sqrt{(a + \frac{1}{4})})} \right\}$ .

Other Solutions may be seen in the British Oracle.

## II. QUESTION 789, by Mr. Paul Sharp, of Biddenden.

In  $56^\circ 19' 38''$  north latitude, the sun's amplitude was observed to be  $26^\circ 34''$  more than his altitude at 6 o'clock: Required the said amplitude, altitude, and the day when the observation was made?

*Answered by Mr. Wm. Terril, of Redruth.*

Put  $b$  and  $d$  for the sine and cosine of  $56^\circ 19' 38''$  the latitude;  $s$  and  $c$  for the sine and cosine of  $26^\circ 34'$  the difference between the amplitude and altitude at six o'clock; and  $x$  and  $y$  for the sine and cosine of the declination. Then by spherics,  $1 : b :: x : bx$  the

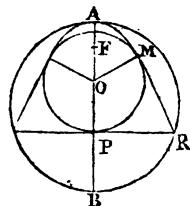
sine of the altitude, and  $d : 1 :: x : x \div d$  the sine of the amplitude, and which must be equal to  $b'c'x + s \sqrt{(1 - b'^2x^2)}$  the sine of the sum of  $26^\circ 34'$  and the altitude. Hence  $x = ds \div \sqrt{(1 - 2bcd + b'^2d^2)} = .3983480$  the sine of  $23^\circ 28' 30''$  the declination, shewing the day of the month to be the 21st of June. Hence  $bx = .3315122$  the sine of  $19^\circ 21' 38''$  the altitude, and  $x \div d = .7184573$  the sine of  $45^\circ 55' 38''$  the amplitude.

### III. QUESTION 790, by Mr. Tho. Robinson, of Biddick.

Given the diameters of the circles circumscribing and inscribed in a parabola equal to 10 and 8; to find the parabola.

*Answered by Mr. John Whitton, of Hull.*

Let  $F$  be the focus of the parabola,  $O$  the centre of the inscribed circle, &c. Put  $a = AB$  the diameter of the circumscribing circle,  $c = 2OM = 2OP$  that of the inscribed one, and  $x = AF$  the focal distance. Then will  $PB = 4x$  the parameter,  $OF = a - \frac{1}{2}c - 5x$ , and (by Emerson's Con. book 3, Pr. 13)  $OF \times BP = OM^2$ , or  $4ax - 2cx - 20x^2 = \frac{1}{4}c^2$ ; and hence  $x = \frac{1}{20}(2a - c \pm 2\sqrt{(a^2 - ac - c^2)})$ . But in the question the numbers are given too near an equality, for from the last expression it is evident that the greatest value of  $c$  will be when  $a^2 - ac - c = 0$ , or  $c = \frac{1}{2}a \times (\sqrt{5} - 1) = a \times .61803$ , &c. so that if  $a$  be 10, then  $c$  cannot exceed  $6.1803$ , &c.



*The same by Mr. John Brinkley, of Harleston School.*

Put  $a$  and  $b$  for the radii of the circumscribing and inscribed circles,  $x = AP$ , and  $y = PR$ . Then by the property of the circle  $x^2 + y^2 = 2ax$ , and by the property of the parabola  $2xy - y^2 = 2bx$ ; the sum of these two gives  $x^2 + 2xy = 2ax + 2bx$ , or  $x + 2y = 2a + 2b^*$ ; consequently  $y = a + b - \frac{1}{2}x$ , which substituted in the equation  $x^2 + y^2 = 2ax$ , &c. we get  $x = \frac{2}{3} \times (3a + b) \pm \frac{4}{3} \sqrt{(a^2 - ab - b^2)}$ , from which every thing may be found when the numbers are consistently proposed.

\* *Corol.* Hence it appears that, in any parabola, the sum of the axis and bounding double ordinate, is equal to the sum of the diameters of the circumscribing and inscribed circles.

### IV. QUESTION 791, by Master John Brinkley, at Harleston.

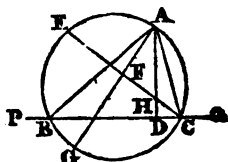
From a given point  $A$  to draw two right lines  $AB$  and  $AC$  to meet a right line  $PQ$  given in position, so that the rectangle under  $AB$  and  $AC$

may be of a given magnitude, and the angles  $ABC$  and  $ACB$  have a given difference.

*Answered by Mr. Rob. Hartley, of Daresbury.*

**Construction.** Draw  $AD \perp PQ$ ; make  $\angle DAG =$  the given difference and produce  $AG$  till  $GA \times AD =$  the given rectangle; on the diameter  $AG$  describe the circle cutting  $PQ$  in  $B$  and  $C$ ; then, drawing  $AB$  and  $AC$ ,  $ABC$  is the triangle required.

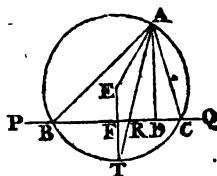
**Demonstration.** Draw  $CHFE \perp$  the diameter  $AFG$ . Then arc  $AE =$  arc  $AC$ , therefore  $\angle ABC = \angle ACE$ , and consequently  $ECB = ACB - ABC$  the given difference of the angles; but the right-angled triangles  $AFH$ ,  $CDH$  have the  $\angle$ s at  $H$  equal, and therefore  $FAD = FCD$  the given difference by construction. Also the given rectangle  $= GA \times AD = BA \times AC$ , Simpson's Geometry 3, 25.



*The same by Mr. Brinkley, the Proposer.*

Draw  $AD \perp PQ$ ; make  $\angle DAR =$  half the given difference, producing  $AR$  till  $AR \times AT =$  the given rectangle. Make  $TFE \perp PQ$ , and  $AE = ET$ , or  $\angle EAT = \angle ETA$ ; with centre  $E$  and radius  $ET$  or  $EA$  describe a circle cutting  $PQ$  in  $B$  and  $C$ . Draw  $AB$ ,  $AC$ , and it is done.

**Demonstration.** Since  $FT$  is  $\perp BC$ , and passes through the centre, therefore  $BF = FC$ , and  $BT = TC$ , and consequently  $AT$  bisects the  $\angle BAC$ ; then (by Simpson's Geometry 3, 26)  $AB \times AC = AR \times AT =$  the given magnitude by construction. Also by a known theorem  $\angle ACB - ABC = 2DAR =$  the given difference by construction.



V. QUESTION 792, by Mr. Rob. Phillips, of St. Agnes.

Required the area of a curve whose equation is  $y = x^{\frac{1}{2}} \times$  hyperbolic logarithm of  $\left\{ (a + x^{\frac{3}{2}}) \div (a - x^{\frac{3}{2}}) \right\}$ , supposing that when  $y = 0$ ,  $x = 0$ .

*Answered by the Rev. Mr. John Hellins.*

The fluxion of the area is  $yx = x^{\frac{1}{2}} \dot{x} \times$  h. l. of  $((a + x^{\frac{3}{2}}) \div (a - x^{\frac{3}{2}}))$  the fluent of which is  $\frac{2}{3} x^{\frac{3}{2}} \times$  h. l.  $((a + x^{\frac{3}{2}}) \div (a - x^{\frac{3}{2}})) -$  fluent of  $\frac{2}{3} x^{\frac{3}{2}} \times$  fluxion h. l.  $((a + x^{\frac{3}{2}}) \div (a - x^{\frac{3}{2}}))$ . But the fluxion of

h. l.  $((a + x^{\frac{3}{2}}) \div (a - x^{\frac{3}{2}}))$  is  $(3ax^{\frac{1}{2}}x) \div (a^2 - x^3)$ , which being multiplied by  $\frac{2}{3}x^{\frac{3}{2}}$ , and the correct fluent of it subtracted from the former quantity, we have  $\frac{2}{3}x^{\frac{3}{2}} \times$  h. l. of  $((a + x^{\frac{3}{2}}) \div (a - x^{\frac{3}{2}})) + \frac{2}{3}a \times$  h. l. of  $(a^2 - x^3) \div a^2 =$  area required.

*The same by Mr. Nathan Parnel.*

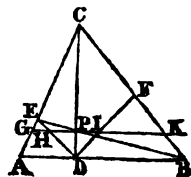
Put  $L =$  hyp. log. of  $(a + x^{\frac{3}{2}}) \div (a - x^{\frac{3}{2}})$ ; then we shall have  $y = Lx^{\frac{1}{2}}x = Lx^{\frac{1}{2}}x + \frac{2}{3}x^{\frac{3}{2}}L - \frac{2}{3}x^{\frac{3}{2}}L$ , the fluxion of the area of the curve; now the fluent of the sum of the two first terms is  $\frac{2}{3}Lx^{\frac{3}{2}}$ ; and because the fluxion of  $L$  is  $= 3ax^{\frac{1}{2}}x \times (a^2 - x^3)^{-1}$ , we have  $\frac{2}{3}Lx^{\frac{3}{2}} = 2ax^{\frac{1}{2}}x \times (a^2 - x^3)^{-1}$ , the correct fluent of which is  $\frac{2}{3}a \times$  h. l.  $a^2 \div (a^2 - x^3)$  and consequently  $\frac{2}{3}Lx^{\frac{3}{2}} - \frac{2}{3}a \times$  h. l.  $a^2 \div (a^2 - x^3) =$  area of the curve.

VI. QUESTION 793, by Lieut. Glenie, of the Engineers.

In any rectilinear triangle ABC, if any one (AB) of the sides be assumed, to determine a point P, through which if a right line be drawn parallel to that side, the differences of the parts of this line and the perpendicular distance of P from the assumed side, shall have to each other, the duplicate ratio of the parts themselves.

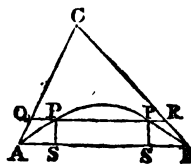
*Answered by the Proposer Lieut. Glenie.*

Draw  $CD \perp AB$ , and  $DE$  to bisect the  $\angle ADC$ ; join  $BE$  cutting  $CD$  in  $P$ , the point required. For by parallel lines,  $GH : HP :: AD : DB$ ; and (drawing  $DIF$  to bisect  $\angle BDC$ ) ( $PI$  or)  $HP : IK :: AD : DB$ ; therefore by equality  $GH : IK :: AD^2 : DB^2 :: GP^2 : PK^2$ .



*The same answered by Mr. John Whitton, of Hull.*

There may be found innumerable points P having the required property, and the locus of P is a certain curve APPB, the height of whose vertex above AB is  $= \frac{1}{4}$  of the height of C above AB. For let QR be any such line,  $\parallel AB$ , to be divided in P so, that  $PQ = PR$  shall be to  $PR = PS$  as  $PQ^2 : PR^2$ . Then I say this will be effected by dividing QR so that the rectangle  $PQ \times PR$  may be  $= PS \times QR$ . For the extremes and means being proportional, we have



these two proportions  $\left\{ \begin{array}{l} QR : PR :: PQ : PS \\ QR : PQ :: PR : PS \end{array} \right\}$  then by division and  $\left\{ \begin{array}{l} PQ : PR :: PQ - PS : PS \\ PQ : PR :: PS : PR - PS \end{array} \right\}$  hence by equality  $PQ^2 : PR^2 :: PQ - PS : PR - PS$ , as required.

VII. QUESTION 794, by Mr. Michael Taylor.

To determine a place on the earth, where a degree of the meridian is equal to a degree of the equator; the ratio of the axis to the equatorial diameter being that of 229 to 230.

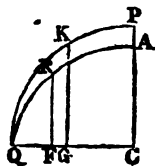
*Answered by Mr. Mich. Taylor.*

By page 43, Simpson's Dissertations, a degree of the meridian is very nearly  $= de \div (d^2 - bs^2)^{\frac{1}{2}}$  where  $e$  = degree of the equator,  $d = \frac{2}{3}\frac{1}{2}\frac{8}{9}$ ,  $b = d^2 - 1$ , and  $s$  = the sine of the latitude. And this, by the question, must be  $= e$ . Hence  $s = \sqrt{\frac{d^2 - d^2}{b}} = \sqrt{\frac{d^2 - d^2}{d^2 - 1}} = \sqrt{\frac{1 + b - (1 + b)^{\frac{1}{2}}}{b}} = \sqrt{(\frac{2}{3} + \frac{1}{9}b - \frac{5}{81}b^2 + \frac{1}{243}b^3 \&c.)} = \sqrt{\frac{2}{3}} \times (1 + \frac{1}{12}b - \frac{1}{240}b^2)$  nearly,  $= .8170890$  = the sine of  $54^\circ 47' 40''$ , the latitude required.

*The same by Mr. Geo. Sanderson.*

Let  $\kappa, \mathbf{r}$  be two places on the sphere and spheroid having the same latitude;  $KG$  and  $RF \perp QC$ . Put  $r = 230 = QC$ ,  $a = 229 = AC$ ,  $d^2 = r^2 - a^2$ ,  $x = FC$ ,  $z = GC$  the cosine of the latitude to radius  $r$ .

The lengths of small arcs on the ellipse and circle are as the radii of curvature; and this in the sphere is  $r$ , and in the spheroid is  $(1 \div ar^2) \times (r^2 - d^2 x^2)^{\frac{1}{2}} = r$  where the degrees are of equal lengths in both; and hence  $d^2 x^2 = r^2 - (ar^2)^{\frac{2}{3}}$ . But because



the tangents at  $\kappa$  and  $\mathbf{r}$  will be parallel, we shall have  $x^2 = \frac{r^2 z^2}{a^2 r^2 + d^2 z^2}$

$=$  (from the former)  $(r^2 - a^{\frac{2}{3}} r^{\frac{1}{3}}) \div d^2$ . And hence  $z \div r = (a \div d) \sqrt{((r \div a)^{\frac{2}{3}} - 1)} = .5765191$  = cosine of  $54^\circ 47' 38''$ . To which add and subtract  $30'$ , and the degree from latitude  $54^\circ 17' 38''$  to lat.  $55^\circ 17' 38''$  is nearly equal to a degree on the equator.

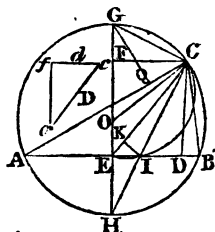
VIII. QUESTION 795, by Amicus.

Given the difference of the sides, the line drawn from the vertex to

bisect the base, and the ratio of the difference of the segments of the base to the diameter of the circumscribing circle; to construct the triangle.

*Answered by Mr. Geo. Sanderson.*

**Construction.** Let  $CE$  be the given bisecting line,  $d$  half the given difference of the sides,  $d$  to  $D$  in the given ratio of the difference of the segments to the diameter of the circumscribing circle, and  $D$  the hypotenuse of a right-angled triangle  $qfc$  having  $d$  for one of the sides of the right angle. On  $CE$  describe the segment of a circle to contain an angle equal to half the angle  $foc$  plus a right angle; on  $CE$  take  $EK : d :: d : CE$ ; erect the perpendicular  $KI$  to meet the circle in  $I$ ; draw  $EI$  and  $GH$  perpendicular to it; also draw  $CH$  and  $CG$  perpendicular to it; on the diameter  $GH$  describe a circle meeting  $EI$  in  $A$  and  $B$ ; join  $AC$  and  $BC$ , and  $ABC$  is the triangle sought.



**Demonstration.** Let fall the perpendiculars  $CD$ ,  $CF$ , and draw  $CO$  to  $O$  the centre of the circle. Because  $CG$  is  $\perp$   $CH$  (by construction) the circle passes through  $C$ , and  $CO$  is a radius; the  $\angle COF = 2\angle CHF$  ( $2\angle IHE$ ) (Euc. 20, 3); but  $IHE$  plus a right angle  $= CIE = foc$  plus a right angle (by construction) therefore the triangles  $FOC$ ,  $foc$  are similar, and  $FC : CO :: fc : co :: 2FC : GH$  where  $2FC = 2ED = AD - DB$ . Again, because  $CH$  bisects  $\angle ACB$ , (Euc. 3, 6, and 17, 5)  $AC + CB : AB :: AC - CB : AI - IB$  ( $2EI$ ), and by a well-known theorem  $AC + CB : AB :: AD - DB$  ( $2ED$ );  $AC - CB$ , whence  $2ED : AC - CB :: AC - CB : 2EI$ , theref. the rectangle  $EI \times ED =$  a square on half the difference of the sides; but, by the similar  $\triangle s EIK$ ,  $ECD$ , rectangle  $EI \times ED = EK \times EC = d^2$  (by construction), therefore  $2d = AC - CB$  as required.

**Scholium.** The  $\angle FOC = CBA - CAB$ , therefore the question is in effect the same as the 24th in *The Mathematician*.

And nearly in this manner was the construction given by the proposer, *Amicus*, who also remarks that, if the sum instead of the difference of the sides be given; then in the  $\triangle EGC$ , are given the base  $EC$ , the opposite  $\angle EGC$ , and the area; for letting fall  $GQ \perp AC$ , it is known that  $AQ$  is half the sum of the sides, therefore  $AG^2 = GE \cdot FH$  is given, and consequently  $GE \times FC$  double the said area, whence the  $\triangle EGC$  is given, and thence  $ACB$  as before.—A similar remark was made by Mr. *N. Parnel*, who also premised these two lemmas, 1st, if the angles at the base of the  $\triangle ACB$  be both acute, then the rectangle  $2ED \times 2EI = (AC - CB)^2$ ; and 2nd, that the rectangle  $4EG \times HF = (AC + CB)^2$ .

#### IX. QUESTION 796, by Nauticus.

Having given the right ascension and declination of any fixed star,

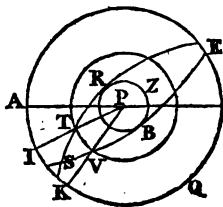
it is required to determine the greatest and least right ascensions that another star can have, whose declination is given, so that it may be seen on the same vertical circle with the former, in a given latitude.

*Answered by Plus Minus.*

Since no two stars can come to one azimuth in a given latitude, unless the great circle passing through them cut, or at least touch, the parallel of latitude; draw two great circles through the given star touching the given parallel of latitude, then they will cut the given parallel of declination in the points which have the greatest and least right ascension required.

*The same answered by Nauticus.*

In the annexed projection, let the primitive  $AEQ$  be the equinoctial,  $P$  its pole,  $s$  the given star,  $tv$  the parallel of declination of the other star whose greatest and least right ascensions are sought,  $RBZ$  the path of the vertex of the given place. Through  $s$  draw two great circles to touch  $RBZ$  and cutting the parallel  $tv$  in  $\tau$  and  $v$ ; draw  $PTI$  and  $PVK$ ; so shall  $AI$  or the  $\angle API$ , and  $AK$  or the  $\angle APK$ , evidently be the greatest and least right ascensions required,  $A$  being the first point of Aries.

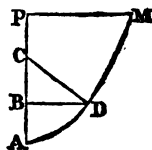


X. QUESTION 797, by Mr. Alex. Rowe, of Reginnis.

Required the nature and quadrature of the curve into which a flexible wire 11.3052 feet long must be bent, so that a ring of heavy metal being put thereon, and the wire revolved about an axis at right angles to the horizon, with a given velocity, the said ring shall rest equilibrio; the abscissa being to the ordinate as 3 to 2.

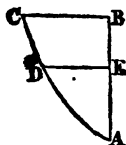
*Answered by Mr. John Whitton.*

Let  $AM$  be the required curve,  $AP$  its axis,  $PM$  and  $BD \perp AP$ , and  $DC \perp$  to the curve  $ADM$  at  $D$ . Then, since the angular velocity is given or constant, the centrifugal force, in direction  $BD$ , at any point  $D$  of the curve, will be as  $BD$ , which resolves into the two forces  $BC$ ,  $CD$ ; of which the latter is destroyed by the curve, being perp. to it; therefore the former, or subnormal  $BC$ , is a constant quantity, being directly opposite and equal to the force of gravity; and consequently the curve will be a parabola. And, its length being given, by means of the general expression for its length, we obtain  $AP = 9$ ,  $PM = 6$ ; and hence the area  $APM = 54$ .



*The same by the Rev. Mr. John Hellins.*

Let  $ADC$  be the curve into which the wire must be bent,  $D$  any point in it, and  $DE \perp AB$  the axis. Call  $AE, x$ ;  $DE, y$ ;  $AD, z$ ;  $32\frac{1}{2}, f$ ;  $3.1416, p$ ; and the time of one revolution, in seconds,  $n$ . Then, by mechanics,  $fxz^{-1}$  = the force of gravity in direction of the curve at  $D$ ; also  $4p^2yn^{-2}$  = the centrifugal force at  $D$  in direction  $ED$ , and therefore in direction of the curve it will be  $4p^2yyn^{-2}z^{-1}$ ; and this must be equal to the above opposite force arising from gravity, therefore  $n^2fx = 4p^2yy$ , and the fluents give  $n^2f = 2p^2y^2$ , or  $ax = yy$ , (putting  $fn^2 \div 2p^2 = a$ ), which is the equation of a parabola. Now the length of the curve of a parabola, whose abscissa is 3, and semi-ordinate 2 feet, is  $= \sqrt{10} + \frac{1}{3}h.l.$   $(3 + \sqrt{10}) = 3.7684$  feet; hence  $11.3052 \div 3.7684 = 3$ ; therefore  $3 \times 3 = 9 = AB$ , and  $3 \times 2 = 6 = BC$ . Consequently  $fn^2 \div 2p^2 = a = 36 \div 9 = 4$ , and  $n = p \sqrt{(2a \div f)} = 1.5667$  seconds  $= 1\frac{1}{2}$  seconds very nearly.



*Corollary 1.* The centrifugal force at  $c$ , the end of the bounding ordinate, or top of the curve,  $4ppy \div nn = 2fy \div a$ , is  $= 3f$ , three times the force of gravity.

*Corollary 2.* The force by which the ring is urged in direction of the curve at  $c$ , both by gravity and the centrifugal force, is  $f \div \sqrt{(1 + a \div 4x)} = 3f \div \sqrt{10}$ .

*Corollary 3.* The pressure of the ring against the curve at  $c$ , is  $= \sqrt{(ff + 9ff)} = f\sqrt{10}$ , or the pressure is to the weight of the ring as  $\sqrt{10}$  to 1.

*Scholium 1.* In like manner may be solved a problem, agitated in in the year 1770, between some gentlemen of my acquaintance, in the county of Devon, which I could wish the Editor would give me leave to insert here: it is this: *A hollow cylinder whose diameter is 4 feet, and depth 1 foot, being placed with its axis perpendicular to the horizon, filled with water, and made so to revolve about its axis that the water shall make one revolution every 4''; Query how much water will flow over it?* Here,  $n$  being  $= 4$ , and  $y = 2$ , we have  $fn^2 \div 2pp = 26.0732$ ,  $x = yy \div a = 0.15341$ , and  $\frac{1}{2}pxyy = 0.96393$  of a solid foot,  $= 7.2107$  gallons wine measure, the quantity sought.

*Scholium 2.* By help of the parabola we may determine the point in any curve revolving about an axis perpendicular to the horizon with a given velocity, on which a ring being put, it shall neither approach to, nor recede from the axis: or, without farther trouble, from the equation  $fx = 4ppyy \div nn$ , or  $ax = 2yy$ , by writing for either  $x$  or  $y$  and its fluxion the values obtained from the equation of the curve. For example, let the curve be a circle whose radius is  $r$ ; then  $2rx - xx = yy$ , and  $rx - xx = yy$  which being written for  $yy$ , and the whole divided by  $2x$ , we have  $\frac{1}{2}a = r - x = \sqrt{(rr - yy)}$ ;

where it is obvious that, if  $\frac{1}{2}a$  be equal to, or greater than  $r$ , the ring cannot remain in equilibrio at any distance from the axis; and that in all other cases it will be holden in equilibrio by the two contrary forces above-mentioned, at the extremity of the ordinate drawn perpendicular to the axis at the distance of  $\frac{1}{2}a$  below the centre of the circle.

XI. QUESTION 798, by Mr. Nathan Parnel.

From two given right lines it is required to cut off two equal parts so, that the two remainders may be to each other as the squares of the whole lines respectively?

*Answered by Mr. N. Parnel.*

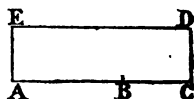
*Analysis.* Suppose AB to be the greater of the two given lines, BC = BE the less, and BD the part cut off from each of them; then by hypothesis AD : CD :: AB<sup>2</sup> : CB<sup>2</sup>, and consequently AC (AD — CD) : CD :: AE × AC (AB<sup>2</sup> — CB<sup>2</sup>) : CB<sup>2</sup>, therefore AE × DC = BC<sup>2</sup>; and because AE and BC are given, DC is also given; hence this



*Construction:* In the right line AE take AB the greater of the two given lines, BC = BE the other, and then take CD : CB :: CB : AE, and it is done. The demonstration is evident from the analysis.

*The same by Mr. James Adams.*

*Construction.* On AC the sum of the two given lines, AB, BC, make the rectangle AD = the rectangle AB × BC; or take CD a fourth proportional to AC, AB, BC; so shall CD be the part sought.



*Demonstration.* By con.  $\left\{ \begin{array}{l} AC : BC :: AB : CD \\ AC : AB :: BC : CD \end{array} \right\}$   
by division and  $\left\{ \begin{array}{l} AB : BC :: AB - CD : CD \\ AB : BC :: CD : BC - CD \end{array} \right\}$  hence (Simp. Geom. 4, inversion)  $\left\{ \begin{array}{l} AB : BC :: AB - CD : BC - CD \\ AB : BC :: CD : BC - CD \end{array} \right\}$  11, &c. AB<sup>2</sup> : BC<sup>2</sup> :: AB — CD : BC — CD.

XII. QUESTION 799, by Mr. Rob. Hartley, of Daresbury.

Suppose a field in form of a semi-parabola, whose abscissa is 9 chains, and semi-ordinate 6 chains; and let the land along the ordinate be worth after the rate of 100 pounds an acre, and from thence uniformly to decrease along the abscissa in such a manner, that every equal part of any one semi-ordinate may be of the same value, so as to be worth 60 pounds an acre at the vertex: It is required to divide the said field into two equivalent parts by a line drawn parallel to the abscissa; and to determine the mean price per acre of the whole field?

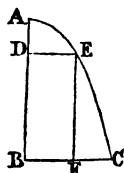
*Answered by the Rev. Mr. Hellins.*

The extreme prices per chain are 10 and 6 pounds, their difference is 4, which put =  $a$ ; also put  $9 = b$ ,  $6 = c$ , any abscissa =  $x$ , its ordinate =  $y$ , and the value of the area =  $v$ . Then  $cc \div b = 4 = a$ ,  $yx = x \sqrt{ax}$  = fluxion of the area, and  $c + axb^{-1}$  = the rate of the value of the land at  $y$ ; therefore  $yx \times (c + axb^{-1}) = x \sqrt{ax} \times (c + axb^{-1}) = \dot{v}$ , and  $x \sqrt{ax} \times (\frac{2}{3}c + \frac{2}{3}axb^{-1}) = xy \times (\frac{2}{3}c + \frac{2}{3}axb^{-1}) = v$ ; which, when  $x$  becomes =  $b$ , is  $bc \times (\frac{2}{3}c + \frac{2}{3}a) = 302.4l. = 302l. 8s.$  the value of the field. And this divided by  $\frac{2}{3}bc$ , the area, gives  $c + \frac{2}{3}a = 8.4l.$  for the mean price per chain, or 84l. per acre.

Now from what is done above we find the value of ADE to be  $y^3a^{-1} \times (\frac{2}{3}c + \frac{2}{3}y^2b^{-1})$ , (see the following figure), and the value of the rectangle BE is = area BE  $\times$  mean rate between DE and BF = DE  $\times$  DB  $\times (\frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}c + \frac{1}{2}axb^{-1}) = \frac{1}{2}y \times (b - x) \times (a + 2c + y^2b^{-1}) = by \times (c + \frac{1}{2}a) - y^3a^{-1} \times (c + \frac{1}{2}y^2b^{-1})$ , and their sum  $by \times (c + \frac{1}{2}a) - y^3a^{-1} \times (\frac{1}{3}c - \frac{1}{10}y^2b^{-1})$  = half the value of the field: in numbers  $72y - \frac{1}{10}y^3 - \frac{1}{30}y^3 = 151.2$  which equation has two affirmative roots, one negative, and two impossible; but  $y = 2.17314$  chains is the only one that can answer the conditions of the question.

*The same answered by Mr. N. Parnell.*

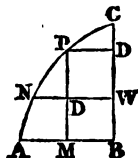
Let  $a = AB = 9$ ;  $b = BC = 6$  chains;  $c = 4$  pounds, the value per square chain which the land along BC is worth more than at A;  $d = 6$  pounds, the price per square chain at A; and  $x = DE = BF$ . Then by the parabola we have  $b^3 : x^3 :: a : ax^2b^{-2} = AD$ ; hence  $a : AD :: c : cx^2b^{-2}$ , the value per chain which the land along DE is worth more than at A; consequently  $d + cx^2b^{-2}$  = the value along DE; and because  $(a - ax^2b^{-2}) \times \dot{x}$  = the fluxion of the space AEFB, it is evident that  $(\frac{1}{2}c + d + \frac{1}{2}cx^2b^{-2}) \times (a - ax^2b^{-2}) \times \dot{x} = \frac{1}{2}ac\dot{x} + ad\dot{x} - acx^2b^{-2} - \frac{1}{2}acx^4b^{-4}$  = the fluxion of the value of the same space; the fluents of which are  $\frac{1}{2}acx + adx - \frac{1}{3}adx^3b^{-2} - \frac{1}{10}acx^5b^{-4}$  = the value of the space AEFB, which when  $x = b$  becomes  $\frac{2}{3}abc + \frac{2}{3}abd = 302.4$  pounds, the value of the whole field ABC; and since the area ABC is =  $\frac{2}{3}ab$  square chains =  $\frac{1}{15}ab$  acres = 3.6 acres, we have  $(\frac{2}{3}abc + \frac{2}{3}abd) \div \frac{1}{15}ab = 6c + 10d = 84$  pounds, the mean price per acre. Again, when the value of the space AEFB is equal to the value of EFC, we shall have  $\frac{1}{2}acx + adx - \frac{1}{3}adx^3b^{-2} - \frac{1}{10}acx^5b^{-4} = \frac{1}{3}abc + \frac{1}{3}abd$ , or 25920r



$-180x^3 - x^5 = 54432$ ; here  $x = 2.173$ , and hence the field is easily divided.

*The same by Mr. John Brinkley.*

Put  $AB = 6 = b$ ,  $BC = 9$  chains  $= a$ ,  $d = 100$  pounds,  $c = 60$  pounds,  $\bullet = 1$  acre  $= 10$  square chains,  $w = CD$ , and  $x = cw$ . Then  $PD = a^{-\frac{1}{2}}bw^{\frac{1}{2}}$ ,  $NW = a^{-\frac{1}{2}}bx^{\frac{1}{2}}$ , and  $NO = a^{-\frac{1}{2}}b \times (x^{\frac{1}{2}} - w^{\frac{1}{2}})$ . But  $a : d - c :: a - x : (a - x) \times (d - c) \times a^{-1}$  = the decrease of the value per acre at NW, therefore  $d - (a - x) \times (d - c) \times a^{-1}$  = the value per acre at NW; and as the fluxion of the area PNO is  $a^{-\frac{1}{2}}bx \times (x^{\frac{1}{2}} - w^{\frac{1}{2}})$ , the fluxion of the value of PNO will be  $(d - (a - x) \times (d - c) \times a^{-1}) \times (x^{\frac{1}{2}} - w^{\frac{1}{2}}) \times a^{-\frac{1}{2}}br^{-1}x$ ; then taking the correct fluents by considering that when  $x = w$ , the fluent is  $= 0$ , and putting  $x = a$ , we get



$$\frac{b}{r} \times \frac{3d + 2c}{15} 2a - \frac{d + c}{2} \sqrt{aw} + \frac{10ac + 3dw - 3cw}{30a} w \sqrt{\frac{w}{a}} =$$

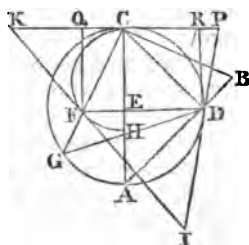
the value of PAM; which when  $w = 0$  becomes  $3d + 2c \times \frac{1}{15}abr^{-1} = 302.4$  pounds = the value of the whole field ABC; and hence the mean price per acre is  $302.4 \div \frac{1}{15}ab = 84$  pounds. Also by putting the above value of PAM = the value of  $\frac{1}{2}ABC$ , the equation in numbers is  $1620w^{\frac{1}{2}} - 45w^{\frac{3}{2}} - w^{\frac{5}{2}} = 1701$ , where  $w = 1.1793$ .

### XIII. QUESTION 800, by Mr. George Sanderson.

Given the sides AC and CB, and the product of the base AB and cube of the perpendicular CD a maximum; to determine the triangle ACB by an equation not exceeding a quadratic, and to construct the triangle.

*Answered by the Proposer, Mr. Sanderson.*

Imagine the thing done, and ACB the required triangle, AC and CB the given sides, and that the base  $AB \times \perp CD^3$  is a maximum. On the longest side CA take CH : CA :: CB<sup>3</sup> : CA<sup>3</sup>, and on the diameters CA and CH describe the semicircles CDA and CFH; and conceive the line DEF drawn  $\perp$  to CH cutting it in E and the semicircle in F, and CF joined. Because CDA is a right angle, the semicircle passes through D; therefore  $CE \times CA = CD^2$  (Eu. 8. 6. Cor.), and  $CE \times CH = CF^2$ , therefore  $CF^2 : CD^2 :: CH : CA :: CB^3 : CA^3$ , whence  $CF : CD :: CB : CA$ ,



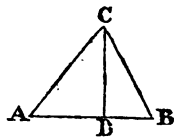
and because the  $\angle CDF = CAB$ , the  $\triangle s$   $CDF$  and  $CAB$  are similar (Eu. 7, 6.); and (by 19, 6.)  $CA^2 : CD^2 :: CD \times \frac{1}{2}AB$  ( $\triangle ACB$ ) :  $CE \times \frac{1}{2}DF$  ( $\triangle FCD$ ), therefore  $2CA^2 \times \triangle FCD = CD^2 \times AB$  a maximum per question, and consequently  $\triangle FCD$  a maximum, because  $2CA^2$  is a given (constant) quantity; but the point  $H$  is given, because  $CA$  and  $BC$  are given; therefore the question is reduced to this, to find a point  $E$  in  $CH$  so that drawing  $DEF \perp$  to  $CH$ , the  $\triangle FCD$  may be a maximum. By the method given at page 46 last year's Diary\*, draw the tangents  $KCF$ ,  $KFI$ , and  $FPI$  so that the distance of the points of intersection  $IK$  and  $IP$  may be bisected by the points of contact  $F$  and  $D$  and the  $\triangle FCD$  is a maximum. For draw  $FQ$  and  $DR \perp KCF$ . The rectangle  $DQ$  is  $= 2 \triangle DCF$ ; and its height being half the height of the  $\triangle KIP$  by construction, &c. therefore  $DQ$  is the greatest parallelogram that can be inscribed in the  $\triangle KIP$  (Simpson on the Max. and Min. Theor. 8 and Cor.), and much more is it greater than any other inscribed in the semicircles, since the angles of all such others that are in the semicircles fall short of the tangents  $IK$ ,  $IP$ ; therefore the  $\triangle DCF$  is a maximum. The point  $D$  being found, draw  $ADB$ , to which from  $C$  apply the other given side  $CB$ , so shall  $ABC$  be the triangle required.

*Algebraically.* Put  $CE = x$ ,  $CH = a$ ,  $CA = b$ ; then  $\sqrt{(bx - x^2)} = DE$ , and  $\sqrt{(ax - x^2)} = FE$ , therefore  $x \sqrt{(bx - x^2)} + x \sqrt{(ax - x^2)}$  = the rectangle  $DQ$  a maximum, in fluxions and reduced  $8x^2 = 9x(a + b) - 9ab$ , &c. the same as page 47 last year's Diary.

*Scholium.* The circle  $CDA$  being completed, and  $CF$  produced to meet it in  $G$ , and  $DG$  drawn; the  $\triangle CDF$  is similar to the  $\triangle CGD$ , therefore  $CG^2 : CV^2 :: \triangle CGD : \triangle FCD$ ; so that if the diameter  $CA$  be given, and the ratio of  $CG$  to  $CD$  ( $CD$  to  $CF$ ), then  $CH$  is given; and if the  $\triangle CGD$  be given or a maximum, the similar  $\triangle FCD$  is given or a maximum. Whence it is manifest that question 768 may be constructed by the above method, and that the correction at page 46 last year is true, it being constructed on the same principles. And therefore that industrious corrector of pretended errors, the Compiler of the Mathematical Part of the 'T. and C. Magazine, in the year 1782, has only exposed his own ignorance in objecting to it.

*Another solution by Mr. Robert Dowden.*

Put the given sides  $AC$  and  $BC$  equal to  $a$  and  $d$ , and the  $\perp CD = x$ . Then  $AD = \sqrt{(a^2 - x^2)}$ , and  $BD = \sqrt{(d^2 - x^2)}$ ; therefore  $CD^2 \times AB = x^2 \sqrt{(a^2 - x^2)} + x^2 \sqrt{(d^2 - x^2)}$  = a maximum, the fluxion of which being made equal to nothing, and reduced, we obtain  $8x^2 - (a^2 + d^2) \cdot 9x^2 = -9a^2d^2$ , a quadratic equation, which may be easily constructed. Or because this equation is similar to that in page 47 last Diary\*, (only the squares of the quantities here for the simple quantities there), this may therefore be constructed from that in page 46 last Diary.



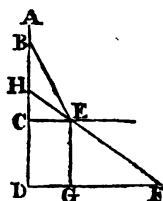
\* See the Solutions to question 768, page 72, 73.

XIV. QUESTION 801, *by Terricola.*

Let A, B, C, D be given points in a vertical line, A the highest, and the rest in succession; and let CE, DF be horizontal lines; then, F being a given point in DF, it is required to find E in CE so that, joining BE, EF, if a heavy body fall freely from A to B, and be then deflected along the plane BE, and thence from E to F along the plane EF, the time of descent to the lowest point F may be the least possible; supposing its motion not to be impeded by striking the planes at B and E.

*Answered by Mr. John Aspland.*

Put  $a = 16\frac{1}{2}$  feet, then the times of descent down AB, AC, AD, are respectively  $\sqrt{(AB \div a)}$ ,  $\sqrt{(AC \div a)}$ ,  $\sqrt{(AD \div a)}$ ; and therefore the times in AC and CD are respectively  $(\sqrt{AC} - \sqrt{AB}) \div \sqrt{a}$  and  $(\sqrt{AD} - \sqrt{AC}) \div \sqrt{a}$ , which call  $m$  and  $n$ ; hence the times in BE and EF are respectively  $m \times BE \div BC$  and  $n \times EF \div CD$  which put  $= p \times BE$  and  $q \times EF$  respectively; hence then  $p \times BE + q \times EF =$  a maximum the fluxion of which being made  $= 0$ , gives us  $p \times CE \div BE = q \times GF \div EF$ , and hence we have this proportion, as sine  $\angle CBE : \text{sine } \angle GEF :: q : p$ , whence the distance CE may be found.



*Corollary.* If  $BC = CD$ , then sine  $CBE : \text{sine } GEF :: n : m :: \sqrt{AD} - \sqrt{AC} : \sqrt{AC} - \sqrt{AB}$ . And if in this case  $AB$  be  $= 0$ , then will sine  $CBE : \text{sine } GEF :: \sqrt{2} - 1 : 1$ .—Also, if  $FE$  be produced to meet  $BC$  in  $H$ ; then  $\angle CHE$  being  $= GEF$ , and sine  $CHE = \text{sine } BHE$ , and sine  $HBE : \text{sine } BHE :: HE : BE$ ; therefore in all cases  $HE : BE :: q : p$  a given ratio; and when  $BC$  is  $= CD$ , it is  $HE : BE :: n : m$ , also when  $AB$  is  $= 0$ , it becomes  $HE : BE :: \sqrt{2} - 1 : 1$  as the difference between the diagonal and side of a square is to the side. So that the problem is reduced to this, in a line  $CE$  given in position to find a point  $E$ , so that from the given points B, F drawing  $BE$ ,  $FE$ , and producing the latter to meet  $BC \perp CE$  in  $H$ ; then  $BE$  to  $HE$  may have a given ratio.

*The same by Mr. James Williams, of Plymouth Dock.*

Put  $AB = a$ ,  $BC = b$ ,  $CD = c$ ,  $DF = d$ ,  $16\frac{1}{2}$  feet (the force of gravity)  $= s$ , and  $CE = x$ ; hence, making  $EG \parallel CD$ ,  $GF = d - x$ , and  $BE = \sqrt{(b^2 + x^2)}$ , also  $EF = \sqrt{(c^2 + (d - x)^2)}$ . Then by the descent of gravity the times of falling through AB, AC, AD, are  $\sqrt{(a \div s)}$ ,  $\sqrt{((a + b) \div s)}$ ,  $\sqrt{((a + b + c) \div s)}$  consequently in BC and CD are  $\sqrt{((a + b) \div s)} - \sqrt{(a \div s)}$  and  $\sqrt{((a + b + c) \div s)} - \sqrt{((a + b) \div s)}$  which put equal to  $m$  and  $n$  respectively, so shall the times in BE and EF be  $(m \div b) \sqrt{(b^2 + x^2)}$  and  $(n \div c) \sqrt{(c^2$

$\frac{1}{2}(d-x)^2$ ), the sum of which must be a minimum, and the fluxion of which put = 0 gives this equation

$\frac{m}{n} \times \frac{x}{\sqrt{(b^2 + x^2)}} = \frac{n}{c} \times \frac{d-x}{\sqrt{(c^2 + (d-x)^2)}}$  which being reduced there finally results a general biquadratic equation.

*Corollary 1.* When  $c$  is in the middle of  $BD$ , then  $b = c$ , and  $m : n :: \sqrt{(a+b)} - \sqrt{a} : \sqrt{(a+2b)} - \sqrt{(a+b)}$ , and the equation is  $\frac{m}{n} \times \frac{x}{\sqrt{(b^2 + x^2)}} = \frac{d-x}{\sqrt{(b^2 + (d-x)^2)}}$ .

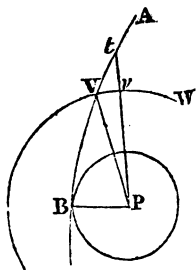
*Corollary 2.* And in that case if also  $AB$  or  $a$  be = 0, the equation will be barely  $\frac{1}{\sqrt{2}-1} \times \frac{x}{\sqrt{(b^2 + x^2)}}$  or  $\frac{(1+\sqrt{2}).x}{\sqrt{(b^2 + x^2)}} = \frac{d-x}{\sqrt{(b^2 + (d-x)^2)}}$ .

#### PRIZE QUESTION, *by Plus Minus.*

By observing the interval of time between two observations, one of two fixed stars on the same azimuth circle, the other of two other fixed stars (or even of the same two stars when that is possible) on the same azimuth circle, the latitude of the place of observation may be determined. It is required to shew how the observer may chuse his star, that the error in the latitude, caused by that which he is liable to commit in judging when the stars are on the same azimuth circle, may be the least possible?

#### *Answered by the Proposer Plus Minus.*

Let  $AB$  be a great circle of the sphere passing through any two known stars,  $P$  the pole of the world,  $wv$  a parallel of latitude, or track of the vertex of an observer, cutting the great circle  $AB$  in  $v$ . Then will the two stars in the great circle  $AB$  be in one azimuth, when the observer's vertex is at  $v$ , supposing them both to lie on the same side of the point  $v$ . But if the observer thinks he sees them in one azimuth when his vertex is at  $v$ , join  $pv$ ,  $pv$ , producing the latter, if need be, to meet the great circle in  $t$ , you will have the angle  $vpv$  equal to the error in the time of his observation, and  $vt$  the corresponding error in the observed latitude, so far as it depends on this single observation. Now if we would inquire how proper this pair of stars is in that parallel of latitude  $wv$ , for the purpose of the observation mentioned in the question, let fall  $pb$  perpendicular to the great circle  $AB$ ; then in the spherical  $\triangle pbv$ , the side  $pb$  and the  $\angle pbv$  remaining constant, a very small variation of the  $\angle wpv$  will be to  $vt$  the corresponding varia-

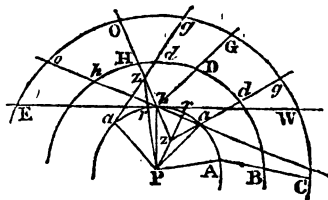


tion of the side  $pv$ , as the tangent  $\angle BVP$  to the sine of  $pv$  (by Cotes's *Æstimatio Errorum*, Theor. 13) that is, the error in time will be to the error in latitude, as the tangent of the azimuth from the pole to the cosine of the latitude of the observer. Now what has been said of this observation is equally true of the other, and of both jointly. Therefore the nearer the observations are made to the prime vertical, the less will be the error in the latitude, and it will be ultimately infinitely less than the error of the time.

*Scholium.* The above solution is for the theorist ; but for the sake of the practical astronomer it may be proper to remark, 1. That the error in latitude cannot be but very small if only one of the observations be made near the prime vertical. 2. That since any two stars which come to one azimuth, will be again in one azimuth on the other side of the prime vertical, and at the same distance from it, and moreover pass each other so very slowly in azimuth when near the prime vertical ; it may be advisable to choose such stars as come to one azimuth twice or only once, or perhaps not at all. 3. If he make choice of two stars in one azimuth at about that distance from the east point northward (in these northern latitudes) and as near as may be to the horizon, he will be sure to have the same stars again in one azimuth at the same distance from the east southward, and of a convenient altitude. 4. That to find the true time of the observations as exactly as may be, he should note the time when he first begins to think the stars are arrived at one azimuth, and also the time when he is first sure they have passed each other, then take the mean for the true time of the observation. If these precautions be observed, this will be found an excellent practical method of finding the latitude of a place, as not depending on the accuracy of any instrument, nor on having a true meridian ; nor is the observation disturbed by the refraction of the atmosphere : it is sufficient to have a clock that will truly measure the time elapsed between the two observations.

*The same answered by Mr. George Sanderson.*

The annexed scheme is a gnomonical projection on the plane of the equator ; the points  $A$  and  $z$ , in the circumference  $Aaza$ , the places of the observer's vertex at the first and second observations ;  $adnd$  and  $cggg$  parallels of the stars declinations. It is manifest that the angle  $zPA$  represents the interval of time in which the observer ought to see the stars  $B$  and  $C$ ,  $H$  and  $O$ , on the same azimuth circles  $ABC$  and  $ZHO$  ; by means of which if the points  $B$  and  $C$  be removed to  $D$  and  $G$ , and through which a great circle be drawn, it will pass through  $z$ , and the stars  $A$  and  $B$  will have apparently moved from the circle  $zDG$  to  $ABC$  in the time the observer's vertex describes the arch  $Az$  : Hence



if the interval of time ( $\angle zpa$ ) be given, the latitude may be found. On the plane of projection put the given stars  $A$  and  $B$  by means of their right ascensions and declinations, as also the stars  $H$  and  $O$ ; project the points  $G$  and  $D$  by means of the given time, through which and  $H$ ,  $O$  describe great circles intersecting in  $z$ ; draw  $pz$  the tangent of the distance of  $z$  from  $p$ , or the complement of the latitude required.

But if the observed interval of time be greater or less than the true time ( $\angle zpa$ ) by the angle  $zpa$ , then  $gdza$  or  $gdaz$  is the great circle (projected on the plane by means of the observed time or  $\angle apa$ ) cutting  $ohz$  in  $z$ ; and if  $pz$  be drawn cutting the circumference  $aaaz$  in  $r$ , then  $rz$  is the error in latitude; and this error will be the least possible when the circles  $zho$  and  $zdg$  coincide with a great circle ( $ohza$ ) passing through  $za$ . For suppose the observed time or  $\angle apa$  less than the true time or  $\angle zpa$ , and the stars  $H$  and  $O$  on the circle  $ohza$ , then  $z$  falls in  $a$ , and  $rz = 0$ . Again if the  $\angle apa$  be greater than  $zpa$ , and the great circle  $zdg$  coincide with  $ohza$ , then the circle  $gdza$  passes through  $z$ , (as is easily demonstrated) and  $rz = 0$ . Now if  $ezw$  be drawn to touch the circle  $aaaz$  in  $z$ , the  $\angle pzo$  ( $wza$ ) is  $= \frac{1}{2} \angle zpa$ , which being considered as very small, the  $\angle ezo$  ( $wza$ ) is nearly equal to the angle it represents on the sphere (Emers. Gnom. Proj. Prob. 12) wherefore the great circle  $ohza$  may be considered in practice as coinciding with the great circle  $ezw$ ; but when the great circle  $ABC$  touches the circle  $aaaz$ , then  $zdg$  coincides with  $ezw$ , and the stars  $B$  and  $C$  (seen at the first observation) will apparently move from  $ezw$  to  $ABC$  in the given interval of time; therefore the prime vertical  $ezw$  may be considered as the great circle on which the observer must choose his stars. Whence it is manifest that the latitude may be found (and the error considered the least possible) by choosing two stars at the same time on the prime vertical; for  $pzw$  being a right angle, a perpendicular ( $pz$ ) let fall on a great circle, drawn through the two stars (projected on the plane) is the cotangent of the latitude required.



*Questions proposed in 1783, and answered in 1784.*

1. QUESTION 803, by Mr. James Williams, Schoolmaster.

Ladies, vouchsafe your kind advice  
To Susan of the vale:  
Richard has met the charmer twice,  
And told his tender tale;  
Henry also pursues the maid  
With every soothing art:  
Richard has pelf, a thriving trade;  
Henry an honest heart.

Were you in lovely Susan's place,  
Say which would you approve,  
With Richard's gold and cares to wed,  
With Henry, only love.

From what's below \* the age of each you'll find ;  
That done, your judgment give, to quiet Susan's mind.

$$\begin{aligned} * \quad x^3y - xy^3 &= 983040 \\ x^4 + x^3y^2 &= 3481600 \end{aligned} \quad \left. \begin{array}{l} \text{Where } x \text{ is Richard's age,} \\ \text{and } y \text{ is Henry's.} \end{array} \right\}$$

*Answered by Amicus.*

The first equation being divided by the second, and  $w$  made  $= \frac{y}{x}$ , we have  $w \times \frac{1-w^2}{1+w^2} = \frac{24}{85} = \frac{3}{5} \times \frac{8}{17} = \frac{3}{5} \times \frac{1-\frac{9}{25}}{1+\frac{9}{25}}$ , where 'tis evident that  $w = \frac{3}{5}$ , and  $x^4 = 3481600 \div (1+w^2)$  by the second given equation, consequently  $x = 40$ , and  $y = 24$ . But the equation above, or which is the same  $\frac{3}{5}x + \frac{3}{5}x^2w - w + w^3 = 0$ , has also another real root affirmative ; for, divided by  $w - \frac{3}{5}$ , it gives  $w^2 + \frac{1}{5}w - \frac{8}{17} = 0$ ,  $w = (\sqrt{769} - 15) \div 34 = .3744372$ ,  $x = 41.79427$ , and  $y = 15.64935$  ; which also answer the conditions of the question.

*The same answered by Mr. L. Evans.*

Substitute  $zx$  for  $y$  in the given equation, and they become  $x^4z - x^4z^3 = a$ , and  $x^4 + x^4z^2 = b$  ; consequently  $x^4 = a \div (z - z^3) = b \div (1 + z^2)$  ; hence  $z^3 - (a \div b)z^2 - z = -(a \div b)$ , the root of which is  $\frac{3}{5}$ . Hence  $x = 40$ , and  $y = 24$ .

II. QUESTION 801, by Mr. J. Jackson, Master of Hutton-Rudby School.

There is a clock whose teeth and wheels are as follows : The great wheel 72 teeth, the pinion that runs in it 8 leaves ; the second wheel 60 teeth, its pinion 5 leaves ; the hour wheel 48, and knut 12. Now there are two pallats, but the pallat, or crown wheel, is lost ; I therefore demand what number of teeth a new crown wheel must have ; the length of the pendulum being 39.2 inches ?

*Answered by Mr. John Scott, of Woolwich.*

From the principles of mechanics, it is evident that the revolution of the great wheel in one hour will be expressed by  $48 \div 12 \times 12$  or  $\frac{1}{3}$  ; and that of the crown wheel thus  $1 \times 12 \times 60 \div 3 \times 8 \times 5 = 36$  : now as the pendulum vibrates seconds, the number of teeth in the crown wheel will be expressed thus  $60 \times 60 \div 36 \times 2 = 50$ , as required.

*The same answered by Mr. George Bewley.*

As the great wheel moves round once in 12 hours, the crown or pallet wheel, by the data, must move round  $9 \times 12 \times 4 = 432$  times in 12 hours, or once in 100 seconds. Now a pendulum of the given length vibrating seconds, and the crown wheel advancing one tooth in two vibrations, we have  $100 \div 2 = 50$  for the number of teeth in the new crown wheel.

*The same by Mr. L. Evans.*

Let  $A$ ,  $B$ , and  $C$  represent the given wheels, acting upon  $a$ ,  $b$ , and  $c$ , their contiguous pinions; also let  $v =$  the number of vibrations of the given pendulum in 12 hours, and  $x =$  the swing wheel sought. Then, by section 1, chapter 2, of the *Artif. Clock Maker*,

$$\frac{ABC}{abc} \times 2x = v : \text{hence } x = \frac{abcv}{2ABC} = 50 \text{ the answer.}$$

III. QUESTION 805, by Mr. Paul Sharp, of Biddenden.

Of all the common parabolas whose area is equal to 400, quere that which will circumscribe the greatest circle.

*Answered by Mr. J. Clarke, of Salford School.*

It is a well-known theorem (and easily demonstrated) that the diameter of a circle inscribed in the common parabola, is equal to the difference between the parameter and the greatest double ordinate. Therefore putting  $x$  to denote the absciss of the required parabola, we have from the given area,  $9000 \div x =$  the parameter, and  $300 \div x$  the ordinate, hence  $300 \div x - 45000 \div x^2$  the radius of the inscribed circle, must be a maximum; this being fluxed and reduced, there arises  $x = \sqrt{450} = 15\sqrt{2} =$  the absciss of the required parabola; and consequently,  $10\sqrt{2} =$  the semi-base, and  $\frac{20}{3}\sqrt{2} =$  the radius of the inscribed circle; that is 21.213203, and 14.14213, and 9.42809, respectively.

Mr. James Williams, adds these corollaries, viz. 1. That the greatest or bounding ordinate of any parabola is equal to the distance on the axis between that ordinate and another ordinate drawn from the point of contact of the inscribed circle. 2. That the radius of the greatest circle that can be inscribed in a parabola whose area is given, is equal to the parameter of the parabola. 3. That the ordinate to the axis drawn from the point of contact, bisects the radius of the circle on the axis. And 4th, that the parameter, ordinate, and absciss, of a parabola having its area given and its inscribed circle a maximum, are respectively as the numbers 4, 6, 9.

## IV. QUESTION 806, by Mr. Joel Lean, of Gwennap.

In the reciprocal roots  $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}}$  &c. infinitely, the sum of all the terms in the odd places, is to the sum of those in the even places, namely  $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}$  &c. :

$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}}$  &c. ::  $\sqrt{2} - 1 : 1$ . Quere the demonstration.

*Answered by Mr. Joseph French.*

Multiplying the latter series, or that of even terms, by  $\sqrt{2}$ , we have  $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{4}} + \sqrt{\frac{1}{6}} + \sqrt{\frac{1}{8}} + \sqrt{\frac{1}{10}} + \sqrt{\frac{1}{12}}$ , &c. which is evidently the sum of both the two series. Putting therefore, for shortness,  $a$  for the former series, and  $b$  for the other, we shall have  $b\sqrt{2} = a + b$ ; hence  $b\sqrt{2} - b = a$ , and  $a : b :: \sqrt{2} - 1 : 1$ .

*Corollary.* In like manner it may be shewn in general that  $1 + \frac{1}{3^n} + \frac{1}{5^n}$  &c. :  $\frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n}$  &c. ::  $2^n - 1 : 1$ . And here

while  $n$  is greater than 1, then  $2^n - 1$  is greater than 1 ;

when  $n$  is = 1, then  $2^n - 1$  is = 1 ; and

when  $n$  is less than 1, then  $2^n - 1$  is less than 1 ;

And accordingly the first series is greater, equal, or less than the second series, respectively as  $n$  is greater, equal, or less than 1.

*The same by Mr. James Clarke, of Salford School.*

If we divide the first or whole series by  $\sqrt{2}$ , the result will evidently be  $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{4}} + \sqrt{\frac{1}{6}}$  &c. (the terms in the even places): consequently it will be as  $\sqrt{2} : 1 :: \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{4}} + \sqrt{\frac{1}{6}}$  &c. :  $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{4}} + \sqrt{\frac{1}{6}}$  &c. and therefore (dividendo)  $\sqrt{2} - 1 : 1 :: \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{4}} + \sqrt{\frac{1}{6}}$  &c. :  $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{4}} + \sqrt{\frac{1}{6}}$  &c.

## V. QUESTION 807, by Mr. John Brinkley, at Harleston.

In a plane triangle, given the vertical angle, the side of the inscribed square, and the rectangle of one side and its segment adjacent to the base made at the point of contact of the inscribed square. To determine the triangle.

*Answered by the Proposer, Mr. John Brinkley.*

*Construction.* Draw the right line AB equal the side of the given inscribed square, on which describe the circular segment ARB to contain the given vertical angle ; and parallel to AB draw MN so that the



seeking the roots of the above equation. For putting the secant  $\sqrt{(1 + 4t^2)} = s$ , we get  $s^3 + \frac{8}{3}s^2 - 5s + \frac{8}{3} = 0$ , and the greatest affirmative root  $= 1.3510299$ , answering to  $47^\circ 45'$  as before; and the other is  $.676753$ , which, being less than radius 1, is impossible, as is likewise the 3d root which is  $= -2.9172609$ .

*The same, by the Proposer Mr. Alex. Rowe.*

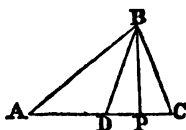
Let  $x = \text{sine of the latitude}$ ; then by spherics  $1 : \frac{1}{2} :: \sqrt{(1 - x^2)} \div x$  (cotangent latitude) :  $\sqrt{(1 - x^2)} \div 2x = \text{tangent declination}$ ; and by the nature of sines, &c. the cosine of double the declination or the cosine of the altitude will be  $(5x^2 - 1) \div (3x^2 + 1)$ ; therefore  $(5x^2 - 1) \div (3x^2 + 1) : x :: 8 : 9$ ; hence  $24x^3 - 45x^2 + 8x + 9 = 0$ ; which solved gives  $47^\circ 45'$  for the latitude, and hence  $24^\circ 26'$  the declination which is too great, as the greatest declination is now only  $23^\circ 28'$ : which was owing to the ratio of 8 to 9 being taken too great. But if it be taken 10 to 11, the equation will become  $30x^3 - 55x^2 + 10x + 11 = 0$ , the root of which is  $x = .7600158$  the sine of  $49^\circ 28'$  the latitude, and hence the declination is  $23^\circ 9'$ , answering to the 11th of June.

VII. QUESTION 809, by Mr. John Aspland.

Given one side, the difference between the square of the other side and the square of the base, and the difference of the segments of the base made by a perpendicular from the vertical angle, of a plane triangle: to determine the triangle.

*Answered by Amicus.*

Produce the given difference AD of the segments of the base till the rectangle ACD = the square of the given side + the given difference of the squares of the other two, then make DB = CB = the given side, and draw AB; so shall ABC be the required triangle. For letting fall the perpendicular BP, we have  $AB^2 - BC^2 = AP^2 - PC^2 = \text{rect. DAC}$ , therefore  $\text{rect. ACD} = AC^2 - \text{DAC} = AC^2 - AB^2 + BC^2 =$  (by construction)  $BC^2 +$  the given difference of the squares of the other two sides; take away  $BC^2$  common, and  $AC^2 - AB^2 =$  their given difference.

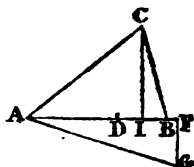


*The same by Mr. Robert Bowmas.*

On any line take  $AF = AC$  the given side, and raise  $FG \perp AF$  and  $= d$  the side of a square equal the given difference of the squares of the other two; draw AG, then (by Simpson's Geometry 5, 18) take AB such, that the rectangle  $AB \cdot (AB + AD) = AG \cdot AD$  being the given

difference of the segments; bisect  $AD$  with the perpendicular  $IC$ , apply  $AC$ , and draw  $BC$ , so shall  $ABC$  be the required triangle.

*Demonst.*  $AC$  = the given side, and  $AD$  = the given difference of the segments by construction, so it only remains to prove that  $AB^2 - BC^2 = D^2$ . In order to which,  $AB \cdot (AB + AD) = AG^2 = AF^2 + FG^2 = AC^2 + D^2$ , or  $AB^2 + AB \cdot AD = AC^2 + D^2$ ; but, by the triangle,  $AC^2 - BC^2 = AB \cdot AD$ , therefore  $AB^2 - BC^2 = D^2$ .



*An Algebraic Solution by the Rev. Mr. John Hellins.*

Put  $a$  = the given side,  $bb$  = the difference of the squares of the other two,  $d$  = the difference of the segments,  $x$  = the base, and  $y$  = the other side: Then  $y^2 - a^2 = \pm dx$ , by Euclid 3, 36, and  $x^2 - y^2 = \pm b^2$  per question, from which equations we get  $x^2 \mp dx = a^2 \pm b^2$ , an equation which exhibits all the four cases of this problem.

*Observation 1.* Since, by the conditions of the question,  $d$  can never be greater than  $x$ , it is evident that the least value of  $x^2 - dx = a^2 - b^2$ , is 0; or that, in this case,  $b$  must not be given greater than  $a$ .

*Observation 2.* Since  $x^2 + dx$  can never be less than  $2d^2$ , this is the least value of  $a^2 - b^2$  in that case.

VIII. QUESTION 810, by the Rev. Mr. John Hellins.

A gentleman, remarkable for odd experiments, having a cask of wine whose content was 100 gallons, ordered a hole to be made in it, and at that instant a supply of water to keep it constantly full; out of which ran one gallon of mixture every 10 seconds. Now admitting that the liquor was always equally mixed throughout the cask, he desires to know how much wine was left in it one hour after the hole was made.

*Answered by Mr. James Young.*

Let  $x$  be the gallons of wine left in the cask after  $t$  seconds. Then  $a$  ( $10''$ ): 1 gallon ::  $t$ :  $ta^{-1}$  = the gallons of mixture run out in the time  $t$ ; also  $b$  (100):  $ta^{-1}$  ::  $x$ :  $-x$ ; therefore  $tx = -abx$ , or  $t = -abx^{-1}$ ; and the fluents are  $t = -ab \times \text{hyp. log. } x$ . But when  $t = 0$ ,  $x = b$ , therefore the correct fluents are  $t = ab \times (\text{hyp. log. } b - \text{hyp. log. } x)$ ; consequently  $\text{hyp. log. } x = \text{hyp. log. } b - t \div ab = \text{hyp. log. } 100 - 3.6$  ( $t = 1 \text{ hour} = 3600''$  per quest.) =  $1.0051702$  the number to which  $\text{hyp. log.}$  is  $x = 2.73237$  gallons = the wine left in the cask.

*Corollary.* All the wine will never be run out; for when  $x = 0$ ,  $t$  is infinite.

*Mr. H. Clarke also answered it otherwise, as follows.*

Put  $100 = a$  the content of the cask, and let  $d$  denote the quantity of wine run out, or water run in, the first instant of time; then by the nature of the question, will the quantities of wine discharged, in the successive instants, or indefinite particles of time, be expressed by  $d, \frac{da - d^2}{a}, \frac{da^2 - 2da + d^2}{a^2}, \frac{da^3 - 3d^2a + 2d^3}{a^3}, \&c.$  the

sum of which series continued to an indefinite number ( $n$ ) of terms, is  $(a^n - (a - d)^n) \times a^{1-n}$ ; which is therefore an approximate expression for the whole quantity of wine discharged; where the greater  $n$  is taken, the nearer the truth will it approach. Suppose  $n = 3600$  (the seconds in 1 hour, the given time), then will  $d = 1$ , and consequently  $(a^n - (a - d)^n) \times a^{1-n} = 97.27$  gallons, therefore 2.73 gallons is the quantity of wine remaining in the cask.

IX. QUESTION 811, by Lieut. Glenie, of the Engineers.

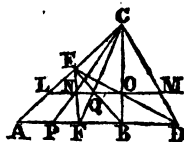
From the right angle B of any right-angled triangle let BE be drawn to any point E in the opposite side AC; draw CD perpendicular to AC, meeting AB produced in D; join DE meeting BC in O; draw OL parallel to AB, meeting BE in Q: then shall the locus of every point Q be a right line. Required the demonstration?

*Answered by Mr. John Whitton.*

The figure being described as directed, draw CQF. Then, because of the parallels LO, AD, we have  $AF : FB :: LQ : QO :: AB : BD$ ; but AB and BD are both given, and therefore both the ratio of LQ to QO and of AF to FB are given or constant. And consequently the locus of Q is the right line CF dividing AB in the given ratio of AB to BD: and that whether the angles ABC, ACD be right angles, as in the question, or of any other magnitudes.

*Amicus*, after a similar demonstration and remark, also farther observes, that, if EF be drawn cutting LO in N, then because the ratio of LQ : QN or AB to BF is given, the locus of the point N is a right line CF passing through C, and the line AD is so divided that  $BD : BA :: FB : FA :: FF : FA, \&c.$  and the line LM is divided in O, Q, N, in the same manner.

And *Mr. G. Beck* renders the problem still much more general; for having as above found  $AB : BD :: LQ : QO$ , he infers that therefore the locus of Q will be a line of the same order with the line BOC when AE



is any right line given in position, and  $BE$  and  $DE$  right lines inflected to it, and moving around from the two poles  $B$  and  $D$ ; so that if  $BEC$  be a right line, then the locus of  $q$  will be a right line; if that be a conic section, so will this; and if the former be a line of the  $n$ th order, so also will the latter. And farther, that if  $BEC$  be a right line and  $AE$  any curve, the locus of  $q$  will also be a curve of the same order. And this he illustrates by some examples, but the subject is too copious for the limits of the Diary.

X. QUESTION 812, by Mr. Robert Hartley.

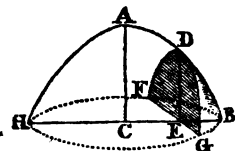
Let the perpendicular height of a hay-rick, in form of a paraboloid, be 8 yards; the circumference of its base 36 yards; the weight of a cubic yard, every where of the same density as that at the centre of the base, 200 pounds; and the perpendicular height of a section, made by part of it being cut away by a plane parallel to the axis, 4 yards: It is required to determine the weight of the whole hay-rick, and the weight of the part taken away; supposing the density to be every where as the perpendicular depth.

*Answered by the Rev. Mr. John Hellins.*

Let  $ABH$  represent the paraboloid, and  $EFCD$  a vertical section of it. Put  $AC = 8 = a$ ,  $200lb. = b$ ,  $CE = CH = 5.729579 = r$ ,  $DE = x$ , and  $EF = EG = y$ .

Now by the question  $a : b :: x : a^{-1}bx$  = the density at  $E$ ; in like manner the density at any other point in  $FE$  will be  $a^{-1}b \times$  the perpendicular height above it; and as all these perpendiculars make up the area of the parabola  $FED$ , therefore the sum of the densities in the whole line  $FE$  will be  $= a^{-1}b \times \text{area } FED = a^{-1}b \times \frac{1}{2}xy$ ; conseq. the fluxion of the value of all the densities in  $FED$  will be  $\frac{1}{2}a^{-1}bxy\dot{x} = \frac{1}{2}abr^{-4}y\dot{y}$ , the fluent of which gives  $\frac{1}{15}abr^{-4}y^5$  for all the densities in  $FED$ ; and this drawn into the fluxion of  $BE$  or of  $-\sqrt{(r^2 - y^2)}$ , gives  $8aby^5 \div 15r^4 \sqrt{(r^2 - y^2)}$  for the fluxion of the weight of the slice  $BDFE$ ; the fluent of which, or the weight of that slice will be  $\frac{1}{6}abr \times BG - ((15r^4 + 10r^2y^2 + 8y^4) \div 90r^4) \times aby \sqrt{(r^2 - y^2)}$ : And in the present case, in which  $x = \frac{1}{2}a$ , or  $y^2 = \frac{1}{2}r^2$ , and  $BG$  is  $\frac{1}{2}$  a quadrant, this expression becomes  $(.7854 - \frac{1}{15}) \times \frac{1}{6}abr^3 = 455.78lb.$  the weight of the slice.

And when  $y = r$ , the same expression becomes  $.7854 \times \frac{1}{6}abr^3 = 13750.99$  for half the weight of the solid, the double of which or 27501.98lb. is the weight of the whole paraboloid.

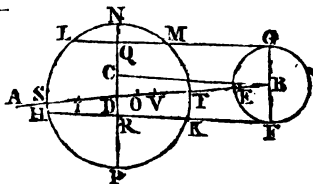


XL. QUESTION 813, by Mr. Nathan Parnel.

Given the magnitude and position of two circles; to draw two chords of the greater circle, so that they shall both be parallel to each other, have a given ratio, and both touch the less circle.

*Answered by Mr. John Hampshire.*

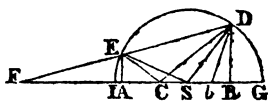
*Construction.* Join the centres B, D of the given circles, and on BD produced take SA, SI, TV, TE each equal the radius BG of the less circle, so that AE is the sum of the two given diameters, which cut in o (by prob. 7. Wales's Determinate Sect.) so that  $OI : OE :: OV : OA :: m^2 : n^2$  the duplicate of the given ratio. With the hypotenuse DB, and perpendicular DC = DO make the right-angled triangle DCB; then parallel to CB draw the tangents LQMG, HKEF, and the thing is done.



*Demonstration.* Produce DC both ways, and draw FBG parallel to it. From the equals DN, DT take the equals DC, DO and CQ (BC), VT, and there remains QN = OV; also by taking the equals QN, OV from the equals NP, AV, there remains QP = AO; and from QP, AO taking QR (FG), AT, there remains RP = IO; and conseq. OE = NR. Whence, by construction, and the property of the circle,  $OI : OE$  or  $RP : RN$  or  $BN^2 : OA \cdot OV$  or  $PQ \cdot QN$  or  $LQ^2 :: m^2 : n^2$ , or  $HK : LM :: m : n$ .

*The same answered by Mr. John Knowles.*

*Construction.* In the indefinite line FG make AB = the diameter of the less circle; and take FA : FB and CA : CB as  $m : n$ , each in the given ratio of the chords; also by Simpson's Geometry 5, 18, add CS to CF so that FS . SC may be = the square of the radius of the greater circle; with which radius, and centre S, describe that circle, cutting the perpendicular AE in E; draw FE to cut the circle again in D; join DB; and AE, BD shall be the semi-chords required.



*Demonstration.* Join c and E, c and D, s and E, s and D. Then by construction  $SF : SE :: SE : SC$ , and the  $\angle$  at s common, therefore the triangles SPE, SEC are equiangular, or  $\angle SEF = SCE$ , and their sup.  $FCE = SED = SDE$  because  $SD = SE$ . Then because  $\angle FCE = FDS$  (SDE), and  $\angle F$  common, the  $\triangle$ s FCE, FDS are equiangular, conseq.  $FC : FE :: FD : FS$ , or  $FD \cdot FE = FS \cdot FC = FS \cdot (FS - SC) = FS^2 - FS \cdot SC$ . But by the nature of the circle  $FD \cdot FE = FG \cdot FI = FS^2 - SE^2$ . Therefore  $SE^2 = FS \cdot SC$  as by construction.

Again, by construction,  $FS : SD :: SD : SC$ , and the  $\angle s$  common, therefore the  $\triangle s FSD, CSD$  are equiangular, and the  $\angle SCD = SDF = FCE$  from above. Therefore the  $\triangle s CAE, CBD$  are equiangular, having  $SCD = FCE$ , and  $\angle s A$  and  $b$  right;

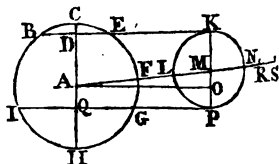
and therefore  $CA : CB :: AE : bD :: FA : FB$ ;

but by constr.  $CA : CB :: - - - FA : FB$ ;

therefore  $FB : BC :: FB : bC$ , and by div.  $FC : CB :: FC : bC$ . Consequently  $b$  coincides with  $B$ ,  $BD \parallel AE$ , and  $AE : BD :: CA : CB$ , the given ratio by construction.

*The same by Mr. John Brinkley.*

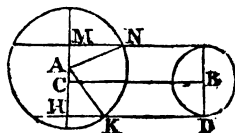
Draw  $AFLMN$  through the centres  $A$  and  $M$  of the given circles, producing it to  $R$  so that  $LR : LN :: m^2 + n^2 : m^2 - n^2$  ( $m : n$  being the given ratio), and again producing it (by Simp. Geom. 5, 18), to  $S$  so that  $LS.SR = AF^2 - LM^2$ ; then on  $AM$  as hypotenuse and  $MO = RS$ , make the right-angled  $\triangle AMO$ . Produce  $MO$  to the circumference at  $K$  and  $P$ , and perpendicular to  $KP$  draw  $BDEK$  and  $IQGP$ ; then  $BE$  and  $IG$  are the chords required.



For perpendicular to  $BE$  and  $IG$  draw the diameter  $CDAQH$ . Now by construction  $AF^2 - LM^2 = LS.SR = LR.RS + RS^2$ ; or taking away  $RS^2$  from each,  $AF^2 - LM^2 - RS^2 = LR.RS : LN.RS :: m^2 + n^2 : m^2 - n^2$  by construction. But  $QG^2 + DE^2 = 2AF^2 - AD^2 + AQ^2 (= KO^2 + OP^2 = 2KM^2 + 2OM^2 = 2LM^2 + 2RS^2) = 2AF^2 - 2LM^2 - 2RS^2$ , also  $QG^2 - DE^2 = AD^2 - AQ^2 = 2LN.RS$ . Hence by exposition  $QG^2 + DE^2 : QG^2 - DE^2 :: m^2 + n^2 : m^2 - n^2$ ; therefore by comp. and div.  $QG^2 : DE^2 :: m^2 : n^2$ ,  $QR : IG : BE :: m : n$ .

*The same algebraically, by Mr. Thomas Robinson.*

Put  $b$  = greater radius  $AN$ ,  $c$  = the less  $BD$  or  $CH$  or  $CM$ , and  $x = AC$ . Then  $AH = c + x$ ,  $AM = c - x$ ,  $MN^2 = b^2 - (c - x)^2$  and  $HK^2 = b^2 - (c + x)^2$ . Hence per question  $b^2 - (c - x)^2 : b^2 - (c + x)^2 :: m^2 : n^2$ , and hence  $x^2 + ((m^2 + n^2) \div (m^2 - n^2)) \cdot 2cx = b^2 - c^2$ .



## XII. QUESTION 814, by Plus Minus.

A conic section being given on a plane, it is required to describe the same in perspective, the positions of the eye and of the plane of the draught being given.

N. B. The latter is, as usual, supposed to be perpendicular to the plane on which the given section is described.

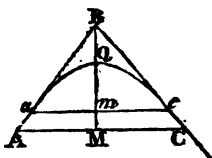
**Answered by Plus Minus.**

Let fall a perpendicular  $ov$  from the eye to the plane of the draught, which call  $d$ ; from the point  $v$  where it meets that plane (which is called the point of view) let fall a perpendicular  $va$  to the fundamental line (or line of intersection of the two planes mentioned in the question), which call  $a$ . Then find an equation for the given conic section, having the vertex of the abscissa ( $z$ ) in that point  $A$ , and the abscissa itself extended along the fundamental line; this done, substitute  $ax \div (a - y)$  for  $z$ , and  $dy \div \{a - y\}$  for  $v$  (the perp. ordinate of the given conic section) and the resulting equation will be that of the perspective appearance required, the abscissa and its vertex being common to both curves and the ordinate perp. to the abscissas.

**Corollary.** The perspective appearance of any algebraic curve will always be a curve of the same order; for the substitution of  $ax \div (a - y)$  for  $z$ , and of  $dy \div (a - y)$  for  $v$  cannot raise the equation to a higher degree, or depress it to a lower.

*The same answered by Amicus.*

It is known that the representation of a conic section on a plane, is also a conic section ; the demonstration being the same as that for the genesis of curves by shadows. Therefore find the representation of five points of the given curve, and through them as directed by writers on conics describe a conic section, and it will be the representation required. Or, otherwise, let  $AC$  be the representation of an ordinate parallel to the plane of the picture,  $ac$  another ; on the line  $mm$  that bisects them find the representation of a point  $q$  in the curve, and describe the section  $aaqqcc$ . Or, let  $AB, BC$  be the representation of two tangents to the first-named ordinate which consequently are tangents to the representation ; bisect  $AC$  in  $M$ , join  $BM$ , and in it let  $q$  be the point that is immediately between the eye and the object ; through the vertex  $q$  and points of contact  $AC$  describe the section as directed by writers on conics, and it will be the representation required.



**XIII. QUESTION 815, by Mr. Henry Clarke.**

### The sum of the infinite series

$$\begin{aligned} & \frac{1}{2}x - \left(\frac{3}{2.4} + \frac{1}{3.4}\right) \times x^{n+1} + \left(\frac{3.5}{2.4.6} + \frac{3}{3.2.6} + \frac{1}{5.6}\right) \times x^{2n+1} \\ & - \left(\frac{3.5.7}{2.4.6.8} + \frac{3.5}{3.2.4.8} + \frac{3}{5.2.8} + \frac{1}{7.8}\right) \times x^{3n+1} + \\ & \left(\frac{3.5.7.9}{2.4.6.8.10} + \frac{3.5.7}{3.2.4.6.10} + \frac{3.5}{5.2.4.10} + \frac{3}{7.2.10} + \frac{1}{9.10}\right) \end{aligned}$$

$\times x^{4n+1} - \&c. \left\{ \frac{ax^{1-\frac{1}{2}n} + x^{1-n}}{\sqrt{(1+x^n)}} - x^{1-n} \right\}$   $\left\{ \begin{array}{l} a \text{ being the circular arc,} \\ \text{is expressed by} \end{array} \right. \left\{ \begin{array}{l} \text{rad. 1, tang. } x^{\frac{1}{2}n}; \text{ and} \\ n \text{ any positive number whatever.} \end{array} \right.$  Required the investigation?

*Answered by Amicus.*

Putting the tangent  $x^{\frac{1}{2}n} = z$ , and dividing both sides of the equation by  $x^{1-n}$ , the finite side becomes  $(ax^{\frac{1}{2}n} + 1) \div \sqrt{(1+x^n)} - 1 = (az + 1) \div \sqrt{(1+z^2)} - 1$  and the infinite side thrown into fluxions will evidently be the product of the two following series, viz.

$$z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} \&c. \text{ and } z \times : 1 - \frac{3z^2}{2} + \frac{3 \cdot 5z^4}{2 \cdot 4} - \frac{3 \cdot 5 \cdot 7z^6}{2 \cdot 4 \cdot 6} \&c.$$

$= az \times (1+z^2)^{-\frac{1}{2}} =$  the arc  $a \times$  fluxion of its sine  $z \div \sqrt{(1+z^2)}$ ; the fluent by Simp. Flux. Art. 342, corrected to be  $= 0$ , when  $z = 0$ , is  $(az + 1) \div \sqrt{(1+z^2)} - 1$ , which is the very expression above, as required to be investigated.

*Note.* When  $z = 1$ , the sum of the series is  $= (a + 1 - \sqrt{2}) \div \sqrt{2}$ .

*The same answered by Mr. Geo. Sanderson.*

Put  $s$  for the sum of the infinite series, and multiply the whole by  $x^{\frac{1}{2}n-1}$ , then  $\frac{1}{2}x^{\frac{1}{2}n} - \left(\frac{3}{2 \cdot 4} + \frac{1}{3 \cdot 4}\right)x^{\frac{3}{2}n} + \left(\frac{3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{3}{3 \cdot 2 \cdot 6} + \frac{1}{5 \cdot 6}\right)x^{\frac{5}{2}n} \&c. = sx^{\frac{1}{2}n-1} = \frac{a}{\sqrt{(1+x^n)}} + \frac{1}{x^{\frac{1}{2}n}} \sqrt{\frac{1}{1+x^n} - \frac{1}{x^{\frac{1}{2}n}}}$  (by No. 327, page 23, Clarke's Supp. to Lorgna's Series,  $x$  there being  $= x^{\frac{1}{2}n}$  here, and is the tangent of an arc expressed by  $a$ ), which is equal to  $(a + x^{-\frac{1}{2}n}) \div \sqrt{(1+x^n)} - x^{-\frac{1}{2}n}$ . Divide by  $x^{\frac{1}{2}n-1}$  then  $s = (ax^{1-\frac{1}{2}n} + x^{1-n}) \div \sqrt{(1+x^n)} - x^{1-n}$ , as was to be investigated,

#### XIV. QUESTION 816, by Nauticus.

In determining the true right ascension of the moon's centre, as seen from the centre of the earth, from the observed time of her limb's transiting the meridian; it is required to shew whether her semi-diameter in right ascension should be deduced from her horizontal semi-diameter, or from her horizontal semi-diameter increased on account of her altitude.

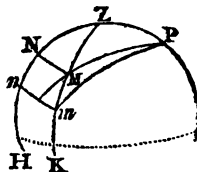
*Answered by the Rev. Mr. John Hellins.*

It is well known to astronomers, that the observed right-ascension of the limb of the moon, where that limb touches the meridian, is the

same as would be seen, at that moment, at the centre of the earth. Now to find the correction of right-ascension on account of the moon's semi-diameter, conceive a plane to pass through the centres of the earth and moon, and the poles of the world, at the moment the moon's limb touches the meridian; then in a right-angled spherical triangle will be given the base = the complement of the moon's true declination, the perpendicular = the moon's semi-diameter as seen from the centre of the earth, to find the angle at the base; which is to be added to, or subtracted from, the right-ascension of the limb, according as the western or eastern limb was observed. And since the perp. of the given triangle is always = the angle which the moon's semi-diameter subtends at the earth's centre, it is evident that the *horizontal diameter pro tempore* is to be used.

*The same answered by Mr. George Sanderson.*

Let  $pzn$  represent the meridian,  $zmk$  a vertical circle passing through the moon's centre,  $p$  the pole,  $z$  the zenith. If  $n$  be the apparent place of the point in the moon's limb touching the meridian, and  $m$  that of her centre in the vertical circle  $zmk$ ; then because the parallax does not alter the true vertical position of an object, their true places  $N$  and  $M$  will be in the same verticals, but nearer the zenith (see la Caille's Astron. art. 405 and cors.); therefore  $NM$  is the true semi-diameter as seen from the earth's centre, and its right-ascension is expressed by the  $\angle NPM$ , being less than the  $\angle nrm$ , which is the right ascension increased on account of her parallax. Whence it is manifest, that the right ascension must be deduced from the true semi-diameter  $NM$ .



PRIZE QUESTION, by Mr. George Sanderson.

A gentleman would have a triangular garden, whose side  $AC$  shall be 8 chains; and in the base  $AB$  are to be two gates,  $v$  and  $e$ , the one ( $e$ ) at the extremity of a gravel walk from the opposite angle  $c$ , and at the distance of 12 chains from  $A$ , and the other ( $v$ ) at the extremity of another walk also drawn from the angle  $c$  perpendicular to  $AB$ : moreover he would have the triangle  $ABC$  such, that a rectangle under  $2AD$  and the distance  $EB$  may be equal to a square on  $CE$ ; and, as there is to be a fish pond at  $e$ , that the angle  $ABC$  be the greatest possible. Now as his Surveyor is unacquainted with the method of fluxions and the conic sections, he begs the assistance of your correspondents for a geometrical construction, and to furnish the trigonometrical calculation of the area of the whole triangle  $ABC$ .

*Answered by Amicus.*

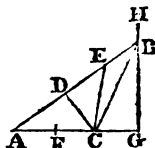
Produce the given side  $AC$  to  $x$  till  $2Ax \cdot AC = AC^2 + AE^2$ ; through



*Note.* If  $AC$  be greater than  $AE$ , then  $q$  falls between  $A$  and  $c$ , in which case the  $\angle ABC$  must be a given quantity, as there can be no maximum.

*The same by Mr. Reuben Robbins, Teacher of Mathematics.*

*Construction.* Bisect the given side  $AC$  in  $F$ , and produce  $AC$  till  $FG$  be a 3d propor. to  $2AC$  and the given distance  $AE$ ; in the indefinite perpendicular  $GH$  take  $GB$  a mean proportional between  $AG$  and  $GC$ ; join  $AB$  and  $BC$ ; and in  $AB$  take  $AE$  of the given length, also let fall the  $\perp$   $CD$ . Then is  $ABC$  the required triangle, and  $D$ ,  $E$  the two gates.



*Demonstration.*  $AC$  is = the given side by construction, and  $GB$  being a mean between  $AG$  and  $GC$ , therefore  $GH$  will (by Simpson's Geom. 3, 22,) be a tangent to a circle passing through  $A$ ,  $B$ ,  $C$ , and the  $\angle ABC$  a maximum (by Simp. Alg. pr. 44, page 358). Again  $AE^2$  being (by construction)  $= 2AC \cdot FG = 2FG \cdot AC = AG^2 - CG^2 = AB^2 - BC^2$  (by Simp. Geom. 2, 9, and cor. 2), therefore (by Simp. Geom. 2, 10),  $2AD \cdot AB = AC^2 + AB^2 - BC^2 = AC^2 + AE^2$ , and (by the same)  $2AD \cdot AE = - - - - - AC^2 + AE^2 - CE^2$ ; therefore by subtraction  $2AD \cdot EB = CE^2$ , as required.

*Calculation.*  $FG = AE^2 \div 2AC = 9$ ,  $AG = FG + AF = 13$ ,  $CG = FG - FC = 5$ ,  $BG = \sqrt{AG \cdot GC} = \sqrt{65} = 8.062258$ ,  $AB = \sqrt{AG^2 + BG^2} = 15.297058$ ,  $BC = \sqrt{CG^2 + BG^2} = 9.486832$ ,  $AD = AG \cdot AC \div AB = 6.798692$ .

*An Algebraic Solution by Mr. Robert Abbatt.*

Put  $AC = a = 8$ ,  $AE = b = 12$ ,  $AD = x$ ,  $DB = y$ ; then  $DE = b - x$ ,  $EB = x + y - b$ , and  $CD = \sqrt{(a^2 - x^2)}$ , and  $CE = \sqrt{(a^2 + b^2 - 2bx)} = \sqrt{(c^2 - 2bx)}$ , by making  $c^2 = a^2 + b^2 = 208$ . Hence per question  $2x \times (x + y - b) = c^2 - 2bx$ . And by trigon.  $DB : DC :: 1$  (rad.):  $\sqrt{(a^2 - x^2)} \div y = \text{tangent } \angle B$  a maximum. From 1st equation  $y = (c^2 - 2x^2) \div 2x$  which value substituted in the maximum gives  $2x\sqrt{(a^2 - x^2)} \div (c^2 - 2x^2)$  the fluxion of which made = 0, gives  $x = ac \div b \sqrt{\frac{1}{2}} = \frac{2}{3} \sqrt{104} = 6.798692 = AD$ . Therefore  $y = 8.498366 = DB$ ,  $EB = 3.297058$ ,  $CD = 4.21637$ ,  $AB = 15.297058$ ,  $CE = 6.695622$ , tangent  $\angle B = .4961389$  answering to  $26^\circ 23' 16''$ , and the area of the triangle  $= 3ac \cdot \text{Or} \cdot 35.98$  p.



*Questions proposed in 1784, and answered in 1785.*

I. QUESTION 818, by Mr. Wm. Purver.

Old Joan, a mark of envious time,  
Long past the period of her prime;

Yet for a beauty still would pass,  
 And oft consults her looking glass;  
 Her silver locks play on her neck,  
 Vermilion gives a blushing cheek.  
 But judge, Diarians, from below,  
 How vain is such external show.  
 Ingenious artists, pray produce  
 A' better mirror for her use;  
 As other mirrors can't maintain  
 Convincing proofs that she is vain.

$$\begin{aligned} x^2 + 2xy + y^2 &= 238840 & \text{To find } x = \text{her height in feet,} \\ x^2 + xy^2 + y^2 &= 14791776 & \text{and } y = \text{her age in years.} \end{aligned}$$

*Answered by Amicus.*

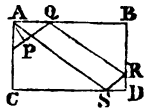
It is evident, at first sight, that  $y$  is much greater than  $x$ ; divide therefore the symbolic side of the 2d equation by that of the first, and the quotient is  $y$ , with a negative remainder: divide the numeral side in the same manner, and the quotient, so as to leave a negative remainder, is 62; which substituted for  $y$  in the first equation, by a quadratic  $x$  is found = 4: which two numbers answer the conditions of the question. If they had not, this value of  $x$  must have been substituted in the 2d equation, whence a nearer value of  $y$  would have been obtained by a quadratic, and thence a nearer value of  $x$ . And so on to any degree of exactness.

II. QUESTION 819, *by Mr. Richard Tattam, jun.*

Given the four sides of a billiard table  $ABDC$ , namely the length 10, and the breadth 6 feet; a ball in  $P$  being struck, touched the sides  $AB$  in  $Q$ ,  $BD$  in  $R$ ,  $DC$  in  $S$ , and went into the pocket  $A$ , the distance of  $A$  from  $S$  being 10 feet. Required the least distance of  $P$  from the pocket  $A$ ?

*Answered by Mr. L. Evans, of Compton.*

Let  $AS$  be made = 10 =  $AB$ ; then because the angles of incidence and reflection are equal, make  $\angle DSR = \angle CSA$ , and  $\angle BRQ = \angle DRS$ , and  $\angle AQP = \angle BQR$ , also  $AP \perp PQ$ ; so shall  $Q R S A$  be the path of the ball, and  $AP$  the nearest distance at the beginning.



*Calculation.* All the triangles  $ACS$ ,  $SDR$ ,  $RBQ$ ,  $QAP$  being similar, and  $AS = 10$ , and  $AC = 6$ ; therefore  $CS = 8$ , and  $SD = 2$ . Hence  $CS : CA :: SD : DR = 1\frac{1}{2}$ , and hence  $BR = 4\frac{1}{2}$ ;  $CA : CS :: BR : BQ = 6$ ,  $AQ = 4$ ;  $AS : AC :: AQ : AP = 2\frac{2}{3}$ , required.

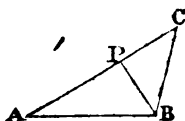
III. QUESTION 820, *by Mr. John Brinkley.*

Given the base, the sum of the squares of the other two sides, and

the rectangle of the segments of one of those sides made by a perpendicular falling from the opposite angle, to determine the triangle.

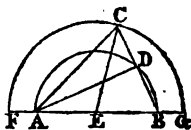
*Answered by the Rev. Mr. Robert Bownas.*

The rectangle  $AC \cdot CP$  is given, being equal to half the excess of  $AC^2 + CB^2$  above  $AB^2$ ; therefore  $PC$  is known, because  $AP \cdot PC$  is given by the question. Hence if  $PC$  be produced till  $CP \cdot PA =$  the given rectangle of the segments, the indefinite perpendicular  $PB$  erected, and  $AB$  applied, the triangle  $ABC$  will be that required.



*The same by Mr. James Williams, of Plymouth Dock.*

*Analysis.* Let  $ABC$  be the required triangle. Bisect  $AB$  in  $E$ , and join  $CE$ , which will be given, because  $AC^2 + BC^2 = 2AE^2 + 2CE^2$ . Again, with centre  $E$  describe the two semicircles  $ADB$ ,  $FCG$ , and draw  $AD$ , which will be  $\perp BC$  because the  $\angle ADB$ , in a semicircle, is a right angle: then the rectangle  $AG \cdot GB = (BC \cdot CD =) BD \cdot DC + DC^2$ ; hence  $CD$  is given, because both the rectangles  $AG \cdot GB$  and  $BD \cdot DC$  are given.



Wherefore, having described circles with the two given radii  $EA$ ,  $EF$ , from any point in the one apply  $CD$  to the other of the given length, which produce to meet  $ADB$  again in  $B$ ; draw the diameter  $AB$ , and join  $AC$ ; so shall  $ABC$  evidently be the required triangle.

*An Algebraic Solution by Mr. Thomas Cock, of Cirencester.*

Put  $AB = a$ ,  $BD \cdot DC = b^2$ ,  $AC^2 + BC^2 = c^2$ , and  $CD = x$ ; then  $BD = b^2 \div x$ ,  $BC = x + b^2 \div x$ ,  $2BC \cdot CD = 2x^2 + 2b^2$ ; hence (by Simp. Geom. 10, 2,)  $AB^2 + 2BC \cdot CD = AC^2 + BC^2$ , or)  $a^2 + 2x^2 + 2b^2 = c^2$ , and  $x = \sqrt{(\frac{1}{2}c^2 - \frac{1}{2}a^2 - b^2)}$ .

IV. QUESTION 821, by the Rev. Mr. John Hellins.

If the sursolid equation  $x^3 + qx^2 - rx^2 - t = 0$ , has two roots equal to each other, then shall one of them be a root of this quadratic,

$$x^2 + \frac{2qq}{5r}x + \frac{5t}{3r} - \frac{4q}{15} = 0. \text{ Query the demonstration.}$$

*Answered by Amicus.*

It has long ago been proved by Hudde, Prestet, Mac Laurin, and many others, that if the given equation has two equal roots, one of them is also a root of  $5x^3 + 3qx - 2r = 0$ : They therefore direct to take the greatest common measure of this equation and the given



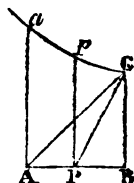
and  $EF = y$ ; then by the principles of geometry  $x : z :: d : y$ , or  $xy = dz$ ; and  $b^2 + z^2 - y^2 = x^2 - d^2$ ; and  $z : x :: d : a + y :: y : d$ , or  $y^2 + ay = d^2$ ; hence  $y = \sqrt{(d^2 + \frac{1}{4}a^2)} - \frac{1}{2}a$ . Also from the 1st equation  $z = x \div d$ , which substituted in the 2d, and reduced, it is  $x = d \sqrt{((b^2 + d^2 - y^2) \div (d^2 - y^2))}$ .

VI. QUESTION 823, by Mr. C. Cockerhill.

AB and BC are two right lines perpendicular to each other: now suppose a body to move from A along AB, and that its velocity at any point P is inversely as the right line PC; in what time after its first setting out will it arrive at B, supposing its velocity at the commencement of motion to be one foot per second, AB equal 100 yards, and AC equal 150 yards?

*Answered by Amicus.*

Since the velocity of the body, moving from A to B, is at any point P inversely as CP, the elementary time at P must be as  $CP \cdot AP$ ; or, describing, to the semi-diameter BC, the equilateral hyperbola  $apc$ , and erecting  $Aa$  and  $pp \perp AB$ , since  $Aa = AC$ , and  $pp = PC$ , therefore  $CP \cdot AP = pp \cdot Aa$ , and the time of describing AP is as the area  $aAPpa$ , and that of moving from A to B as the hyperbolic area  $aABC = A = \frac{1}{2}AB \cdot AC + \frac{1}{2}CB^2$ , hyp. log. of  $(AB + AC) \div BC$ ; and as the velocity per second is  $\frac{1}{3}AC \div BC$  yards, the time is  $3A \div AC = 267''$ , 034 as required.



*The same by the Rev. Mr. Bownas.*

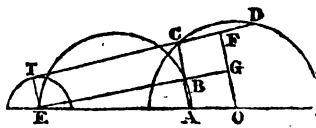
Put  $BC = a$ ,  $AB = b$ ,  $AC = c$ , and  $PB = x$ ,  $v$  the velocity at P, and  $t$  the time. Then is  $v = c \div \sqrt{(a^2 + x^2)}$ , consequently  $t = x \div v = x\sqrt{(a^2 + x^2)} \div c$ ; and  $t$  corrected when  $x = b$  gives  $\frac{1}{2}b + (a^2 \div 2c) \times \text{h. l. } ((b + c) \div a) = 268'' = 4' 26''$  the time required.

VII. QUESTION 824, by Mr. Nathan Parnel.

Given the magnitude and position of two circles; to draw a chord CD of the greater, to touch the less at T, so that CT and TD shall have a given ratio.

*Answered by Mr. George Sanderson.*

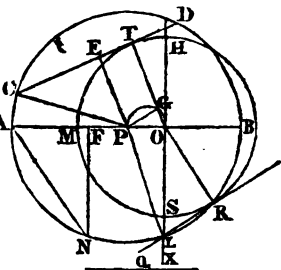
Suppose the thing done, and TCD the required line. Join the centres E, O, of the given circles; and draw  $EBG \parallel$ , and  $TE, ABC, OGF \perp TD$ . Then because of parallel lines,  $TF = EG$ ,  $TE = FG$ ; and, by similar  $\Delta$ s,  $TC (EB) :$



$CF(BG) :: EA : AO$ , therefore the point A is given, and the  $\angle EBA = TCA$  being right; whence, on EA describe the semicircle ABE, and (by Apol. Incl.) draw ABC cutting the circles in B and C so that  $CB = TE$ , and through C draw TCD to touch the less circle in T, and it is done.

*The same by Mr. Nathan Parnell.*

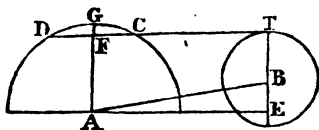
**Construction.** Through the centres O, P of the given circles draw the diameter AB, which divide at M in the given ratio; draw  $HOX \perp AB$ , and take  $sx : sl :: ml : hs$ ; apply  $AN = BM - AM$ , draw  $NF \perp AB$ , and take  $q : hs :: af : bf$ ; then on OP describe the semicircle PGO, and draw OGT, so that  $GO^2 + GO \times q$  may be  $PO^2 - sx \times q$ ; lastly draw  $CTD \perp OT$ , and it is done.



**Demonstration.** Draw  $PE \perp CD$ , and  $PG \perp OT$ ; draw also the tangent LR, and the radii OR, PL. Then, by construction,  $GO^2 + GO \times q = PO^2 - sx \times q$ , or  $q \times (sx + go) = PO^2 - GO^2 = PG^2 = ET^2$ ; and, consequently, by construction,  $af : bf :: q : hs :: q \times (sx + go) = ET^2 : hs \times (sx + go) = hs \times sx + hs \times go$ ; but, by construction,  $hs \times sx = hl \times ls = LR^2 = OL^2 - OR^2 = PC^2 - PO^2 - OT^2$ , because  $PC = PL$ , and  $OT = OR$ ; therefore  $hs \times sx + hs \times go = PC^2 - PO^2 - TO^2 + 2TO \times go$ ; and because  $2TO \times go - TO^2 = GO^2 - TC^2 = GO^2 - PE^2 = PO^2 - PG^2 - PE^2$ , it is evident that  $hs \times sx + hs \times sg = PC^2 - PE^2 - PG^2 = CE^2 - TE^2$ ; wherefore  $CE^2 - TE^2 : TE^2 :: AB - AF = BF : AF$ , and by comp.  $CE^2 : TE^2 :: AB : AF :: AB^2 : AF \times AB = AN^2$ , or  $CE : TE :: AB = MB + MA : AN = MB - MA$  by const. hence by comp. division, &c.  $CT (= CE + ET) : TD (= CE - ET) :: 2BM : 2AM :: BM : AM$  the given ratio by construction.

*The same by Mr. Thomas Robinson, of Biddick.*

Let A and B be the two centres,  $AE \parallel$  and  $AF$  and  $TBE \perp DCB$  the required line. Since TC to TD is a given ratio, their half sum and half difference, namely FT to FD or FC will be in a given ratio, which let be that of m to n. Put also the radius  $AG = a$ , radius  $BT = b$ ,  $AB = c$ , and  $AF = ET = x$ . Then  $BE = x - b$ ,  $AE^2 = FT^2 = c^2 - (x - b)^2$ , and  $FC^2 = a^2 - x^2$ ; wherefore  $c^2 - (x - b)^2 : a^2 - x^2 :: m^2 : n^2$ ; hence, multiplying extremes and means, &c. we have  $(m^2 - n^2)x^2 + 2n^2bx = m^2a^2 + n^2b^2 - n^2c^2$ ; the root of which quadratic gives AF, &c.



VIII. QUESTION 825, by Mr. Isaac Dalby.

To determine how the latitude and variation of the needle can be

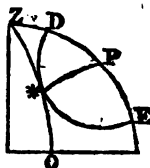
found, by observing two azimuths of the same known fixed star with the azimuth compass.

*Answered by Mr. Isaac Dalby.*

If the azimuth of a circumpolar star, or one whose co-declination is less than the co-latitude of the place of the same name as the declination be taken on each side of the meridian when at the greatest, from the elevated pole; then it is evident that *half the sum* of these two azimuths will be the true azimuth from the north, or south, according as the latitude is north or south, and that *half the difference* will be the variation of the needle. Now supposing, at the time of observation, an hour circle to pass through the star at the point where the azimuth circle touches its parallel of declination, there will be formed a right-angled triangle, where there is given a leg = the star's polar distance, and the opposite angle = the azimuth; whence, as *sine azim. :  $\phi$ 's polar dist. :: radius : sine hypot. the co-lat. sought.*

*The same by Mr. Alexander Rowe, of Reginnis.*

Let  $D\phi E$  represent the parallel of declination of any known object, as a star, &c. whose declination is greater than the latitude of the place of observation. From the zenith draw the quadrant  $zo$  to touch the parallel of declination in  $\phi$ ; then  $z\phi P$  will be a right angle, and the  $\angle \phi z P$  the greatest azimuth that such an object can make in that latitude. On both sides of the meridian let this greatest azimuth be taken; then the horizontal middle of these two observations will be the true meridian, and shews the variation of the needle. And to find the latitude in the right-angled  $\triangle z\phi P$  are given the  $\angle \phi z P$  the azimuth, and  $\phi P$  the co-declination; whence *sine  $\phi z P$  : sine  $\phi P$  :: radius : sine  $zP$  the co-latitude.*



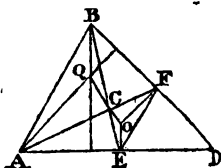
#### IX. QUESTION 826, by Atticus.

In every triangle, the intersection of the perpendiculars let fall from the angular points on the opposite sides, the centre of gravity of the triangle, and the centre of the circumscribing circle, are all in the same straight line; and the distances of those points from one another, are in a given ratio.

*Answered by Amicus.*

From  $A$  and  $B$  let fall perpendiculars to the opposite sides intersecting in  $q$ ; parallel to them, through the middle points  $E$ ,  $F$  of the sides, draw  $xo$ ,  $yo$  intersecting in  $o$ ; join  $EF$ , and  $AF$ ,  $BE$  intersecting in  $c$ : then it is well known that  $c$  is the centre of gravity of the triangle  $ABD$ , and  $o$  that of its circumscribing circle, also  $AB \parallel$  and  $= 2EF$ , and

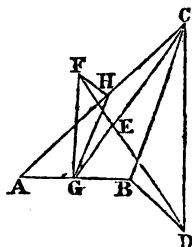
therefore  $BC = 2CE$ ,  $AC = 2CF$ , and by reason of the parallel lines the triangles  $AQB$ ,  $FOE$  are equiangular, consequently  $AQ = 2OF$ ,  $BQ = 2EO$ , and  $BQ : OE :: BC : CE$ , therefore the two triangles  $BCQ$ ,  $ECO$  having the angles at  $B$  and  $E$  equal; and the sides about them proportional, must be similar, and conseq. the  $\angle BCQ = \angle ECO$ ; therefore  $Q, C, O$ , are in a right line; and because  $BC = 2CE$ , therefore  $QC = 2CO$ .



*The same by Atticus, the Proposer.*

Let  $D$  be the intersection of perpendiculars drawn from the angular points  $A, B, C$ , on the opposite sides;  $E$  the center of gravity of the  $\triangle ABC$ , and  $F$  the centre of a circle passing through  $A, B, C$ : then shall  $D, E, F$  be in a right line; and  $DE$  to  $EF$  in a given ratio.

Bisect  $AC$  in  $H$ , and draw  $DB, DC, CEG, FG, FH$ , and  $GH$ . From the nature of the centre of gravity  $AB$  is bisected in  $G$ ; and, from the nature of the circumscribing circle,  $FG$  will be  $\perp AB$ , and  $FH \perp AC$ . Because  $GH$  is  $\parallel BC$ ,  $GF$  to  $DC$ , and  $FH$  to  $BD$ , the  $\triangle BCD, HGF$  are equiangular, and  $CD : GF :: CB : GH$ ; but  $BC = 2GH$ , therefore  $DC = 2GF$ . Again, in the  $\triangle GEF, CED$ , because  $CE = 2EG$  (from the nature of the centre of gravity), and  $CD = 2GF$ ,  $CD : CE :: GF : GE$ ; and  $\angle DCE = \angle FGE$  because  $CD$  is  $\parallel FG$ : therefore the  $\triangle GEF, CED$  are equiangular, and the  $\angle FEG = \angle CED$ . But  $CEG$  is a straight line, therefore  $FED$  is also a straight line. Also, because of the similar  $\triangle GEF, CED$ ,  $DE : EF :: CE : EG :: 2 : 1$ , that is, in a given ratio.



**X. QUESTION 827, by Mr. George Beck.**

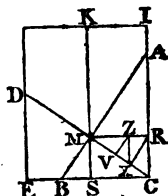
A youth unskill'd in ancient lore,  
Who ne'er was smit with love before,  
Is now involved in deep distress;  
He loves a fair young Platonesse \*.  
Her snowy neck and beaming eye,  
The powers of mimic art defy;  
Who would not learn geometry!  
To find what skill her ardent love has,  
She gives this problem to the novice:  
"Turn down a page i'th' Almanac,  
The corner doubling to the back,

\* Alluding to the famous motto  $\epsilon\delta\iota\sigma\alpha\gamma\epsilon\omega\mu$ , &c. which Plato set over the door of his academy.

The crease must be a minima;  
 But mind you use no algebra:  
 Ye fair who solve each arduous prize,  
 Assist us now, or Strephon dies.

*Answered by Amicus.*

Draw  $ms \parallel$  to the edge  $AC$  bisecting the end  $CE$  of the leaf in  $s$ , bisect  $SE$  in  $B$ , on  $BC$  (diameter) describe a semicircle cutting  $ms$  in  $M$ , draw  $BMA$  which will be the crease required. For  $MC = MD$  is  $\perp AB$ , therefore  $c$  will be doubled to  $D$ : draw  $ME \parallel$  to  $CE$ ,  $EX$  to  $AB$ ,  $XZ$  to  $AC$ , and  $ZV$  to  $AB$ ; then, by similar  $\Delta s$ ,  $AB : RM = CS : RM : ZV$ , and because  $RM$  is given,  $AB$  will be a minimum when  $ZV$  is a maximum;  $RM^2 : MX^2 = \text{rectangle } RMZ : ZX^2 = \text{rectangle } BZM : ZV^2$ , which is therefore a maximum when the solid  $ZR \cdot ZM^2$  is so, that is, by Theor. 17 of Simpson on Maxima, when  $ZM = 2ZR$ , and consequently  $SC = 2SB$  as by construction.



Nearly in the same manner is the solution given by Mr. *George Sanderson*, who also adds this Note: Because  $AB = 3MB$ , therefore  $AC^2 = 2BC^2$ ; but  $BC = \frac{1}{2}EC$ ; consequently if  $IC$  be less than  $\frac{1}{2}EC\sqrt{2}$ , or  $8IC^2$  less than  $9EC^2$ , then  $KS$  will be the least crease, or the front edge will coincide with the back.

*The same Analytically by Mr. John Gough, of Kendal.*

It is evident that  $CMN$  is bisected at right angles by the crease  $AMB$ , and if from  $s$ , the middle of  $EC$ , a perpendicular  $SK$  be drawn, it will always pass through  $M$ : then the equiangular  $\Delta s$   $BSM$ ,  $BCA$  give  $BS^2 : BC^2 :: BM^2 = BS \times BC : BA^2 = BC^2 \div BS = x^2 \div (x - \frac{1}{2}a)$ , putting  $EC = a$ , and  $BC = x$ . This in fluxions and reduced, gives  $x = \frac{1}{2}a$ .

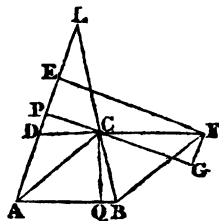
XI. QUESTION 828, by Mr. John Bonnycastle.

It is required to give a geometrical demonstration of the common rule for finding the area of a trapezium inscribed in a circle, from the four sides being given, as the demonstrations that I have hitherto seen cannot be called geometrical.

*Answered by Amicus.*

If there be any where taken a right line  $P =$  the perimeter of the trapezium  $ABCD$ , another  $Q =$  a mean proportional to  $P - 2AD$  and  $P - 2BC$ , and a third right line  $R =$  one to  $P - 2AB$  and  $P - 2CD$ , the rule geometrically expressed is, that a rectangle under  $Q$  and  $R$  is equal to 4 times the area of the trapezium. But  $P - 2BC = AB + BC + AD - DC =$  the sum of  $AD - DC$  and  $AB + BC$ , in like manner  $P - 2AD =$  the difference of  $AB + BC$  and  $AD - DC$ . By Simp. Geom.

2, 6, and 7, the rectangle under  $P - 2DC$  and  $P - 2AD =$  the difference of the squares on  $AB + BC$  and  $AD - DC$   
 $= AB^2 + BC^2 - AD^2 - DC^2 + 2AB \cdot BC + 2AD \cdot DC = q^2$ ; and in like manner it appears that  
 $r^2 = 2AB \cdot BC + 2AD \cdot DC - AB^2 - BC^2 + AD^2 + DC^2$ . Draw  $AC$  and  $BF \parallel$  to it and meeting  $DC$  produced in  $F$ , draw also  $CQ \perp AB$ ,  $FE$  and  $GCF \perp AD$  produced, and  $FG \parallel AE$ . Then the  $\triangle s$   $ABC$ ,  $AFC$  being between the same parallels, are equal, and  $\triangle AFD =$  the trapezium, which being in a circle, the  $\angle BCF = \angle DAB$ ,  $\angle DFB = \angle DCA = \angle DBA$ , and the  $\triangle s$   $DAB$ ,  $BCF$  similar, therefore  $2AD \cdot CF = 2AB \cdot BC$ , and  $2AB \cdot BC + 2AD \cdot DC = 2AD \cdot DF$ , moreover  $\angle CFG = \angle FDC = \angle QBC$ , and therefore  $\triangle s$   $CQB$ ,  $CGF$  similar, and  $QB : GF :: CB : CF :: AD : AB$ , or  $AB \cdot QB = GF \cdot AD$ : but  $AC^2 = AD^2 + DC^2 + 2AD \cdot DF = AB^2 + BC^2 - 2AB \cdot BQ$  ( $2AD \cdot GF$ ), and by equal addition  $AB^2 + BC^2 - AD^2 - DC^2 = 2AD \cdot (DF + GF) = 2AD \cdot DE$ , consequently  $2AD \cdot DE + 2AD \cdot DF = q^2$ , and  $2AD \cdot (DF - DE) = r^2$ , consequently the rectangle  $q \cdot r =$  one under  $2AD$  and a mean proportional to  $DF + DE$  and  $DF - DE$ , but that mean proportional is  $= EF$ , wherefore rectangle  $q \cdot r = 2AD \cdot EF = 4 \triangle ADF = 4$  trapezium.



*Otherwise.* Produce  $AD$ ,  $BC$  till they meet in  $L$ , then the  $\triangle s$   $DLC$ ,  $BLA$  being similar, are in the ratio of  $DC^2$  to  $AB^2$ , and by division  $DC^2 : AB^2 - DC^2 :: \triangle DLC : \text{trapezium } ABCD$ . Also

$AB : CD :: \begin{cases} AL + BL : CL + DL \\ AL - BL : CL - DL \end{cases}$ ,  $AB + DC : DC :: AD + BC : CL + DL$   
 $AB - DC : DC :: \begin{cases} AB + BC + AD - DC : CL + DL + DC \\ BC + AD + DC - AB : CL + DL - DC \end{cases}$  where the

$AB + DC : DC :: \begin{cases} AB + DC + AD - BC : DC + CL - DL \\ AB + DC + BC - AD : DC + DL - CL \end{cases}$  factum un-  
 der the 4 last consequences giving the known rule for 4 times the  $\triangle DLC$ , and that under their antecedents the rule for 4  $\times$  trapezium, and the one rule: the other  $:: DC^2 : AB^2 - DC^2 :: \triangle : \text{trap.}$  their truth is manifest.

### XII. QUESTION 829, by Plus Minus.

How many ways is it possible to lay a pavement (that is, to complete a space round a point) with regular polygons?

*Answered by Plus Minus, the Proposer.*

*Lemma.* There cannot be more than 6 polygons, nor fewer than 3 used at once, to complete the space round a point. Not more than 6, because 6 angles of the triangle (the smallest angle of any polygon) are equal to 4 right ones. Not fewer than 3, because one angle of any polygon is less than 2 right ones. Nor can there be more than 3 sorts of polygons used at once; because the 3 sorts whose angles are smallest, when one of each are added together, make  $60^\circ + 90^\circ$

$4 \cdot 108^\circ = 256^\circ$ , and if to this you add the next greater ( $120^\circ$ ) the sum will be  $378^\circ$ , which is greater than 4 right angles.

This premised, let  $x$ ,  $y$ , and  $z$ , be the number of sides in the 3 sorts; then will  $\frac{2x-4}{x} + \frac{2y-4}{y} + \frac{2z-4}{z}$  be the number of right angles in the sum of one angle of each sort; and this, if we use only 3 polygons, must be equal to 4. Hence  $x = 2yz \div (yz - 2x - 2y)$ ; but this, by the nature of the question, must be a whole number. Now if we suppose that  $z = 3$ , we shall have

$\{ x = 42, 24, 18, 15, 12, 10, 9, 8, \text{ or } 7, \}$ ; but if we suppose  $y = 7, 8, 9, 10, 12, 15, 18, 24, \text{ or } 42, \}$ ; but if we suppose  $z = 4$ , we shall have  $x = 20, 12, 8, 6, \text{ or } 5$ ;  $y = 5, 6, 8, 12, \text{ or } 20$ . If  $z = 5$ , we shall have  $x = 20, 10, 5, \text{ or } 4$ ;  $y = 4, 5, 10, \text{ or } 20$ . Lastly if  $z = 6$ , we have  $x = 12, 6, \text{ or } 4$ ;  $y = 4, 6, \text{ or } 12$ .

If we use 4 polygons, there must be two of one of the sorts; let that be of the sort  $z$ . Then  $\frac{2x-4}{x} + \frac{2y-4}{y} + \frac{4z-8}{z} = 4$ ; hence  $x = yz \div (yz - z - 2y)$ , a whole number; Where if  $z = 3$ , we have  $x = 4, 6, 12$ ;  $y = 12, 6, 4$ : But if  $z = 4$ , then  $x = 3, 4, 6$ ;  $y = 6, 4, 3$ .

If 5 polygons be used, there must be either 2 of two sorts, and one the other; or else 3 of one sort, and one of each of the other. In the

first case let  $\frac{4x-8}{x} + \frac{4y-8}{y} + \frac{2z-4}{z} = 4$ ; so shall  $x =$

$\frac{4yz}{3yz - 4x - 2y}$  a whole number; and if  $z = 3$ , then  $x = 3 \text{ or } 4$ ,

and  $y = 4 \text{ or } 3$ . In the 2d case let  $\frac{6z-12}{z} + \frac{2x-4}{x} + \frac{2y-4}{y}$

$= 4$ ; hence  $x = \frac{2yz}{3yz - 2x - 6y}$  a whole number, and if  $z = 3$ ,  $x = 3, 4, 6$ ;  $y = 6, 4, 3$ .

If 6 polygons be used, they must be all triangles. So that a pavement may be laid 10 ways with 3 polygons, thus, 3, 7, 42; 3, 8, 24; 3, 9, 18; 3, 10, 15; 3, 12, 12; 4, 5, 20; 4, 6, 12; 4, 8, 8; 5, 5, 10; 6, 6, 6.

With 4 polygons 4 ways, thus, 3, 3, 4, 12; 3, 3, 6, 6; 3, 4, 4, 6; 4, 4, 4, 4.

With 5 polygons 2 ways, thus, 3, 3, 3, 4, 5; 3, 3, 3, 3, 6.

With 6 polygons one way, viz. all triangles. So that there are 17 ways of laying a pavement with regular polygons; but without regard to the order in which they may be placed.

### XIII. QUESTION 830, by Mr. George Sanderson.

Given the vertical angle, and the sum of the base and perpendicular, to construct the triangle when the sum or difference of the sides is a maximum.



Simp. Trig.  $AE \times Ai = AG^2$ ; and, by similar  $\Delta s$ ,  $qh^2 : qh \times hk :: AE \times Ai (AG^2) : qA \times AK$ , and therefore  $qA$  may be taken of any length less than  $hq$ ,  $AG$  than  $hf$ , which is its greatest limit, because  $qK$  is a given quantity; and when  $Aq = hq$ ,  $Ai$  is  $= AE = hf$ , and consequently the  $\perp FE = 0$ .

*Analytical Solution, by Mr. Thomas Robinson, of Biddick.*

Put  $s$  = sum of base and perpendicular,  $c$  = cotangent of half the vertical  $\angle$ ,  $z$  = sum or difference of the sides, and  $x$  = the base. Then  $s - x$  = the perpendicular, and by prop. 13 or 14 Simp. Trig.  $2sx - 2x^2 : (z + x)(z \oslash x) :: 1 : c$ ; hence  $x^2 \oslash x^2 = 2csx - 2cx^2$ , and  $x^2 = 2csx \oslash 2cx^2 + x^2$  a maximum, which in fluxions, and reduced, gives  $x = cs \div (2c \oslash 1)$ , viz.  $x = cs \div (2c - 1)$  or  $= cs \div (1 - 2c)$  according as the sum or difference of the sides is a maximum.

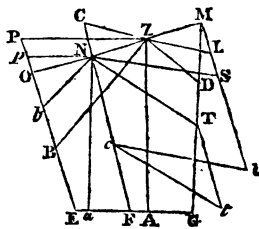
XIV. QUESTION 831, by Mr. Isaac Dalby.

Having four right lines given in position; to find a point from which if four other right lines are drawn to meet the former in given angles, the sum of two of them, and the rectangle of the other two, may be respectively given: A geometrical construction is required?

The above is prob. 12, Newton's Arith. edit. 1720. Or prob. 23 of the later editions.

*Answered by Amicus.*

*Construction.* From any point  $c$  in one of the lines  $CF, GD, EG, BE$ , given in position, as  $CF$ , drawn  $ct, cl$  each = the given sum, and making the required angles with  $CF, DG$ ;  $\parallel$  to  $CF$  draw  $tr, lm$  meeting  $DE$  in  $T$  and  $M$ ; draw  $NT \parallel ct$  meeting  $CF$  in  $N$ , through which draw  $mo$ ; draw  $nb, na$  making the required angles with  $BE, EG$ ; also draw  $np \parallel EG$ ; by Simp. Geom. 5, 18, produce the given line  $EO$  to  $P$  till the rectangle  $OPE$ : the given one, in the given ratio of rectangle  $OPE$ : rectangle  $anb$ , and through  $P \parallel$  to  $EG$  draw  $pz$  meeting  $OM$  in  $z$ , the point required.

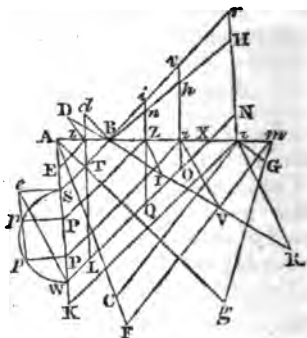


For draw  $ZA \parallel na, ZD \parallel ct, ZC \parallel cl, ZB \parallel nb$ , and they make the given angles by construction; draw  $NS \parallel cl$ , and consequently equal to it, being in the same parallels  $NC, ML$ , and for the same reason, continuing  $CZ$  to  $L, ZL = ZD$  because  $NS = NT$ ; therefore  $CZ + ZD = cl$  the given sum. Moreover, by similar  $\Delta s$ ,  $op : dn :: of : bz$ , and  $pe : na :: fe : za$ , therefore compoundedly rectangle  $OPE$ :  $bna :: OPE$ :  $BZA$  the given rect. by construction.

*Scholium.* If the diff. or the ratio, of  $ZD, ZC$ , or *plus minusve quam in ratione*, be given, the locus of  $z$  will still be a right line, and the construction very little different.

*The same, by Mr. Isaac Dalby, the Proposer.*

Imagine the thing done, and let  $AC$ ,  $DR$ ,  $NL$ ,  $mg$  be the four lines given in position, and  $z$  the point required from which  $ZN$ ,  $ZG$ ,  $ZR$ , and  $ZC$  are drawn to make given angles with the lines given, then  $ZG + ZC$ , and  $ZR \times ZN$  are given. Take  $mF$ ,  $Ag$  each equal to  $ZG + ZC$ , and draw  $mF$ ,  $Ag \parallel ZC$ ,  $ZG$  respectively, then if  $mA$  be drawn it will pass through the point  $z$ ; For, by similar  $\Delta s$ ,  $Am : Ag :: zm : ZG$ , and  $Am : mF :: ZA : ZC$ , and because  $mF = Ag$ ,  $zm : ZG :: ZA : ZC$ , that is  $zm : ZA :: ZG : ZC$ , and, by compos.  $zm + ZA (Am) : ZG + ZC :: ZA : ZC :: Am : mF$ , hence by equality  $ZG + ZC = mF = Ag$ , therefore  $mA$  is the locus of the point  $z$ . In  $ZN$  produced take  $zr = ZR$ , and draw  $Br$ , then  $ZN \times zr (ZR)$  is given in magnitude; draw  $BH \parallel NL$ , then  $NH$  is given; and because  $ZN \times zr$  is given, and  $zr, zH$  are in a given ratio, the rectangle  $zH \times ZN$  is given in magnitude, for  $zr : zH :: ZN \times zr : ZN \times zH$ . Supposing  $z$  to fall between the points of intersection  $x, B$ , then drawing  $zv, zo \parallel ZR, ZN$  respectively, and producing  $oz$  to  $v$ , it follows by similar  $\Delta s$  that  $zv = zv$ ; and because  $oz \times zv (zv)$  is given in mag. therefore  $zh \times zo$  is given in mag. for  $zv : zh :: oz \times zv : oz \times zh$ . In like manner if  $z$  be taken on the other side of  $B$ , and  $ZL, ZD$  drawn  $\parallel ZN, ZR$  respectively, and in  $ZL$  produced,  $zd$  be taken  $= ZD$ , and  $hd$  be drawn, because of the similar  $\Delta s$   $ZBT, Zbh, Zbd, Zbv$  ( $HB$  being produced) and  $ZD \times ZL$  being given in mag. the rectangle  $ZL \times ZT$  is given in mag. for  $zd : ZT :: ZL \times zd (ZD) : ZL \times ZT$ ; therefore the rectangles  $ZL \times ZT, zh \times zo, ZH \times ZN$  are given in magnitude and equal to each other; hence this

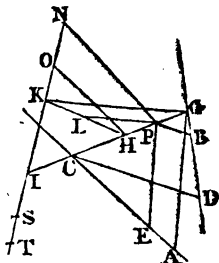


**Construction.** Having drawn the locus  $Am$ , through  $B$  draw  $HS \parallel NL$ , and  $AW \parallel ZN$ , upon  $sw$  describe a semicircle,  $\perp$  to  $sw$  take  $se$  the side of a square  $= ZL \times ZT$ , draw  $ep \parallel AW$ , and from  $p, p$ , where it cuts the semicircle, draw  $pp, pp \parallel es$ , join  $ew$ , and take  $ew, sk$  each  $= ew$ , then draw  $kz, pz, pz, ez \parallel NL$ , and  $z, z, z, z$  will be four points answering the conditions of the problem. For drawing  $ZD, ZI, ZV, ZR \parallel ZR, ZV, ZR$ ; and  $EW \times ES (ZL \times ZT) = sk \times wk (zh \times zn) = es^2 = pp^2 = sp \times pw (nz \times zq) = wp \times ps (oz \times zh)$  therefore the construction is manifest.

Here it is evident that there will be four cases when  $ep$  cuts the semicircle, 3 when it touches it, and only 2 when it falls without, because then  $z$  will not fall in  $ax$ ; and when  $ew$  is greater than  $Aw$ , the problem is impossible. If the difference of  $zc, zc$ , instead of the sum, was given, the construction will be similar to the above, because in that case the locus of  $z$  is a right line.

*Another Answer to the same by the Rev. Mr. Bownas.*

Let  $GD$ ,  $CA$  be the given lines which those are to meet whose sum is given; apply  $GA$ ,  $CD$  each equal the said sum, and meeting  $CA$ ,  $GD$  at  $A$  and  $D$  in the given angles respectively; join  $CG$ : Also make the  $\angle$ s  $GKH$ ,  $HOI$  respectively equal to those, which the two lines, whose rectangle is given, are to make with the other two given lines  $HL$ ,  $NI$ . Take  $GH : GK :: IS : Q$  the side of a square equal to the given rectangle, and  $HI : HO :: IT : Q$ ; then determining the point  $P$  so that the rectangle  $PI \cdot PH =$  the rectangle  $IS \cdot IT$ , it will be the point sought.



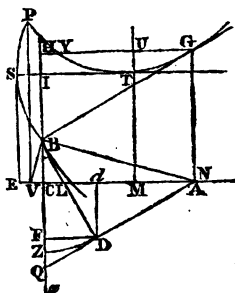
For drawing the lines as required, by similar  $\triangle$ s  $PH : PL :: GH : GK ::$  (by construction)  $IS : Q$ , and  $PI : PN :: HI : HO ::$  (by construction)  $IT : Q$ , hence by mult.  $PH \cdot PI : PL \cdot PN :: IS \cdot IT : Q^2$ , but the antecedents are equal by construction, therefore the consequents must be so too, that is  $PL \cdot PN = Q^2$ . Again, by similar  $\triangle$ s  $CD : CG :: PB : PG$ , and  $AG = CD : CG :: PE : PC$ , therefore  $PB : PE :: PG : PC$ , and by compos.  $PB + PE : PE :: PG + PC = CG : PC :: AG : PE$ , consequently  $PB + PE = AG$ .

## PRIZE QUESTION, by Amicus.

If two bodies  $A$  and  $B$ , connected by a string or otherwise, at the same invariable distance from each other, move, the one  $A$  along a given right line with a given uniform celerity, the other  $B$  so, that its velocity in the direction of the connecting line  $AB$ , may always be equal to that in a direction perpendicular to it. I demand the asymptote, equation, quadrature, and rectification of the path of  $B$ , its centre of curvature, and the quadrature of the path of that centre.

*Answered by Amicus, the Proposer.*

Suppose, when the body  $A$  is at  $A$  moving with an uniform celerity along the right line  $EN$ , that the other is at  $B$ ; draw  $BV \perp BA$ , then because, by the question the velocity of  $B$  in direction  $BV =$  that in direction  $BA$ , the line  $BL$  bisecting the  $\angle ABV$  must be a tangent to the path of the body at  $B$ . Draw  $AD \perp BL$  produced, and meeting  $BC$  ( $\perp$  to  $EN$ ) produced in  $Q$ ; draw  $FD \parallel CA$ , and  $dd$  to  $BQ$ , and with  $BD$  radius describe the arc  $DZ$ . Let  $P$  be the place of  $B$  when it is at rest, the distance thereof from  $EN$  being then  $= AB$ . Make the square  $ESTM$  whose diagonal  $MS = AB$ ,  $s$  being the place of  $B$



when A is at M, and consequently ES a tangent to the path at s. Let PYTG be the evolute, BG ( $\perp$  DB) the radius of curvature at B, and let CB produced cut ST in I and GH,  $\parallel$  to ST, in H. Then will  $SI = EC = x$ ,  $ID = y = IF - BF$ ,  $AB = a$ ,  $BD = AD = YM = IC = a\sqrt{\frac{1}{2}} = c$ ,  $FD = cd = u$ , and  $\angle dAD = FBD$ , therefore  $dA = BF = w$ , and  $vd = FD = CF = u$ ,  $IF = c + u$ ,  $y = c + u - w$ , curve  $SB = x$ ; then by similar  $\Delta s w : u :: \dot{y} = \dot{u} - \dot{w} : \dot{x}$ . and  $u^2 + w^2 = c^2$ ,  $u\dot{u} = -w\dot{w}$ , whence  $\dot{x} = -\dot{w} - \dot{u} + (c^2 \div w^3)\dot{u}$ , and the fluent corrected by considering that at s,  $u = 0$ , and  $w = c$ , gives  $x = c - w - u + \frac{1}{2}CL$ , where  $L = \text{hyp. log. of } ((c + u) \div (c - u))$  which becomes infinite when  $u = c$ , therefore producing  $cx$  till  $ca = BD$ , a line drawn through  $a \parallel$  to  $EN$  will be the asymptote of the path. Moreover the fluxion of the area  $SBI = cx + ux - wx = cx - 2u\dot{u} - (c^2 \div w)\dot{w}$ , and  $cx - u^2 + c^2$  h. l. of  $(c \div w)$  is the quadrature of  $SBI$ . Again  $u : c :: \dot{x} : \dot{z} = (c \div u)\dot{x} = (c \div w)(\dot{u} - \dot{w})$ , hence  $z = zD + c$  h. l. of  $(c \div w) = SB$ . And then by the known rules we shall obtain the radius of curvature  $BG = (c \div w)(w + u)$ , or  $BF : BD :: CA : BG$ , therefore by similar  $\Delta s BG = AQ$ , but they are both perpendicular to  $BD$ , and consequently parallel, therefore  $CA = BG$ , and letting fall  $GN$ , the points A and N coincide, and  $GN = HC = BQ = (c^2 \div w)$ ,  $MA = x + w + u - c = \frac{1}{2}CL$ , its fluxion  $= (c^2 \div w^3)\dot{u}$ , that of the quadrature of  $TGAM = \frac{c^4\dot{u}}{w^3} = \frac{c^4}{w^2} \cdot \frac{u^2\dot{u} + w^2\dot{u}}{w} = \frac{c^4}{w^2}(w\dot{u} - u\dot{w})$ , fluent  $= \frac{c^2u}{w} = BQ \cdot FD = CH \cdot cd$  the quadrature  $TGAM$  of the evolute. But when B falls in the curve  $SP$ ,  $BI = BF - IF$ ,  $IF = IC - FC$ , therefore  $BI = w + u - c$ ,  $\dot{y} = \dot{w} + \dot{u}$ ,  $\dot{x} = (u \div w) \cdot (\dot{w} + \dot{u})$ ,  $\dot{z} = (c \div w) \cdot (\dot{w} + \dot{u})$ , radius of curvature  $= (c \div w) \cdot (w - u)$ , or a fourth proportional to  $BF$ ,  $BD$ , and  $w - u$ , but here  $w - u = AC$ ; and a perpendicular being let fall from  $x$ , it will be found that the quadrature cut off by it  $= CH \cdot cd$  as before, and consequently  $MT$  produced bisects  $YC$  in  $U$ . Moreover in  $SP$  it will be found that  $x = c + u - w - \frac{1}{2}CL$ ,  $z = zD - c$  h. l. of  $(c \div w)$ , and quadrature  $= u^2 - c^2$  h. l. of  $(c \div w) - cx$ , as required.

*Scholium.* If the velocities of B in directions EA and BV be in any other constant ratio, the  $\angle ABD$  will still be constant, and so will, therefore, the ratio of  $BD = c$  to  $DA$ , let  $1 : n :: c : DA :: BF = w : dA = nw :: FD = cd = u : dD = FC = nu$ ,  $BI = nu \mp w \pm c$ ,  $x = n(c - w) \mp u \pm \frac{1}{2}CL$ ,  $z = n \cdot zD \pm c$  h. l. of  $(c \div w)$ , quadrature  $= (n^2 - 1)A \pm cx \mp nu^2 \pm nc^2 \cdot X$  h. l. of  $(c \div w)$ ; radius of curvature  $= (c \div w) \cdot (nw \pm u)$ , and quadrature of the evolute  $= c^2u \div w$ . Where A is the area of the circular segment  $FDZ$ , and the lower sign takes place only between s and P. And when  $n = 0$ , the curve is that to which AB is the constant tangent.

N. B.—Only part of the curve above ST is drawn, the other part which forms with  $SP$  a cusp at P and branches out the other way towards ST produced as an asymptote, being omitted for want of room, and to avoid confusion in the scheme: nor does any new difficulty hence arise, for the values of the abscissa, radius of curvature,

&c. are precisely the same as in the portion *sp*, only with their signs changed, because the ordinate *is* beyond *p* continually diminishes. The curve can on no supposition have more than two infinite branches, and never can have any centre whatever.

*Questions proposed in 1785, and answered in 1786.*

I. QUESTION 833, by Mr. Thomas Leigh.

To find the diameter of a balloon, being a spherical shell of copper, one hundredth of an inch thick, which shall just float in air, when filled with inflammable air 10 times lighter than common air; Supposing copper to be 9 times heavier, and air 800 times lighter than water.

*Answered by Amicus.*

Copper, by the question, is 7200 times heavier than air, consequently the solidity of the globe =  $7200 \times \text{shell} + \frac{1}{10} \times \text{concavity}$  =  $7200 \text{ globe} - 7199.9 \text{ concavity}$ , or  $7199 \text{ globe} = 7199.9 \text{ concavity}$ ; therefore  $\sqrt[3]{7199} : \sqrt[3]{7199.9} :: \text{globe's radius minus } .01 : \text{globe's radius}$ ; and  $\sqrt[3]{7199.9} - \sqrt[3]{7199} : \sqrt[3]{7199} :: .01 : \text{radius of the concavity} = 241.36 \text{ inches} = 20\frac{1}{2} \text{ feet nearly.}$

*The same answered by Mr. James Williams.*

Put  $.02 = a$ , and  $.5236 = b$ , also the external diameter of the balloon =  $x$ ; then its internal diameter is =  $x - a$ , its whole solidity =  $bx^3$ , and its internal capacity =  $(x - a)^3 \times b$ ; consequently the solidity of the shell =  $(x^3 - (x - a)^3)b$ . And since the proportional weights of the two airs and copper are as the numbers 1 and 10 (*c*) and 72000 (*d*); therefore that the balloon may just float, we have  $(x - a)^3 + (x^3 - (x - a)^3) d = cx^3$ , or  $(x - a)^3 (d - 1) = x^3 (d - c)$ ; hence  $(x - a) \sqrt[3]{d - 1} = x \sqrt[3]{d - c}$ , or  $x \sqrt[3]{71999} - x \sqrt[3]{71990} = a \sqrt[3]{71999}$ . From which  $x = 480.0419 \text{ inches} = 40 \text{ feet nearly.}$

II. QUESTION 834, by Mr. Alex. Rowe, of Reginnis.

Mongolfier, with his air-balloon,  
Steering between the earth and moon,  
Observes with wonder, as he flies,  
The setting sun appear to rise;  
And sees him take his evening's nap  
In gentle Thetis' watery lap,

A quarter \* after lying fame  
 Had seen him kiss the buxom dame.  
 Tell me, ye Philomaths profound,  
 How high above earth's spacious round,  
 The daring artist took his flight,  
 Soaring beyond frail mortal's sight,  
 Far in the liquid fields above,  
 And rivalling the bird of Jove ;  
 Or him, whose fate, as legends fain,  
 Gave name to the Icarian main.

\* Of an hour. This happened at Paris in latitude  $48^{\circ} 50' 0''$ , on April 1, the sun's declination  $5^{\circ} 0'$ . The radius of the earth is supposed 4000 miles.

*Answered by Mr. Isaac Darby.*

To the time of sun-setting at Paris 6h. 22m. 58s. 8th. add 15m. the sum is 6h. 37m. 58s. 8th. answering to  $99^{\circ} 29' 32''$ . Now to find the sun's zenith distance ; in a spherical triangle there are given two sides =  $85^{\circ}$  and  $41^{\circ} 10'$ , (namely the sun's polar distance and the co-latitude) and the included angle =  $99^{\circ} 29' 32''$  ; whence the third side or zenith distance is =  $92^{\circ} 26' 16''$ , and consequently the sun's depression below the horizon was =  $2^{\circ} 26' 16''$ , or the distance from Paris on the arc of a great circle where a tangent drawn from the sun to the balloon would touch the surface of the earth. Hence if that point of contact and the balloon and the centre of the earth be joined by three lines, there will be formed a right-angled triangle ; in which are given a leg = 4000 miles, and the angle at the centre =  $2^{\circ} 26' 16''$  ; therefore as  $\cos. 2^{\circ} 26' 16'' : \text{radius} :: 4000 : 4003 \cdot 624$  the hypotenuse ; or distance of the balloon from the earth's centre ; and consequently 3.624 miles is the height of Mons. Mongolfier above the earth's surface.

### III. QUESTION 835, by Mr. L. Evans.

On a certain day in the spring quarter, 1782, and in North latitude, I found the difference of the latitude of the place and the meridian altitude of the sun, to be  $14^{\circ} 29' 12'' \cdot 6$  ; and the difference of the said latitude and his altitude at 6, to be  $34^{\circ} 30' 47'' \cdot 4$ . When and where was this observation made ?

*Answered by Mr. John Jackson, of Hutton-Rudby School.*

Let  $x$  = the latitude of the place in degrees, &c. also  $a = 14^{\circ} 29' 12'' \cdot 6$ , and  $b = 34^{\circ} 30' 47'' \cdot 4$  ; then  $x + a$  = sun's altitude at 12, and  $x - b$  = his altitude at 6. Then, in a right-angled spheric triangle, we have the perpendicular =  $x - b$ , and the angle opposite =  $x$ , to find the hypotenuse, which will be the declination ; thus, as

$\sin. x : \sin. (x - b) :: \text{radius } (1) : \sin. (x - b) \div \sin x = \sin. \text{ declin.}$   
 Then, by the common rule for finding the latitude, we have  $x = 90^\circ - (x + a) + \text{degrees to } \sin. (\sin. (x - b) \div \sin. x)$ , and  $2x = \text{deg.}$   
 $\sin. (\sin. (x - b) \div \sin. x) = 75^\circ 30' 47'' \cdot 4$ . This equation is very easily resolved by a table of sines; from whence we find  $x = 45^\circ 30' \text{ N.}$  the latitude of the place, and the declination  $= 15^\circ 29' 12'' \cdot 6 \text{ N.}$  answering to May the 2d.

*The same by Mr. L. Evans, of Compton-Beauchamp.*

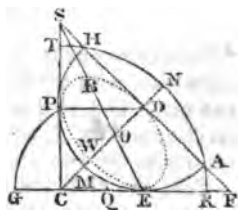
Let  $a$  and  $b = \text{sine and cosine of } 14^\circ 29' 12'' \cdot 6$ ,  $s$  and  $c = \text{sine and cos. } 34^\circ 30' 47'' \cdot 4$ , also  $x$  and  $y = \text{sine and cos. latitude required.}$   
 Then, by the rules for the sums and difference of arcs,  $bx + ay = \text{sine of the meridian altitude,}$   $by - ax = \text{its co-sine,}$   $cx - sy = \text{sine of altitude at six, and } (bx + ay)x - (by - ax)y = \text{sine of the declination.}$  Then, in the right-angled spheric triangle,  $x : cx - sy :: 1 : (bx + ay)x - (by - ax)y$ ; hence  $bx^3 + 2ax^2y - bxy^2 = cx - sy$ ; from which  $y$  being expelled, it is  $x^6 + (as - bc - 1)x^4 + (\frac{1}{4}bc + \frac{1}{4}b^2 + \frac{1}{4} - as)x^2 = \frac{1}{4}s^2$ , or in numbers  $x^6 + 1 \cdot 6560589x^4 + \cdot 7415155x^2 = \cdot 08025764$ . Hence  $x = \cdot 7134115$  the sine of  $45^\circ 30' \cdot 79$  the latitude, and the declination  $= 15^\circ 30' +$ , answering to the 2d of May.

#### IV. QUESTION 836, by Mr. Tho. Robinson, of Biddick.

In a given quadrant of a circle it is required to inscribe the greatest semicircle it will contain, and within the semicircle the greatest ellipse; and to determine the relation between the axes of the ellipse and the radii of the quadrant and semicircle.

*Answered by Mr. Isaac Dalby.*

It is evident that the centre of the greatest semicircle  $HPEA$  that can be inscribed in a quadrant  $TCR$ , will be in the line  $CN$  which bisects the  $\angle TCR$ ; and that the circumference of the semicircle will touch the radii of the quadrant; therefore if  $n$  be the centre of the semicircle, we have, by the nature of the circle,  $(CN + CD) \times (CN - CD) = DN^2$  or  $DP^2$  or  $PC^2$ , that is  $CN^2 - CD^2 = CP^2$ ; but  $CD^2 = 2CP^2$ , therefore  $CN^2 = 3CP^2$ . Hence this construction is evident:—Take  $ca = \frac{1}{3}CN$  or  $\frac{1}{3}cu$ , bisect  $ca$  in  $q$ , with centre  $q$  describe  $aq$ , draw  $PD \parallel CR$ , then  $p$  is the centre of the semicircle.

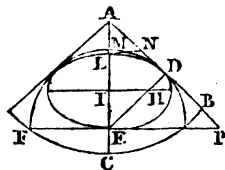


For the greatest ellipse in the semicircle. It is evident that  $p$  will be the extremity of the conjugate diameter, and that the ellipse must touch the circumference in two points. Now if  $cr$ ,  $pn$  are produced, then  $pc = ns$ , and  $pc = rs$ ; and it appears from the *Schol. theor. 8*,

*Stimp. Max. and Min.* that if  $MPHD$  be the quadrant of a circle, or  $WPD$  a semi-ellipse, they will be the greatest that can be inscribed in the  $\triangle cns$  when they bisect  $cs$  in the point of contact  $p$ ; therefore the ellipse will touch the radii in the same points  $p$  and  $x$  as the semicircle; and consequently  $sp$ ,  $sd$  are tangents to the ellipse. Hence this construction is obvious: Draw  $se$  through the middle of  $pd$  and  $cf$ ; take  $ow = on$ , and  $ob = oe$ , then about the diameters,  $be$ ,  $dw$  describe the ellipse required.

*The same by Mr. Thomas Robinson, of Biddick.*

First, putting  $r = AB$  or  $AC$  the radius of the quadrant, and  $x = EF$  or  $ED$  or  $AD$  that of the semicircle; then  $AE^2 = 2x^2$ , and  $EF^2 = AC^2 - AE^2$ , or  $x^2 = r^2 - 2x^2$ , and  $3x^2 = r^2$ , or  $x = r\sqrt{\frac{1}{3}}$  the radius of the semicircle. Again, putting  $z = AL = LN$ , by Conic Sections  $IL^2 = LN \times EP$  or  $LN \times AE = z \times r\sqrt{\frac{2}{3}}$  the square of the semitransverse axis, and  $EL^2 = (r\sqrt{\frac{2}{3}} - z)^2$  = the square of the conjugate; therefore  $z \times r\sqrt{\frac{2}{3}} \times (r\sqrt{\frac{2}{3}} - z)^2$  is a maximum, or  $\frac{2}{3}r^2z - 2rz^2\sqrt{\frac{2}{3}} + z^3$  a maximum. Its fluxion made = 0 gives  $z = \frac{1}{3}r\sqrt{\frac{2}{3}}$ . Hence  $IL = \frac{1}{3}\sqrt{2}$  the semitransverse axis, and  $IL = \frac{1}{3}r\sqrt{\frac{2}{3}}$  the semiconjugate. So that the radius of the quadrant, the radius of the semicircle, and the two semiaxes of the ellipse, are to one another as the numbers 1, and  $\sqrt{\frac{1}{3}}$ , and  $\frac{1}{3}\sqrt{2}$ , and  $\frac{1}{3}\sqrt{\frac{2}{3}}$ , which numbers are in discontinued proportion, shewing that the two radii have the same proportion as the two axes of the ellipse.



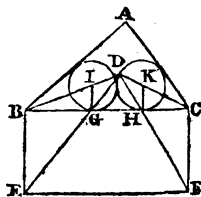
#### V. QUESTION 837, by Omicron.

A cabinet-maker having a triangular plank, wants to cut two circular tea-boards out of it, both equal, and as large as possible. Please to inform him how he must do it with rule and compasses only.

*Answered by Mr. Isaac Saul.*

\* Bisect the angles  $B$  and  $C$ , of the given triangle  $ABC$ , with the lines  $BD$  and  $CD$ ; erect  $BE$  and  $CF \perp$  and  $= \frac{1}{2}BC$ ; draw  $DG \perp$ ,  $DH \perp$ ; and erect  $GI$  and  $HK \perp BC$ , so shall  $I$  and  $K$  be the centres of the two greatest circles required.

For, the circles must evidently touch the longest side  $BC$ : also, by similar  $\triangle s$ ,  $IG : BE :: DG : DE :: GH : (EF \text{ or } ) BC$ ; and as  $BE$  is  $= \frac{1}{2}BC$ , so  $IG = \frac{1}{2}GH$ . In like manner  $HK = \frac{1}{2}GH$ .



#### VI. QUESTION 838, by X + Y.

A Sugar-loaf hangs twirling high \*  
Whose sweets attract a liquorish fly,

\* The cone equilateral, and its slant side 20 inches.

- And whilst he mounts its steep ascent,  
 - The Cone once round its axis went :  
 Tell me when all the journey's done,  
 How far my little Musca's run,  
 And as he winds his spiral way,  
 His course detect, his route display.

*Answered by the Rev. Mr. J. Hellins.*

Put the slant height of the cone =  $h$ , sine of half the vertical  $\angle = s$ ,  $3 \cdot 1416s = c$ , the number of revolutions it makes while the fly moves between the vertex and the base =  $n$ , and any variable distance from the vertex =  $x$ ; then  $h : 2cnh :: x : 2cnx$  = the fluxion of the space described by a point in the circumference of the base of the cone, corresponding to the fluxion of the space described in the vertical plane; and  $h : x :: 2cnx : 2cnh^{-1}x\dot{x}$  = the fluxion of the circumference described at the distance  $x$  from the vertex; consequently, (putting  $2cnh^{-1} = p$ ),  $x\sqrt{1 + ppx} =$  fluxion of the path of the fly upon the cone, the correct fluent of which is  $\frac{1}{2}x\sqrt{1 + p^2x^2} + (1 \div 2p) \times \text{hyp. log. } px + \sqrt{1 + p^2x^2}$ , which, when  $x = h = 20$ ,  $s = \frac{1}{2}$ , and  $n = 1$ , gives 38.897 for the length required.

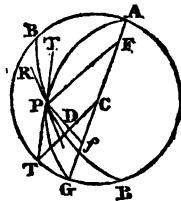
*Amicus*, and *Mr. Dalby*, shew that the curve in question is *Archimedes's spiral*, when the surface of the cone is spread out in a plane, and thence, from the known rectification of that curve, they find the same length as above.

VII. QUESTION 839, by *Mr. J. Turner*.

To describe, geometrically, the representation of a great circle, in the orthographic projection, which shall make a given angle at a given point with another great circle already projected.

*Answered by Mr. I. Dalby.*

Let  $BPB$  be the given great circle or ellipse,  $P$  the given point, and  $PR$  a tangent to it at that point; make the  $\angle RPT =$  given  $\angle$ , from the centre  $c$  draw  $CT$ , and  $PF \parallel CT$ , which, by the property of the ellipse, will pass through one of the foci of the required ellipse; make the  $\angle TRF = \angle TPF$ , then  $RF$  will pass through the other; take  $DF = DP$ ; through  $f$  and  $c$  draw the diameter  $GA$ , then by similar  $\Delta s$ ,  $cf = CF$ , and consequently the points  $f, F$  will be the foci, and  $GA$  the transverse diameter of an ellipse  $APG$  touching  $TR$  in  $P$ , and making the given  $\angle (RPT)$  with  $(BPB)$  the other ellipse. Which is too well known to need a demonstration.

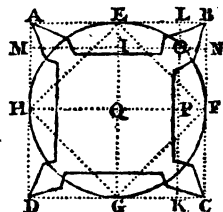


## VIII. QUESTION 840, by Lieut. Glenie, of the Engineers.

An officer one day asking another, what was the figure of a certain work, which was the subject of conversation, received for answer; that it's figure was such, that supposing a circle to be inscribed touching the exterior sides, from which the different parts of it were set off, and from any point in the circumference of the said circle perpendiculars be drawn to these sides, the squares on these perpendiculars would together be always equal to thrice the square inscribed in the said circle. Required the figure of the fortification?

*Answered by Lieut. James Glenie.*

I say the work is a square. For let ABCD be a square, from the sides of which any such regular work is set off and constructed; and from any point o in the circumference of the inscribed circle let perpendiculars OL, OK, OM, ON, be drawn to the said sides. Then (Eu. 4, 2,) the squares on OL, OK are together equal to the excess of the square on LX or AB above twice the rectangle under OL and OK, or IE and IG, or the excess of the square on AB above twice the square on IO or QR. In like manner the squares on OM, ON are together equal to the excess of the square on AB above twice the square on OP or IQ. Wherefore the squares on OL, OK, OM, ON are together equal to the excess of twice the square on AB or eight times the square on EQ above twice the squares on IO and IQ, or twice the square on EQ, that is, six times the square on EQ, or thrice the inscribed square EFGH.



*The same answered by Amicus.*

By Dr. Stewart's General Theorems, prop. 5, twice the sum of the squares of the perpendiculars to the sides of the figure, will be equal to the multiple of the square of the radius of the circle by the number of sides of the figure, together with twice the multiple of the said square of radius by the same number. But the inscribed square = twice the square of radius; therefore twice the sum of the squares of the perpendiculars = the number of sides  $\times \frac{3}{2}$  the inscribed square; and the half of this by the question or the number of sides  $\times \frac{3}{4}$  the inscribed square = 3 times the inscribed square; consequently the number of sides is 4, and the figure of the fortification a square.

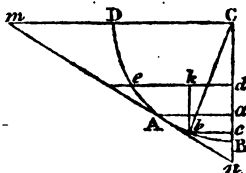
## IX. QUESTION 841, by the Rev. Mr. John Hellins.

Given the weight of a ball of lead, and the length of a thread by

which it is suspended, together with the arch it describes in vibrating ; to find the greatest horizontal force with which it acts on the centre of suspension.

*Answered by the Rev. Mr. Hellins, the Proposer.*

By the force on the centre, I mean both that which arises from the centrifugal force, and from the weight of the ball conjointly. Let  $DABC$  be the quadrant of a circle whose centre is  $c$  ; and let  $Adb$  be half the arc described by the centre of oscillation. Put  $32\frac{1}{2} = f$ , the weight of the ball  $= w$  ; and let  $b$  be the point required, drawing  $Aa$  and  $bc \perp cb$ . Then  $\sqrt{\frac{1}{2}f} : \sqrt{ac} :: f : \sqrt{ac \times 2f}$  = the velocity of the centre of oscillation at  $b$  ; from which the centrifugal force at that point, is found to be to gravity, as  $ac \times 2f \div cb$  to  $f$ , or as  $2ac \div cb$  to 1 ; and therefore  $w \times 2ac \div cb$  will express that tension of the thread which arises from the centrifugal force only. And if to this we add that tension which arises from gravity, or  $w \times cc \div cb$ , we shall have  $w \times (2ac + cc) \div cb$  = the whole force by which the thread is stretched ; the effect of which in a horizontal direction is evidently  $= w \times bc \times (2ac + cc) \div cb^2$  ; which expression, because  $w$  and  $cb$  are constant, is always as  $(2ac + cc) \times bc$  ; or, by writing  $(ca + ac)$  for its equal  $cc$ , we have  $(3ac + ca) \times bc$ , and  $(ac + \frac{1}{3}ca) \times bc$  a parallelogram to which the horizontal force is always proportionate. Take  $ad = \frac{1}{3}ca$ , and draw  $de \parallel Aa$ , and  $bk \parallel bc$ , then the parallelogram  $kc$  will always be proportionate to the horizontal force. The problem then comes to this : In a given segment of a circle to inscribe the greatest parallelogram ; which has been done in several places. See Simpson on the Maxima and Minima.



*Remark 1.* It is evident that, in the above construction, the point  $b$  may coincide with  $A$ , or even ascend above it. To determine the arc  $AB$  where  $b$  coincides with  $A$ , draw  $mn$  to touch the circle in  $A$ , and intersect  $cb$  and  $cb$  produced in the points  $m$  and  $n$ . Then, since in this case  $an = ad = \frac{1}{3}ca$ , by similar  $\Delta$ s the tangent  $an = \frac{1}{3}$  the cotangent  $am$  ; and the arc  $AB$  comes out  $= 30^\circ$ . And hence it follows that, if the arc  $AB$  be either equal to, or less than  $30^\circ$ , the greatest horizontal force is at  $A$ , and is expressed by  $w \times ca \times Aa \div cb^2$ .

*Remark 2.* If the arc  $AB$  be very small,  $ca$  will be nearly  $= cb$ , and the greatest horizontal force will be nearly expressed by  $w \times Aa \div cb$ . *Example.* If the weight of the gridiron pendulum of an astronomical clock be 20lb. and half the arc of its vibration  $2^\circ$ , the greatest horizontal force with which it acts on the centre will be  $20 \times 0.035 = 0.7$ lb. or 11.2 ounces nearly.

## X. QUESTION 842, by Amicus.

In the equation  $(y^3 + x^3 + a^3) \times x^2y - (2a^3 - 2x^3) \times y^3 - 3xyx^2y - xyx^3 = 0$ , to find the relation of  $x$  and  $y$  in finite terms.

*Answered by Amicus, the Proposer.*

Multiply the given equation by  $2y$ , and it gives  $4a^3y^2y - 2a^3x^2y - 2x^3x^2y - 2xyx^2y + 4x^3y^2y - 6xyx^2y + 2y^3x^2y = 2xy^3x^3 + 2x^3x^2y - 2y^3x^3 - 2xyx^3y + 2y^3x^3 + 2xyx^3 + 4x^3y^3y - 2y^3x^3 - 2xyx^3 - 6xyx^3y + 2y^3x^3 + 2y^3x^3y$  where, all the members admitting of complete fluents, we obtain  $a^3y^4 - a^3y^3x^3 = (x^3 + y^3)(xy - yx)^2$ , or putting  $n = x \div y$ ,  $x - yn = a \sqrt{\frac{1-n^3}{1+n^3}}$  and substituting this for  $x - yn$  in the original given equation, we obtain by proper reduction  $x = \frac{1+2n^3-n^6}{(1+n^3)^2} a \sqrt{\frac{1+n^3}{1-n^3}}$ , and thence  $y = \frac{2an}{(1+n^3)^2} \sqrt{\frac{1+n^3}{1-n^3}}$ , and  $\frac{x}{y} = \frac{1+2n^3-n^6}{2n}$ , or  $n^6 - 2n^3 + \frac{2xn}{y} - 1 = 0$ , but squaring  $x - yn = a \sqrt{\frac{1-n^3}{1+n^3}}$ , we obtain  $n^6 - \frac{2xn}{y} + \frac{a^2 + x^3 + y^3}{y^3} \cdot n^3 - \frac{2xn}{y} + \frac{x^3 - a^3}{y^3} = 0$ , and subtracting the former equation from this we get  $-2xyn^3 + (a^3 + x^3 + 3y^3) \cdot n^3 - 4xyn + x^3 + y^3 - a^3 = 0$ , but the original given equation is  $-2xyn^3 + (2a^3 + 2x^3 + 2y^3) \cdot n^3 - 6xyn + 4x^3 - 4a^3 = 0$ , their diff.  $(a^3 + x^3 - y^3) \cdot n^3 - 2xyn + 3x^3 - y^3 - 3a^3 = 0$ , and dividing the latter of them by this difference, and making the remainder  $= 0$ , we shall obtain  $n$  in terms of  $x$ ,  $y$ , and  $a$ , which value substituted in their foresaid difference, gives

$$\begin{aligned} & 2yx^2 + 8a^3xy^3 - 2x^3y^3 - 4y^3x - 2a^3yx^3 - 4a^3yx \\ & x^3 + a^3x^3 - a^3x^3 - a^3 - 3y^3x^3 - 5a^3y^3 - 3x^3y^3 + 5a^3y^3 + 4a^3x^3y^3 + y^3 \\ & = \frac{2xy}{x^3 + a^3 - y^3} \times \\ & x^3 + a^3x^3 - a^3x^3 - a^3 - 3y^3x^3 - 5a^3y^3 - 3x^3y^3 + 5a^3y^3 + 4a^3x^3y^3 + y^3 \\ & 2yx^2 + 8a^3xy^3 - 2x^3y^3 - 4xy^3 - 2a^3yx^3 - 4a^3xy \end{aligned}$$

the relation of  $x$  and  $y$  in finite terms required.

*Scholium 1.* If the fluent first found above be corrected by the addition of  $a^3c^3x^3$ , where  $c$  may be any constant quantity whatever, we shall, by proceeding as above, obtain  $x - yn = a \sqrt{\frac{1-n^3 + c^3n^3}{1+n^3}}$ , and thence  $x$  and the general value will be obtained, by proceeding in the same manner as in the particular one above, where  $c = 0$ .

*Scholium 2.* If the coefficient of  $x^2y$  had been  $y^3 + x^3 + 2a^3$ , in

the original equation, instead of  $y^2 + x^2 + a^2$ , and the first found fluent corrected by the addition of  $a^2x$ , by proceeding as above, it will

$$\text{be found that } x - yn = a \cdot \frac{1 - n^2}{\sqrt{(1 + n^2)}}, x = a \cdot \frac{3n^2 + 1}{(1 + n^2)^{\frac{3}{2}}}, y = \frac{3n + n^3}{(1 + n^2)^{\frac{3}{2}}}, x + y = a \cdot \frac{(1 + n)^2}{(1 + n^2)^{\frac{3}{2}}}, x - y = a \cdot \frac{(1 - n)^2}{(1 + n^2)^{\frac{3}{2}}}, (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = a^{\frac{2}{3}} \cdot \frac{(1 + n)^2}{1 + n^2} + a^{\frac{2}{3}} \cdot \frac{(1 - n)^2}{1 + n^2} = a^{\frac{2}{3}} \cdot \frac{2 + 2n^2}{1 + n^2} =$$

$2a^{\frac{2}{3}}$ , the finite relation of  $x$  and  $y$  in this case; where the equation answers to the curve whose constant tangent applied within a right angle is  $= 2a$ ,  $x$  being the distance of the semi-ordinate  $y$  from the right angle, and taken along a line bisecting that angle; and agrees with the conclusions of *Bernoulli*, and *De l'Hospital*.

#### XI. QUESTION 843, by Mr. Bonnycastle.

Let  $AB$  and  $CD$  be two diameters, drawn at right angles to each other, in the circle whose centre is  $O$ ; then if the radius  $OA$  be bisected in  $E$ , and on  $EA$  there be taken  $EF$  equal to  $EC$ ,  $CF$  will be the side of the inscribed pentagon.

This elegant practical construction is given by Ptolemy in his *Almagest*; but it has been said that Euclid could not have admitted it into the 4th book of his *Elements*, on account of its being impossible to be demonstrated by the principles he had previously established. This assertion, however, is not true, and a demonstration is now required by means of the first 3 books only.

*Answered by Mr. Bonnycastle.*

Make  $AK$  and  $AG$  each  $= FO$ , and draw  $KF$ ,  $KG$ ,  $OK$ ,  $OG$ : then since  $EO = EB$ , and  $EC = EF$ ,  $AO \times AF = FO^2$  (Eucl. 2, 11,)  $= AK^2$  by construction; therefore  $AK$  is a tangent to a circle drawn through  $K$ ,  $F$ , and  $O$  (Eucl. 3, 37,) and consequently the  $\angle AKF = \angle KOF$  (Eucl. 3, 32).

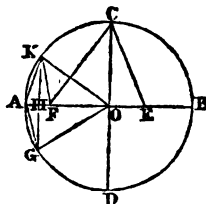
And because  $\angle AFK = \angle KOF + \angle FKO = \angle AKF + \angle FKO = \angle AKO = \angle KAO$ , therefore  $KF = KA = FO$  and  $AF = HF$ , and  $\angle AFK$  or  $\angle KAO = 2\angle KOF = \angle GOK$ .

But  $\angle KOB = 2\angle KAO$  (Eucl. 3, 20,)  $= 2\angle GOK$ ; and, for the same reason,  $\angle GOB = 2\angle GAO = 2\angle GOK$ ; whence  $KG =$  side of a pentagon inscribed in the circle  $ACBD$ .

Again,  $CF^2 = CO^2 + OF^2$  (Eucl. 1, 47,)  $= AO^2 + OF^2$ .

But  $AO^2 = 4AH(4HF) \times HO + OF^2$  (Eucl. 2, 8); and  $OF^2 = 2AH \times AO$  by construction.

Therefore  $CF^2 = 4AH \times HO + 4AH \times AO = 4AH \times (HO + AO) = 4AH \times HB = 4KH^2 = KG^2$ , and consequently  $CF = KG$ .



## XII. QUESTION 844, by the Rev. Mr. Robert Bownas.

A general rule for finding the two equal roots of an equation of any number of dimensions, is this : "Multiply the coefficient of each term in the equation by its index, and dividing the products by the index of the first term, there arises a second rank of coefficients : Divide the respective differences of these two ranks of coefficients by the difference of the first two unequal terms thereof, and there results a third rank : Divide the differences of the last two ranks in like manner by the difference of their first unequal terms, and a fourth rank will be had ; and so on ad libitum, always managing the last two ranks in the same manner in order to a succeeding one ; and one of the equal roots sought will always be a root of such depressed equations." Required the demonstration or investigation, with an illustration of the rule by examples of different equations.

*Answered.*

*Amicus* observes that, "The rule in this question differs no otherwise from the common one mentioned in my solution to question 821, than as the *Italian* method of division in vulgar arithmetic, differs from the common one : as, to any one that will take the trouble to make the trial, will be too evident to need farther illustration here."

And the Rev. Mr. *Bownas* illustrated this rule by various examples wrought out at full length ; but they are too long to be inserted here. He also adds the two following notes to the rule as proposed in the question.

*Note 1.* If any terms in the given equation are wanting, the blanks must be considered as so many terms in the operation.

*Note 2.* The operation may often be rendered less troublesome, by taking the difference of the last rank and some other not immediately preceding it, provided the number of its terms exceed not the index of the highest power of the resulting equation by more than two.

## XIII. QUESTION 845, by Mr. George Beck.

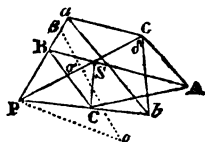
Upon a given base to construct such a triangle, that if the vertical angle, and either of the angles at the base be moved along two straight lines given by position, the locus of the other angle at the base may be an ellipse ; but if both the angles at the base be moved along the said lines, the locus of the vertex may be a right line passing through any given point.

*Answered by Amicus.*

Apply the given base  $ab$  any where within the lines  $ax$ ,  $bx$  given in

position, and make the isosceles triangle  $acb$  whose angle  $c$  may be the supplement of  $ar\delta$ , and the thing is done.

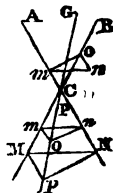
Through  $c$  draw  $rc$ ; then since the points  $r$ ,  $a$ ,  $c$ ,  $b$  are in a circle, and  $ac = bc$ , the angle  $arc = brc$ , and consequently the point  $c$  is always in the same right line, passing through the given point  $r$ , and bisecting the given angle  $ar\delta$ .



Parallel to  $ab$  apply  $bc = bc = ac$  within the given lines, parallel to  $ac$  draw  $ba = ba$ ; then the triangles  $bca$ ,  $bca$  are evidently equal in all respects, and  $bsc$  similar to them; and  $bc$  being given,  $bs = sc$  is given also in length, and is applied within a given angle  $arc$ ; but  $ba$  is also given, wherefore the locus of the point  $\Delta$  is an ellipse; as is elegantly demonstrated (and I should suppose first found out) by Schooten in his *Exercitationes*, lib. 4. And if  $\beta o$  be the position of  $bs$  when perpendicular to  $rc$ , and it be continued to  $o$  till  $\beta o = ba$ , let  $ro$  be drawn, on  $rc$  take  $rd = \beta o$ , then  $r$  being the centre,  $ro$ ,  $rd$  will be a pair of semiconjugates to the ellipse.

*The same answered by Mr. I. Dalby.*

An infinite number of triangles may be made to answer the conditions of the problem. For, let  $mb$ ,  $na$ , be the lines given in position,  $r$  the given point, and  $mn$  the given base; through  $r$  and the point of intersection  $c$ , draw  $rg$ , in which take any point  $p$ , and make the  $\angle mpn = \text{comp. of } \angle mcn \text{ to two right ones}$ ; join  $mn$ , parallel to which draw  $mn = \text{the given base}$ ; draw  $no \parallel np$ , and join  $om$ , then the  $\Delta s nom$ ,  $npm$  are evidently similar: Now if the  $\Delta s mpn$ ,  $mon$  move between the lines  $na$ ,  $mb$ , so that the points  $m$ ,  $n$ ,  $m$ ,  $n$ , are always in those lines, the locus of the points  $p$ ,  $o$ , will be the line  $pg$ ; but if the angular point  $o$ , and either of the other two,  $m$  or  $n$  (as in the upper part of the figure) move along the lines, the third angular point  $m$  will describe an ellipse in all cases, except when the  $\angle m$  is = twice the  $\angle mcn$ , for then it will describe a circle. A demonstration is needless here, because on this property is founded the construction of the elliptical compasses. When the given lines are parallel, it is evident the locus in every case will be a right line.



#### XIV. QUESTION 846, by Mr. George Sanderson.

Given the area of a plane triangle, and the diameter of its circumscribing circle; to determine the sides so, that the ratio of the greater to the less shall be a maximum.

*Answered by Amicus.*

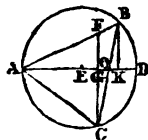
From the solutions to the 768th and 800th questions, (see figure to question 768) it is evident, that no triangle, having the same area

as BAC, and the ratio of whose sides is at the same time greater than that of AB to AC, can be inscribed in the circle; therefore when the area is given, the ratio of the sides is a maximum when BAC is the triangle. To the centre E suppose the radii BE, CE to be drawn; then because EO is  $\frac{1}{2}$  of AO, the triangle BEC =  $\frac{1}{2}$  BAC is given in magnitude, and the two equal sides BE, CE being given, the triangle itself is given, and consequently the triangle BAC.

*The same answered by Mr. Geo. Sanderson.*

Suppose ABC to be the required triangle, inscribed in the given circle ABDC, whose centre is E; and the ratio of the longest side AB to the shortest AC a maximum.

Draw the diameter AD, also CGF and BK  $\perp$  AD. Then the  $\triangle ACF$  is evidently similar to the  $\triangle ABC$ ; and having AC common, and the  $\triangle ABC$  given in magnitude, the  $\triangle AFC$  will be a minimum when the ratio of AB to AC is a maximum; or when AC to AB is a minimum. And hence the investigation and construction will be the same as of question 768, in which AO is =  $\frac{1}{2}$  AD, &c. For whether a triangle be a maximum or a minimum, its base and perpendicular will still be to each other, directly as their increments or decrements.

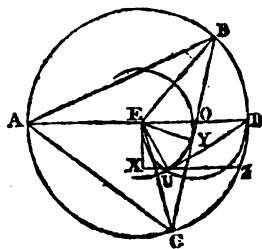


*Or, the same is determined Algebraically thus :*

Put  $a^2 = 2 \triangle ABC$ ,  $d = AD$ ,  $x = AG$ ,  $y = AK$ . Then  $\sqrt{dx} = AC$ ,  $\sqrt{dy} = AB$ ; and consequently  $x \div y$  is a minimum, therefore  $a^2 x \div y = x \sqrt{(dx^2 \div y - x^2)} (AG \times GF) + x \sqrt{(dx - x^2)} (AG \times GC)$  is a minimum, the fluxion of which made = 0, and reduced, gives  $(3dx \div y) \sqrt{(dx - x^2)} + 3d \sqrt{(dx^2 \div y - x^2)} = 4x \sqrt{(dx - x^2)} + 4x \sqrt{(dx^2 \div y - x^2)} = 4a^2 x \div y$ ; whence, dividing by  $3dx \div y$ , we have  $\sqrt{(dx - x^2)} (CG) + \sqrt{(dy - y^2)} (BK) = 4a^2 \div 3d$ , but  $AO \times (CG + BK) = 2 \triangle ABC = a^2$ , therefore  $AO = a^2 \times (3d \div 4a^2) = \frac{3}{4}d$ . Whence the construction as in the Diary for 1781.

*Or it may be otherwise constructed thus :*

Because EO =  $\frac{1}{2}$  AO, the  $\triangle EBC = \frac{1}{2} \triangle ABC = \frac{1}{2} a^2$ . On ED make the rect. EZ = this  $\triangle$  or  $\frac{1}{2} a^2$ , and on ED describe a semicircle cutting xz in U. With centre E and radius EU describe a circle; and through O draw BC touching it in Y. For then, joining AB, AC, EB, EC, EY, the  $\triangle EYC = \triangle EUD$ , because ED = EC, EY = EU, and  $\angle Y = \angle U$ . Theref. the  $\triangle EBC = 2 \triangle EYC = 2 \triangle EUD = \text{rect. EZ} = \frac{1}{2} \triangle ABC$  by construction.



PRIZE QUESTION, *by Plus Minus.*

Let a thread LMF, equal in length to the indefinite ruler LMR, have one end fixed in the given point *r*, and the other to the end *L* of the ruler: Let the ruler move with its other end *R* in the right line *BR* given in position, and its edge passing through the given point *A*; the thread at the same time being stretched close to the edge of the ruler, by means of the pin *M*; it is required to enumerate the curves which the pin *M* may describe, according to all the various positions of the given point *r*.

*Answered by Plus Minus.*

Let fall *AB* and *MQ*  $\perp$  *BR*, also *PH* and *MF*  $\perp$  *AB*, and *FI*  $\perp$  *FM*. Now, calling *AF* *x*, *FM* *y*, *AB* *a*, *HF* *b*, and *AB* *d*, the similar triangles *APM*, *MQB* give this proportion *AP* (*x*) : *AM* ( $\sqrt{(x^2 + y^2)}$ ) :: *QM* (*d* - *x*) : *MR* ( $= MF = \sqrt{(IM^2 + IF^2)}$ )  
 $= \sqrt{(y - b)^2 + (a - x)^2}$ .



$$\text{Hence } xy^2 - \frac{d}{2} y^2 - \frac{b}{d} x^2 y = \frac{a-d}{d} x^2 + \frac{d^2 - a^2 - b^2}{2d} x^2,$$

which is the general equation for all the curves described by this method: And from this equation it may be inferred that if you describe a conic parabola to the axis *AB*, vertex *B*, and focus *A*, the curve will be a defective, redundant, or parabolic hyperbola, according as the point *r* is within, or without, or in the curve of this parabola respectively. Moreover, that if with the centre *A*, and radius *AB*, you describe a circle, the curve will be of the 35th species, if *r* be in the circumference of that circle; of the 36th species, if within it; and of the 34th if without it, provided it be within the parabola. Note, I suppose *r* not to be in the line *AB*, or the same produced; those cases will be considered afterwards.

If *r* be in the curve of the parabola, the figure will be of the 51st species. If *r* be without the parabola, but on the same side of the line *BR* that *A* is, the curve will be of the 4th species. If *r* be in the line *BR*, the curve will be a conic hyperbola with a right line. If *r* and *A* be on different sides of the line *BR*, and *r* nearer to *BR* than *A* is, the curve is of the 7th species; if farther from it, of the 8th species; and if equidistant, of the 25th species. Now let us suppose the point *r* to be somewhere in the line *AB*, or the same produced, and the curves will have a diameter; More particularly, if *r* be between *A* and *B*, the curve will be of the 44th species. If *r* coincide with *A*, the figure coincides with its asymptote, and the locus becomes a right line with a conjugate point. If *r* be on the other side of *A* in respect of *B*, but nearer to it than *B* is, the curve will be of the 43d species; if at the same distance, of the 42d species; if at a greater distance, of the 41st species. If *r* coincide with *B*, the locus is three right lines, two of which coincide. If *A* and *r* be on different sides of *B*, but *r* nearer to *B* than *A* is, the curve will be of the 18th species; but if *r* and *A* be

equidistant from B, it will be of the 30th species; and if F be still farther from B, of the 19th species. In all these various cases the point A is a double point, and a line drawn parallel to BR through the middle point between A and B, is an asymptote of the curve. Lastly, if F remain fixed, and A be infinitely distant, the curve described is a conic parabola. But note, that when the curve is a redundant hyperbola, it will have a diameter, if  $2a' + 4da' + 4a^2b^2 - 14a^3d' + 8ad^2 - 2adb' - 2a^2b' + b' = \pm b \times (bb + 4ad - 4dd)^{\frac{3}{2}}$ .



*Questions proposed in 1786, and answered in 1787.*

I. QUESTION 848, by Mr. J. Hunt, of Stoney Stratford.

For Polly impatient I sigh'd,  
'Twas she I could only approve;  
Each day ev'ry effort I try'd  
To gain her affections and love:  
At last, to relieve my distress,  
And her generous kindness to shew,  
She consented to what I express  
By equations inserted below.

Namely  $\begin{cases} x^2 + xy + xz = 736 \\ y^2 + yx + yz = 160 \\ z^2 + zx + zy = 128 \end{cases}$  Where  $x, y, z$ , denote the letters in the alphabet composing the word.

*Answered by Mr. Matthew Fleck, of Gadlis.*

Add the three given equations together, and the square root of the sum will be  $x + y + z = 32$ . Then divide each of the three given equations by this last equation, and we shall obtain respectively  $x = 23 = w$ , and  $y = 5 = e$ , and  $z = 4 = d$ . So that Miss Polly consented to WED.

*The same answered by Mr. J. Birch, Schoolmaster, of Moulton.*

Divide the former equations by the latter, and we shall have

$$\frac{x}{y} = \frac{23}{5}, \text{ and } \frac{x}{z} = \frac{23}{4}, \text{ and } \frac{y}{z} = \frac{5}{4}.$$

And because the values of  $x, y, z$ , must be whole numbers, within the limits of the alphabet, therefore  $x = 23$ , and  $y = 5$ , and  $z = 4$ . Consequently to WED was what the fair one granted.

II. QUESTION 849, by Mr. Alex. Rowe.

There is a lever of sound dry oak, in form of a square parallelopiped, 10 feet in length, and the side of its square 6 inches, resting upon a

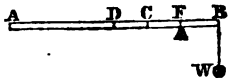
prop or elevated fulcrum at  $\frac{1}{4}$  of a foot from one end : Required to determine what weight must be fixed at the end of the said shortest arm, to keep the lever in equilibrium.

*Answered by Mr. William Gooch.*

It is evident that if the lever extended but  $\frac{1}{4}$  of a foot on each side of the support, it would rest in equilibrio ; but it extends  $8\frac{1}{2}$  feet farther on one side. And as the centre of gravity of a body may be taken for the place of the body, the weight of the said part, viz. of  $8\frac{1}{2}$  feet, or 102 inches, may be deemed in its centre of gravity, which is evidently in the middle of it, or at 5 feet from the fulcrum. But  $6 \times 6 \times 102 \times .0330946 \text{ lb.} = 121 \cdot 52337 \text{ lb.}$  is the weight of the said part: and as the weights are reciprocally as their distances from the fulcrum, it will be as  $\frac{1}{4} : 5 :: 121 \cdot 52337 : 810\frac{7}{15} \text{ lb.}$

*The same by Mr. Thomas Woolston, Master of the Boarding School at Adderbury.*

Let  $AD = DB = 5$ ,  $BF = \frac{1}{4}$ , and  $w =$  the weight sought. Now the solidity of the lever is 4320 cubic inches, and its weight 2106 oz. or  $175\frac{1}{2} \text{ lb. troy} = c$ . Then as  $BF : DF$  or as  $\frac{1}{4} : 4\frac{1}{4} :: c : \frac{1}{3}c = 994\frac{1}{2} \text{ lb. troy}$ , the weight required.



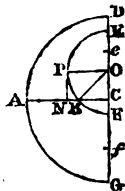
### III. QUESTION 850, by Master W. Gooch, of Harleston School.

There being two eccentric circles, the one within the other, and a diameter to the greater passing through both their centres, the two distances between the circumferences, from the extremities of that diameter, are 5 and 21 ; but the two equal distances between the circumferences, taken upon another diameter of the greater circle, perpendicular to the former, are each 15 : To determine the diameters of the two circles.

*Answered by Amicus.*

On  $AB$ , one of the given equal distances, take  $AN$  equal half the sum of the two unequal ones, and perpendicular to  $AN$  draw  $NP$  equal half their difference ; join  $PB$ , draw  $PO \parallel AB$ , also draw  $BO$  to make the angle  $PBO = BPO$ , so is  $O$  the centre, and  $PO = BO$  the radius of the less circle required.

For, through  $O$  draw  $DE \parallel PN$ , cutting  $AB$  produced in  $C$  ; with the radii  $PO$ ,  $AO$  describe the circles  $EPBF$ ,  $DAG$  ; set off  $ce = cf = CN = BO = PO$  ; then  $Ee$  is evidently  $= Ff$ , and  $De = fg \Rightarrow \frac{1}{2}FG + \frac{1}{2}DE = AN$  half the given sum by construction. And since  $CD = CG = AC$ , and  $OE = OF = ON$ , therefore  $FG - DE = OG - DO = 2oc =$



2<sup>PN</sup> the given difference by construction ; and consequently the circles are those required.

*Calculation.* By the question  $AN = 13$ ,  $BN = 2$ ,  $PN = 8$  ; then  $PN' = PN + BN = 68$ , and since the angle  $PSO = RPO = PBN$  because of the parallels  $AB$ ,  $PO$  ; by similar triangles as  $2BN = 4 : PN :: PN : PO = 68 : 17$  ; and  $AC = 30$ .

*The same Algebraically by Mr. John Dalton, Kendal.*

Put  $x = CB$ . Then is  $x + 15 = CA = CB = CG$ ,  $x + 10 = CX$ , and  $x - 6 = CF$ . And, by the property of the circle,  $CE \times CF = CB^2$ , that is  $(x + 10) \times (x - 6) = x^2$ , or  $x^2 + 4x - 60 = x^2$  ; hence  $4x = 60$ , and  $x = 15$ . Therefore  $DG = 2CA$  or  $2CB = 60$ , and  $EF = 2x + 4 = 34$ , the two diameters sought.

#### IV. QUESTION 851, by Mr. Richard Denning.

Supposing Mr. Sadler with his balloon ascend vertically over Oxford to such a height that the city of London just appears in the horizon ; it is required to determine his height above the earth, with the direct distance between London and Oxford, supposing the latitude of London to be  $51^\circ 31'$ , that of Oxford  $51^\circ 45' 38''$ , and its longitude  $1^\circ 10'$  west of London ; also the radius of the earth 3979 miles.

*Answered by John Rotheram, M. D.*

The two sides  $38^\circ 29'$  and  $38^\circ 14' 22''$ , and included angle  $1^\circ 10'$  of a spherical triangle being given, namely the complements of the two latitudes, and the difference of longitude, the third side or direct distance is found to be  $45' 49''$  ; which, the earth's radius being 3979, is  $53 \cdot 0302$  miles. And as London appeared in the horizon, the height of Mr. Sadler + earth's radius is the hypotenuse of a right-angled plane triangle, one of whose sides is 3979, and the adjacent angle  $45' 49''$  ; then, by similar triangles, as  $1 : \text{nat. sec. } 45' 49'' = 1 : 3979 : \cdot 353$  miles or 621 yards, which is the height of the balloon.

Mr. Thos. Willan, of Gorton, after giving the solution by the common tables of trigonometry, justly observes that, The answer will, in some respect, be different (and I apprehend more accurate) if the method of calculation be used which is recommended in page 168 of Dr. Hutton's Introduction to his Mathematical Tables, lately published ; by which the required side of the spherical triangle is  $45' 53''$  nearly, and the distance of the two places  $53 \cdot 107$  miles ; though in Mr. Sadler's height it makes but little alteration.

#### V. QUESTION 852, by Mr. Stephen Ogle.

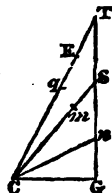
There is a house of three equal stories high ; now a ladder being

raised against it, at 20 feet distance from the foot of the building, reaches the top; whilst another ladder, 12 feet shorter, raised from the same point, reaches only to the top of the second story. It is required to determine the height of the building.

*Answered by Amicus.*

Make  $cg$  = the distance of the given point from the building; perpendicular to which draw  $tg$ , on which take  $gn = \frac{2}{3}$  of the given difference of the lengths of the ladders;  $cn$  being joined, apply  $cs = cn + \frac{2}{3}gn$ , and  $ct = cn + \frac{2}{3}gn$ , so shall  $tg$  be the height required.

For take  $cm = cq = cn$ , and  $cx = cs$ ; then, by construction,  $tq = \frac{2}{3}gn$ , and  $sm = eq = \frac{2}{3}gn$ ; consequently  $tq \cdot sq = tq \cdot sm = gn^2$ , and  $te = \frac{2}{3}gn - \frac{2}{3}gn = \frac{2}{3}gn$  = the given difference of the lengths of the ladders by construction. Moreover (by Simp. Geom. 2, 9,)  $tg^2 - sg^2 = ct^2 - cs^2 = \text{rect. under } te \text{ and } tc + ce$ ; in like manner  $sg^2 - gn^2 = \text{rect. under } sm \text{ and } cs + cm$ ; and  $gn^2 = sm \cdot tq$ , so shall  $sg^2 = \text{rect. under } sm \text{ and } cs + cq + tq$ , or under  $sm$  and  $tc + ce$ ; consequently  $tg^2 - sg^2 : sg^2 :: te : sm :: 5 : 4$ , and by composition  $tg^2 : sg^2 :: 9 : 4$ , or  $tg : sg :: 3 : 2$ .



*Generally.* If  $tg : gs :: m : n$ , then  $gn : te :: mn : m^2 - n^2$ , and  $sm : tq :: n^2 : m^2$ , and the construction as before.

*Calculation.*  $cn = \sqrt{(cg^2 + gn^2)} = 24.644677488$ ,  $tc = cn + tq = 46.244677488$ , and  $tg = 41.6961504$  required.

*Algebraical Solution by Mr. John Cullyer, Assistant at Mr. M'Kain's Boarding School, Bungay.*

Put  $x$  for the length of the shorter ladder. Then is  $x + 12$  the longer, and as  $2 : 3 :: \sqrt{(x^2 - 400)} : \frac{2}{3}\sqrt{(x^2 - 400)}$  the height of the building. Consequently  $\frac{2}{3} \times (x^2 - 400) + 400 = (x + 12)^2 = x^2 + 24x + 144$ ; hence  $5x^2 - 96x = 2576$ , and  $x = 34.24$ ; and the height of the building  $41.696$  feet.

#### VI. QUESTION 853, by Diophantoides.

To find two numbers so, that their difference, the difference of their squares, and the difference of their cubes, may be all square numbers.

*Answered by Mr. James Young, of Prudhoe.*

Let  $x$  and  $y$  be the two numbers. Then must  $x - y$ ,  $x^2 - y^2$ ,  $x^3 - y^3$  be all three squares. Or divide the second by the first, so shall  $x + y$  be a square also. Take the first  $x - y = z^2$ ; then  $x = z^2 + y$ , and the second square or  $x + y = z^2 + 2y$  suppose  $4z^2$ ; hence  $y = \frac{3}{2}z^2$ , and  $x = \frac{5}{2}z^2$ . Then, by the third condition,  $x^3 - y^3$

or  $\frac{9}{8}z^2 = \frac{1}{2}z^2$  must be a square, which it evidently is. Hence then the two numbers are  $\frac{1}{2}z^2$  and  $\frac{5}{2}z^2$ , where  $z$  is any number whatever. And when  $z$  is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. then  $x = 2\frac{1}{2}, 10, 22\frac{1}{2}, 40, 62\frac{1}{2}, 90, 122\frac{1}{2}, 160, 202\frac{1}{2}, 250, \&c.$  and  $y = 1\frac{1}{2}, 6, 13\frac{1}{2}, 24, 37\frac{1}{2}, 54, 73\frac{1}{2}, 96, 121\frac{1}{2}, 150, \&c.$

*The same by Mr. William Cole, of Colchester.*

Let  $x$  and  $y$  represent the two numbers; and assume  $x - y = a^2$ , and  $x + y = m^2 a^2$ ; so shall  $x^2 - y^2$  be  $= m^2 a^4$ , which is always a square. Now  $x = \frac{1}{2}(m^2 a^2 + a^2)$ , and  $y = \frac{1}{2}(m^2 a^2 - a^2)$ , therefore  $x^2 - y^2 = \frac{1}{4}(3m^4 + 1) a^6$  must be a square, and consequently  $3m^4 + 1$  a square: this it is evident will happen when  $m$  is  $= 2$ . Therefore the diff. of the two numbers may be taken equal to any square number at pleasure, and their sum equal to 4 times that number.

Suppose  $x - y = 4$ ; then  $x + y = 16$ ,  $x = 10$ , and  $y = 6$ . Hence  $x - y = 4$ ,  $x^2 - y^2 = 64$ , and  $x^3 - y^3 = 784$  are all square numbers; and these seem to be the least whole numbers the question will admit of.

VII. QUESTION 854, by Lieut. Glenie, of the Engineers.

There is a fortification such, that, if a circle be inscribed, touching its extreme sides; and from any point whatever in the circumference of the said circle perpendiculars be drawn to these sides, the squares on these perpendiculars shall together be equal to the squares on the sides of an equilateral triangle inscribed in the said circle taken together. Required the figure.

*Answered by Mr. J. Wilson, of Colyton.*

The square on the side of an equilateral triangle, is equal to three times the square on the radius of the circumscribing circle; therefore the squares on all the three sides, are equal to 9 times the square on the radius. By the theorem referred to in Amicus's solution to quest. 840, twice the sum of the squares on the perpendiculars to the sides of the figure, is equal to the number of sides  $\times$  3 times the square on radius; that is, 18 times the square on radius  $=$  3 times the number of sides  $\times$  square on radius; therefore 6  $=$  the number of sides, and the figure is a hexagon.

*The same by Mr. Robert Dowden, of Woollavington.*

Call the radius of the inscribed circle  $r$ , the side of the inscribed equilateral triangle  $s$ , and the number of sides of the polygon  $n$ . [Then, by Eucl. 1, 47,  $\sqrt{(s^2 - \frac{1}{4}s^2)} = \sqrt{\frac{3}{4}s^2}$  is the perpendicular; and, by the nature of the circle, as  $\sqrt{\frac{3}{4}s^2} : s :: s : \sqrt{\frac{3}{4}s^2} = 2r$ ; hence  $s^2 = 3r^2$  and  $3s^2 = 9r^2$  is the sum of the squares; then by Stewart's theorem  $9r^2 = \frac{3}{2}nr^2$ ; consequently  $3n = 18$ , and  $n = 6$ ; therefore the figure of the fortification is a hexagon.

## VIII. | QUESTION 855, by X + Y.

To find how long a ball will be in falling to the earth, from the distance of the sun, or 100 millions of miles; as also its velocity when it arrives at the earth: supposing it to be acted on only by the attraction of the earth; and that it is not resisted by the earth's atmosphere.

*Answered by Mr. A. Whitehouse, of Wolverhampton.*

Put  $r = 21$  million feet the earth's radius,  $a = 528000$  million feet the sun's distance,  $x$  any variable distance from the centre of the earth, and  $s = 32\frac{1}{2}$  feet; also  $v$  the velocity, and  $t$  the time. Then  $x^2 : r^2 :: s : sr^2x^{-2}$  the force at  $x$  distance; therefore  $v\dot{v} = -sr^2x^{-2}\dot{x}$ ; and the fluent corrected (by taking  $x = a$  when  $v = 0$ ) is  $v^2 = 2sr^2 \times \left(\frac{1}{x} - \frac{1}{a}\right)$ , and  $v = r\sqrt{\left(\frac{2s}{x} - \frac{2s}{a}\right)} = (\text{when } x = r) r \times \sqrt{\left(\frac{2s}{r} - \frac{2s}{a}\right)} = 36755$  feet or  $6.961$  miles, the velocity at the earth's surface.

$$\text{Again } \dot{t} = -\frac{\dot{x}}{v} = \frac{-\dot{x}}{r\sqrt{\left(\frac{2s}{x} - \frac{2s}{a}\right)}} = -\frac{1}{r}\sqrt{\frac{a}{2s}} \times \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{(a-x)}}$$

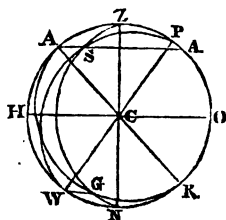
and the correct time  $\int \sqrt{\frac{aa - ar}{2rs}} + \frac{a}{2r}\sqrt{\frac{a}{2s}} \left\{ \times \text{arch} \right\} \sqrt{\frac{r}{a}} =$   
when  $x = r$  is  $t = \left\{ \sqrt{\frac{aa - ar}{2rs}} + \frac{a}{2r}\sqrt{\frac{a}{2s}} \right\} \times \text{to cosine} \left\{ \sqrt{\frac{r}{a}} = \right.$   
 $113$  years,  $227$  days,  $22$  hours,  $14\frac{1}{2}$  minutes, the whole time of descent.

## IX. QUESTION 856, by Mr. John Turner.

The sun's azimuth from the east or west being  $42^\circ 42' 42''$  at the time his altitude was  $49^\circ 30'$ , and the angle at the intersection of the azimuth and hour circles  $28^\circ 2' 15''$ ; to determine, geometrically and by the orthographic projection, the latitude of the place, the sun's declination, and hour of the day, or time from six of the clock; both latitude and declination being north.

*Answered by Mr. Isaac Dalby.*

Let the primitive represent the meridian,  $HO$  the horizon,  $z$  the zenith; describe the azimuth circle or ellipse  $zSN$  at the distance of  $42^\circ 42' 42''$  from the prime vertical  $ZN$ , and draw  $AA$  the parallel of altitude; then the point  $s$  where it cuts  $zSN$  is the sun's place; through  $c$  the centre draw  $AR$ , and in the same manner as the azimuth circles are described let the ellipse  $rgA$  be described to make the  $\angle GRN = 28^\circ 2' 15''$ , draw  $GW \parallel AA$ , and draw the diameter  $WP$ , on



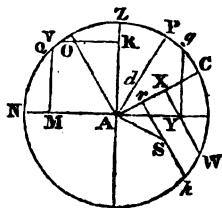
which as a transverse, and through the point  $s$  describe the hour circle or ellipse  $psw$ ; then  $zp$  is the co-latitude,  $sp$  the co-declination, and the  $\angle zps$  the hour angle from noon. For  $AZ$  being the measure of  $zs$ , and  $pz = wn$  that of  $ng$ , and the  $\angle gnr = \angle szp$ , therefore the  $\triangle szp = \triangle gnr$ , consequently the  $\angle zsp$  (the angle made by the hour and azimuth circles)  $= \angle nrg =$  the given angle by construction.

In the triangle  $szp$  there is given the  $\angle szp = 132^\circ 42' 42''$  the azimuth from the north, the  $\angle zsp = 28^\circ 2' 15''$ , and the included side  $= 40^\circ 30'$  the co-altitude; whence  $zp = 36^\circ 56'$  the co-latitude,  $sp = 69^\circ 56'$  the co-declination, and the  $\angle zps = 30^\circ 32'$ , answering to 2h. 2m. 4s. the time from noon.

*Remark.* If the proposer of question 7, in Diary for 1785, meant that the given angle was that made by two great circles on the globe and their orthographic representation required, then, by supposing  $szn$  the given great circle, and  $s$  the given point, the construction will be the same as above. For in my answer in last year's Diary, the great circles or ellipses are inclined in the given angle *when projected*.

*The same by Amicus.*

Let  $z$  be the zenith in the given meridian whose centre is  $A$ , and  $AN$  the horizon. Take  $nv =$  the given altitude,  $zq =$  the given azimuth; draw  $mq \parallel AZ$ , and  $vr$  to  $NA$ ; divide  $vr$  in  $o$  in the same ratio as  $NA$  is divided in  $M$ . Then, by the nature of the orthographic projection,  $o$  is the place of the sun when he has the given altitude and azimuth. We have therefore only to find a point  $p$  in the primitive, through which if a great circle be drawn passing through  $o$ , and making there an angle with that passing through  $z$ ,  $o$ ,  $M$  equal to the given angle, it shall be the hour circle at the time, and  $p$  the north pole. Set off a quadrant along the primitive from  $q$  to  $g$ ; draw  $gy \parallel qm$  cutting the horizon in  $y$ , which therefore will be the pole of the great circle  $zom$ ; draw  $xyw \parallel OA$  cutting  $AC$  (perpendicular to  $OA$ ) in  $x$ , and set off  $wk =$  the measure of the given angle which is to be made at  $o$ ; draw  $kr \parallel wx$ , and divide  $kr$  in  $s$  as  $wx$  is divided in  $y$ ; join  $As$ , and draw  $Ap \perp As$ ; then is  $p$  the north pole required.



For, per projection,  $c, y, s$  are in the great circle whose pole is  $o$ , and  $s$  is the pole of a great circle passing through  $p$  and  $o$ , and the arc  $sy = wk$  is the distance of the poles of the great circles  $po$  and  $zom$ , and is therefore the measure of the angle made by them at  $o$ , which consequently by projection is that given.

*Computation.* Since  $qg$  is a quadrant, therefore  $zg = nq$ , and  $ay = mq$  the cosine of the azimuth, the  $\angle yax = rao$ , and as  $ao : ay :: ar : ax :: or : xy$ ; whence  $xw$ , the arcs  $cw$ , and  $ck$  become known,

and thence also  $kr$ ,  $Ar$ ,  $rs$ ,  $sk$ ,  $As$ ,  $\angle rAs = \angle PAO$ , and conseq.  $PAV = 53^\circ 4'$  the latitude,  $Ad = 20^\circ 4'$  the declination, and  $As = 2h. 2m.$  the time from noon required.

**X. QUESTION 857, by Mr. J. Jackson, of Hutton-Rudby.**

To find the force with which a spherical shell of copper,  $\frac{1}{100}$  of an inch thick, and 100 feet internal diameter, filled with inflammable air 10 times lighter than common air, will ascend, or what weight it will keep in equilibrio; and to what height it will ascend in the atmosphere, without being loaded with any weight, the density of air being  $\cdot 0012$ , of copper 9, and of water 1.

*Answered by Amicus.*

By the question, copper is 7500 times the density of air at the surface; and if this latter = 1, and  $m = 5280$  the feet in a mile, then at  $x$  feet above the surface of the earth, supposing the force of gravity to be uniform, the density  $\left\{ \begin{array}{l} - \\ (4) \end{array} \frac{x}{7m} \right\}$  and, the diameter and weight of the globe being given, we have as the weight of the globe : that of an equal one of air at the surface :: 475 : 1000 nearly, hence  $\left\{ \begin{array}{l} - \\ (4) \end{array} \frac{x}{7} \right\}$  if  $x$  be taken in miles, from which we obtain  $\frac{475}{1000} = \left\{ \begin{array}{l} - \\ (4) \end{array} \frac{x}{7} \right\}$   $x = 3.759$  miles, the height at which the globe would rest in equilibrio. And if  $n =$  the weight of a globe of air at the surface  $100\frac{1}{100}$  feet diameter  $= 1.2 \times .5236 \times (100\frac{1}{100})^3$  ounces, then  $\cdot 525n$  oz. avoirdupois, is the weight necessary to keep the globe at rest at the surface.

But though 3.759 miles is the height at which the globe, after oscillating up and down, would at length rest; yet to find the height to which it would at first rise, other principles are necessary. But we are sorry our limits will not admit the elaborate solution given by Amicus of this curious problem.

*The same by Mr. R. Waugh, of Lanchester, Durham.*

Put  $b = 1200.02$  inches the external diameter of the balloon,  $a = .02$ ,  $n = .5236$ ,  $c = .0000435$  lb. the weight of a cubic inch of common air,  $d = .32656$  ditto of copper. Then  $cnb^3 = 39360$  lb. = the weight of a mass of air of the size of the balloon, and  $dn \times (b^3 - (b - a)^3) = 14773$  lb. the weight of the shell of copper, and  $\frac{1}{10}cn \times (b - a)^3 = 3936$  lb. is the weight of the included gas; therefore 18709 is the whole weight of the balloon, which taken from 39360 leaves 20651 the power of the balloon, or the weight to balance it.

Again, the balloon will ascend and rest at such a height in the air, where it will be of equal weight with the same bulk of it, and therefore where its density is to the density of the surface, as 18709 to 39360, or as 1 to 2.104. Hence, by page 389, Emerson's Fluxions,

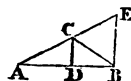
or page 81, Saunderson's Fluxions, we have  $68444 \times \log. \text{ of } 2 \cdot 104 = 22110$  feet or 4 miles nearly, the height required.

**XI. QUESTION 858, by Mr. George Sanderson.**

To divide a given right line into two such parts, that the square of the one part may be to the excess of a given rectangle above the square of the other part, in a given ratio.

*Answered by Mr. William Simpson, jun. of Bath.*

Let AB be the given line. Make BE  $\perp$  AB, and take AB to BE<sup>2</sup> in the given ratio; apply BC the side of a square equal to the given rectangle, and draw CD  $\parallel$  BE; so shall D be the point required.



For by similar triangles,  $AD^2 : DC^2 :: AB^2 : BE^2$ ; but  $DC^2 = BC^2 - BD^2 = \text{the given rect.} - BD^2$ ; therefore  $AD^2 : \text{the given rect.} - BD^2 :: AB^2 : BE^2$ , that is in the given ratio by construction.

*Note 1.* The rectangle must not be given less than the square on the perpendicular from B to AE. And when BC is less than BE, there will be two points D answering the question.

*Note 2.* This problem is the same as, "Given, in a plane triangle, the base, one side, and the ratio of the perpendicular to the alternate segment."

*Algebraically by the Rev. Mr. L. Evans.*

Let  $a =$  the given line,  $c^2 =$  the given rectangle, and  $m$  to  $n$  the given ratio; also  $x =$  the one part, then is  $a - x =$  the other part.

Hence  $x^2 : c^2 - (a - x)^2 :: m : n$ ; therefore

$$x^2 - \frac{2am}{m+n}x = \frac{c^2 - a^2}{m+n}m, \text{ and } x = \frac{am \pm \sqrt{((m^2 + mn) \cdot c^2 - mna^2)}}{m+n}.$$

**XII. QUESTION 859, by the Rev. Mr. John Hellins.**

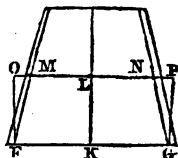
If a copper vessel in form of a frustum of a cone, and open at both ends, or having no bottom, were placed upon a horizontal plane, and quicksilver poured into it; query the greatest quantity it would contain before the quicksilver begun to run below the edges of the base; the dimensions of the vessel being as follows:

Inside top diameter . . . . .	1 inch
Outside ditto . . . . .	1.1
Inside bottom diameter . . . . .	2.
Outside ditto . . . . .	2.1
Perpendicular height . . . . .	6.

*Answered by Mr. John Farey.*

It is proved by the writers on hydrostatics that the pressure of fluids

on the bottoms of their containing vessels (of whatever form), is equal to the pressure of a cylinder, whose base is the bottom of the vessel, and height the perpendicular height of the fluid. Therefore the excess of pressure on the bottom of conical vessels, above the absolute weight of the fluid, must be exerted upwards against the vessel; and when this becomes greater than the weight of the vessel, the fluid will escape at the bottom; therefore the greatest quantity of fluid such a vessel is capable of containing, must be when the excess of pressure is equal to the weight of the vessel. This being premised, let  $x = KL$  the depth of the quicksilver when the vessel contains the most possible,  $a = .7854$ ,  $n = 8.101753$  oz. the weight of a cubic inch of quicksilver, and  $m = 5.208359$  oz. ditto of the copper. By a well known theorem in mensuration



$\frac{1}{3}(1^3 + 2^3 + 1 \times 2) \times 6a = 14a$  the inner solidity and  
 $\frac{1}{3}(1 \cdot 1^3 + 2 \cdot 1^3 + 1 \cdot 1 \times 2 \cdot 1) \times 6a = 15.86a$  the outer ditto,  
 therefore  $1.86a =$  the solidity of the copper,  
 and  $1.86am =$  its weight.

Again, by similar triangles,  $2 - \frac{1}{2}x =$  the diameter  $MN$ , therefore by the same theorem the solidity of the fluid  $FMNG$  will be

$(2^3 + (2 - \frac{1}{2}x)^3 + 2 \times (2 - \frac{1}{2}x)) \times \frac{1}{3}ax$  or  $4ax - \frac{1}{3}ax^2 + \frac{1}{108}ax^3$ ;  
 also the solidity of the cylinder  $FOFG$  is  $4ax$  and  $0$

their difference is  $\frac{1}{3}ax^2 - \frac{1}{108}ax^3$   
 and its weight  $\frac{1}{3}anx^2 - \frac{1}{108}anx^3$ .

Putting these weights equal, gives  $1.86am = \frac{1}{3}anx^2 - \frac{1}{108}anx^3$ ;  
 hence  $x^3 - 36x^2 + 200.88(m \div n) = 0$ , and  $x = 1.94739$  inches,  
 the depth of the quicksilver. Then the greatest quantity of the quicksilver or  $4ax - \frac{1}{3}ax^2 + \frac{1}{108}ax^3$  is  $= .517879$  cubic inches; and its weight  $41.9573$  oz. avoirdupois.

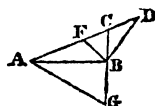
### XIII. QUESTION 860, by Mr. I. Dalby.

The simplest method I have ever seen, of making a pentagon on a given right line, is the following —

Let  $AB$  be the given line, make  $BC$  perpendicular to  $AB$  and equal to half  $AB$ , draw  $AC$ , which produce till  $CD$  is equal to  $CB$ , join  $BD$ , then  $BD$  is equal to the radius of the circumscribing circle. A geometrical demonstration is required.

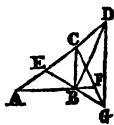
*Answered by Mr. John Ryley, of Beeston.*

Draw  $BF \perp BD$ , so shall  $FD$  be equal  $AB$ . Produce  $CB$  till  $CG = CA$ , and join  $AG$ . Then because the triangles  $CBF$ ,  $CGA$  are isosceles, and the  $\angle C$  common,  $AG \parallel BF$ , and consequently the triangles  $DBF$ ,  $ABG$  equiangular; therefore  $AG : AB :: AB (FD) : BD$ , and since  $AB$  is the radius of the circle about the pentagon whose side is  $AG$  by quest. 843, therefore  $BD$  is the radius to the side  $AB$ .



*The same by Mr. John Sampson, Schoolmaster, at Old-Hutton, near Kendal.*

Draw  $EBG \perp BD$  and  $CFG \perp AD$ , and join  $DG$ . Then is  $DE = AB$ , or  $CB = CD = CE$ , and consequently  $GD = GE$ . The angles  $ACB, CFB$  being equal, their supplements  $BCD, BFG$  are equal; again by taking the  $\angle EBC$  from the right angles  $ABC, EBD$ , there remains the  $\angle CBD = ABE = FBG$ ; hence the triangles  $CBD, FBG$  are similar, and since  $AB = 2BC$ , and  $BC = 2BF$ , therefore  $BD = 2BG$ . Lastly, since  $BG$  is  $\perp BD$  and  $= \frac{1}{2}BD$ , and  $GE = GD$ ,  $ED$  is the side of a pentagon inscribed in a circle of which  $BD$  is the radius, by quest. 11, last year, but  $ED = 2BC = AB$ ; therefore  $AB$  is the side of a pentagon inscribed in a circle of which  $BD$  is the radius.



XIV. QUESTION 861, by Amicus:

To find in finite terms the perfect sum of the infinite series  $\frac{1}{a} - \frac{1}{2c} +$   
 $\frac{1}{2.4a} - \frac{3.3}{2.4.6c} + \frac{5.5}{2.4.6.8a} - \frac{3.3.7.7}{2.4.6.8.10c} + \frac{5.5.9.9}{2.4.6.8.10.12a}$   
 $- \frac{3.3.7.7.11.11}{2.4.6.8.10.12.14c} + \frac{5.5.9.9.13.13}{2.4.6.8.10.12.14.16a} - \&c.$

where  $a =$  the whole fluent of  $xx^{\frac{1}{2}} \cdot (1 - x^2)^{-\frac{1}{2}}$ , and  $c =$  that of  $xx^{-\frac{1}{2}} \cdot (1 - x^2)^{-\frac{1}{2}}$ .

*Answered only by the Proposer Amicus.*

Let  $s =$  the sum of the proposed series,  $q =$  the length of the quadrant of a circle to the radius unity; then it is well known that  $ca = 2q$ , and multiplying the series by  $ca$  we obtain  $\{c - \frac{a}{2} + \frac{c}{2.4} - \frac{3.3a}{2.4.6} + \frac{5.5c}{2.4.6.8} - \frac{3.3.7.7a}{2.4.6.8.10} + \frac{5.5.9.9c}{2.4.6.8.10.12} - \frac{3.3.7.7.11.11a}{2.4.6.8.10.12.14} + \&c. = 2qs.$

Now  $\frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}\sqrt{(1-x^2)}} = \frac{x - 3x^3}{\sqrt{(1-x^2)}} + \frac{3x^{\frac{3}{2}}}{\sqrt{(1-x^2)}}$ , and when  $x = 1$ , the fluent of the first member of this  $= 0$ , and consequently the whole

fluent of  $\frac{x^{\frac{3}{2}}}{\sqrt{(1-x^2)}}$  is  $= \frac{c}{3}$ ; in like manner the whole fluent of

$\frac{5x^{\frac{5}{2}}}{\sqrt{(1-x^2)}} =$  that of  $\frac{7x^{\frac{7}{2}}}{\sqrt{(1-x^2)}}$ , whence that of  $x^{\frac{7}{2}} \times (1-x^2)^{-\frac{1}{2}} = \frac{5c}{3.7}$ , that of  $x^{\frac{9}{2}} \times (1-x^2)^{-\frac{1}{2}} = \frac{5.9c}{3.7.11}$ , of  $x^{\frac{11}{2}} \times (1-x^2)^{-\frac{1}{2}}$

$= \frac{5 \cdot 9 \cdot 13c}{3 \cdot 7 \cdot 11 \cdot 15}$ , &c. Again, that of  $\frac{3x^{\frac{1}{2}}}{\sqrt{1-x^2}}$  or of  $\frac{3x^{\frac{1}{2}} - 5x^{\frac{3}{2}} + 5x^{\frac{5}{2}}}{\sqrt{(x^3 - x^2)}} = \text{that of } \frac{5x^{\frac{5}{2}}}{\sqrt{(1-x^2)}}$ ; hence that of  $\frac{x^{\frac{1}{2}}}{\sqrt{(1-x^2)}}$   
 $= \frac{3a}{5}$ , that of  $x^{\frac{3}{2}} \times (1-x^2)^{-\frac{1}{2}} = \frac{3 \cdot 7 \cdot a}{5 \cdot 9}$ , of  $x^{\frac{5}{2}} \times (1-x^2)^{-\frac{1}{2}}$   
 $= \frac{3 \cdot 7 \cdot 11a}{5 \cdot 9 \cdot 13}$ , &c. Whence, dividing the first term of the series above,  
 by  $c$ , the second term by  $a$ , the third by  $\frac{c}{3}$ , the 4th by  $\frac{3a}{5}$ , the 5th by  
 $\frac{5c}{3 \cdot 7}$ , the 6th by  $\frac{3 \cdot 7a}{5 \cdot 9}$ , &c. it becomes  $1 = \frac{1}{2} + \frac{3}{2 \cdot 4} - \frac{3 \cdot 5}{2 \cdot 4 \cdot 6}$   
 $+ \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}$ , &c.  $= (1+x)^{-\frac{1}{2}}$  when  $x = 1$ ; con-  
 sequently multiplying the second term of this last series by  $x$ , the  
 third by  $x^2$ , the 4th by  $x^3$ , the 5th by  $x^4$ , &c. it is manifestly  $+$   
 $(1+x)^{-\frac{1}{2}}$ , and since the series above is no more than this  $\times$   
 $\frac{x}{x^{\frac{1}{2}} \sqrt{(1-x^2)}}$  and the whole fluent taken, its sum is therefore equal to  
 the whole fluent of  $\frac{x}{x^{\frac{1}{2}} \sqrt{(1-x^2)} \sqrt{(1+x)}}$ , which by form the 60th  
 of Cotes's *Harmoniarum Mensurarum* is  $q\sqrt{2} = 2qs$  above, consequently  
 $s = \sqrt{\frac{1}{2}}$  = the sum of the proposed series.

PRIZE QUESTION, by Mr. Bonycastle.

The initial velocity of a 24lb. ball of cast iron, which is projected  
 in a direction perpendicular to the plane of the horizon, being suppos-  
 ed equal to 1200 feet per second; and that the resistance of the me-  
 dium is constantly as the square of the velocity, and every where of  
 the same density. Required the time of flight, and the height to which  
 it will ascend?

*Answered by Amicus.*

Allowing a cubic foot of cast iron to weigh 7200 oz. and that it is  
 6000 times heavier than air, the diameter of the ball will be .46702  
 parts of a foot; and applying these numbers to the theorems, at art.  
 366 of Mr. Simpson's *Fluxions*, we have, putting  $d$  = the ball's dia-  
 meter,  $3d \times 6000 = 16000d = d$  the space that might have been uni-  
 formly described by the ball in vacuo, whilst its motion is destroyed  
 by the resistance of the medium alone uniformly continued, and if  $b$  =  
 $32\frac{1}{2}$  the force of gravity; then  $a$  = the greatest velocity that could  
 possibly be acquired in falling  $\sqrt{16000db}$  = feet 490.2646 per second,  
 the height of the ascent 7265.808, the time of ascent  $18'' \cdot 027$ , time of

descent  $24''\cdot807$ , and the velocity acquired in the descent  $453\cdot85$  feet per second. There is not the least doubt of the truth of these theorems, provided the value of  $d$  or  $a$  be at first rightly ascertained: that here given is agreeable to all the English, and to Mr. Daniel Bernoulli, and many other foreign mathematicians. But Mr. John Bernoulli (Daniel's father), considering the air as an elastic medium with small interstices between the particles, determines  $d = 4000n$ , and  $a = 245\cdot1323n$  and then the height ascended comes out  $3004\cdot84$ , time  $10''\cdot43477$ , time of descent  $17''\cdot46433$ , and the acquired velocity  $240\cdot1726$ . For a second hypothesis, he supposes the particles to be non-elastic, and then  $d = 8000n$ ,  $a = 346\cdot6697$ , height  $4788\cdot946$ , time of ascent  $13''\cdot89778$ , of descent  $21''\cdot0697$ , and  $333\cdot0503$  the velocity thereby acquired. His theorems differ from Mr. Simpson's in form only, the variation arises wholly from  $d$ , the determination of which, depending on the internal properties of the medium, is doubtful.



*Questions proposed in 1787, and answered in 1788.*

1. QUESTION 863, by Mr. J. Hunt, of Stoney Stratford.

From the following equations, dear Gents, will appear,  
An ornament greatly becoming the fair.

$$x + y + z = 20, \quad x + 2y + 3z = 53, \quad x^2 + y^2 + z^2 = 266.$$

*Answered by Mr. Thomas Woolston, of Adderbury.*

From the second equation take twice the first, and we have  $z = 13 + x$ ; and from three times the 1st equation take the 2d, and we have  $y = 7 - 2x$ : substitute these values of  $y$  and  $z$  in the 3d equation, and we have  $x^2 - \frac{1}{2}x = 8$ : whence we find  $x = 3$ ; and consequently  $y = 1$ , and  $z = 16$ ; therefore the word is CAP.

*The same by Mr. George Roope, of Tring Academy.*

By taking the 1st equation, and the double of it, from the 2d, we get  $y = 33 - 2z$ , and  $x = z - 13$ ; these substituted in the 3d. equation give  $158z - 6z^2 = 992$ . Hence  $z = 16$ ,  $x = 3$ , and  $y = 1$ ; and the ornament is a CAP.

*The same by Mr. J. Scholefield, Schoolmaster at Brumley.*

From the 2d equation take double the first, and  $z = x + 13$ ; take this from the first, and  $y = 7 - 2x$ ; place these values of  $z$  and  $y$  in the 3d equation, which then becomes  $6x^2 - 2x = 48$ ; hence  $x = 3$ ,  $y = 1$ ,  $z = 16$ , and the answer a CAP, an ornament becoming all modest women.

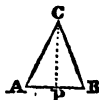
II. QUESTION 864, *by Juvenis.*

In Hawney's Mensuration, and Break's Surveying, the multiplier for finding an undecagon, is 8.51425, and for a duodecagon 9.330125, which last number Hawney has pretended to find. Now as I know some teachers who use these numbers, and who persist in their being right, I hope it will not be thought below Lady D's notice, for the sake of young Tyros who may be led into error, to rectify the mistake, as she has ever stooped to utility, though dressed in ever so humble a garb.

*Answered by Mr. R. Bretherick, of Kirby-Overblow.*

Those who would see this problem resolved in a general manner, for all polygons, may, I presume, have their curiosity abundantly satisfied by perusing the Scholium at pa. 81 of Dr. Hutton's elegant Treatise on Mensuration. But for the sake of those who are not in possession of that book, I have copied one of the methods there laid down.

Suppose ABC to be one of the triangles which constitute any regular polygon : Then, "as radius = 1 : tang.  $\angle CAP = t :: AP : PC :: t \times AP = \frac{1}{2}t$ , supposing AB = 1 ; then  $\frac{1}{2}t$  ( $= AP \times PC$ ) = the  $\triangle ACB$ , and  $\frac{1}{2}nt$  = the polygon ; where  $n$  is the number of sides. So that, by finding the tangent of the  $\angle CAP$ , by the table of tangents, and multiplying it by the number of sides,  $\frac{1}{2}$  of the product will be the multiplier required. Hence we obtain 9.365640 the multiplier for the undecagon, and 11.196152 for the duodecagon.



*The same by Mr. J. Dalton, Teacher of the Mathematics, Kendal.*

In the example for finding the multiplier for a dodecagon, Hawney seems to have fallen into several mistakes. After finding the perpendicular on one of the sides, he multiplies it by .5 or  $\frac{1}{2}$ , half the base or side of the dodecagon, which would give the area of one triangle, or  $\frac{1}{12}$  of the whole ; but by misplacing the decimal point, he in effect multiplies the said area by 10, and concludes he has found the whole area of the dodecagon, when he has only  $\frac{1}{12}$  or  $\frac{1}{2}$  of it. In like manner he has mistaken  $\frac{1}{11}$  of the area of the undecagon for the whole area.

III. QUESTION 865, *by Mr. I. Saul, of Holland, near Wigan.*

Required a general rule for determining the legs of a right-angled triangle, having given the radius of its circumscribing circle, and the distance of the centres of its circumscribing and inscribed circles.

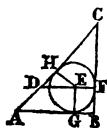
*Answered by Mr. M. Fleck.*

Since the triangle is right-angled, the hypotenuse  $2h$  must be the

diameter of the circumscribing circle, which is given ; then if  $2s$  be the sum, and  $2d$  the difference of the legs, the radius of the inscribed circle will be  $= s - h$ . Then since the given distance  $g$  is the hypotenuse, and  $d$  and  $s - h$  the legs of another right-angled triangle, we have  $g^2 = d^2 + (s - h)^2 = 3h^2 - 2sh$ ; hence  $s = (3h^2 - g^2) \div 2h$ , and  $d = \sqrt{(2h^2 - s^2)}$ . Which being now known, then  $s + d =$  the greater leg, and  $s - d =$  the less.

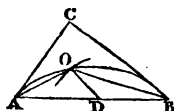
*The same answered by Mr. S. Clement, Schoolmaster, Arundel.*

Put  $AD$  or  $DC = a$ ,  $DE = b$ ,  $HE = y$ , and  $DH = z$ ; then will  $AB = a + y + z$ , and  $BC = a + y - z$ . Now by right-angled triangles  $y^2 + z^2 = b^2$ , and  $(a + y + z)^2 + (a + y - z)^2 = 4a^2$ ; therefore  $2ay + b^2 = a^2$  and  $y = (a^2 - b^2) \div 2a$  the less radius. Then  $z = \sqrt{(b^2 - y^2)}$  becomes known, and consequently the legs  $a + y + z$  and  $a + y - z$ .



*The same answered by Mr. John Burrow, of Bolton Field.*

Suppose that  $ACB$  is the required triangle, and  $o$  the centre of the inscribed circle. Join  $AO$  and  $BO$ , and draw  $OD$  to the middle of  $AB$ . Then because  $C$  is a right angle,  $AB$  is the diameter of the circumscribing circle,  $D$  its centre, and  $DO$  the distance between the centre of the inscribed and circumscribing circles. Moreover, because  $AO$  and  $BO$  bisect the angles at  $A$  and  $B$ , and the sum of those angles is a right angle, the sum of the angles  $OAB$ ,  $OBA$  is half a right angle, therefore the angle  $AOB$  is the supplement of half a right angle. Whence this construction.



On the given diameter  $AB$  describe a segment of a circle to contain the given angle  $AOB$ , and from the middle of  $AB$ , with radius equal to the given distance  $DO$ , describe an arc to intersect the former in  $o$ ; join  $AO$ ,  $OB$ , and make the angle  $OAC = OAB$  and  $OBC = OBA$ : Then the triangle will be constructed.

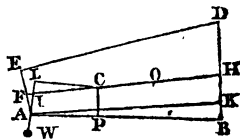
IV. QUESTION 866, by Mr. Wm. Penn, jun. of Chalfont.

If a lever of dry oak, in the form of the frustum of a square pyramid, the length being 30 inches, each side of the greater end 6 inches, and each side of the less end 3 inches, rest upon a prop at 3 inches from the smaller end; it is required to find what weight must be fixed to the extremity of this end, to keep the larger end in equilibrio.

*Answered by Amicus.*

The lever  $AEDB$  and weight  $w$  will rest in any position, provided their common centre of gravity be supported; and because it can then

only be supported by the prop, when the lower side  $AB$  of the lever is horizontal; if  $P$  be the prop, and  $PC$  perpendicular to  $AB$  meet the axis  $FH$  of the lever in  $C$ , draw  $WAL$  parallel to  $CP$  cutting the axis in  $L$ , and  $CL$  parallel to  $AB$  in  $L$ ; then  $L$  may be considered as the place of the weight,  $c$  the common centre of gravity of the weight and frustum, and  $o$  the centre of gravity of the frustum. By the question  $AE = b = 3$ ,  $BD = 6$ ,  $AP = LC = 3$ ,  $FH = AK = 30 = a$ ,  $BK = \frac{3}{2}$ , and  $AK : BK :: 1 : .05 :: 2 : n = .1$ , or  $2BK = na = 3$ ,  $AB = \frac{1}{2}\sqrt{3609}$ ,  $AK : AB :: CL : CP = 3.003747 = c$ ,  $IL = 2IF = \frac{1}{20}CL$ ,  $FI = \frac{1}{40} = .025$ , and  $FC = 3.078747 = g$ . Now in order that  $c$  may be the common centre of gravity of the weight and frustum, it is manifestly necessary that  $ew = (fo - g) \times \text{solidity or weight of the frustum} = (fo - g) \times (\frac{1}{3}n^2a^3 + b^2a + nba^2) = \frac{1}{2}b^2a^2 + \frac{2}{3}na^2b + \frac{1}{4}n^2a^4 - \frac{1}{2}n^2a^2g - b^2ag - nba^2g = \frac{1}{12}b^2a \times (17a - 28g)$ , or  $w = 3174.495$  cubic inches of oak = 91.8538lb. avoirdupois.

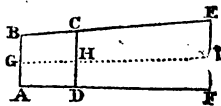


*The same by Mr. John Dalton, Kendal.*

The weights of the parts of the frustum on each side of the prop will be thus, the greater = 17.367lb. the less = 0.862lb. And from the directions of the writers on fluxions, the distance of the fulcrum from the centre of gravity of the greater part is 16.0415 inches, and from that of the less part 1.4524 inches, which distances, multiplied by the respective weights, give the momenta of the two parts, the difference of which is the momentum of the required weight, which divided by 3, its distance from the fulcrum, gives the weight 92.446lb.

*The same by Mr. John Aspland, of Soham.*

$GI : GH :: FF - AB : DC - AB = .3$ . Put  $EF = a = 6$ ,  $DC = b$ ,  $\frac{1}{4}h = 3.3$ ,  $HI = h = 27$ ; then will  $\frac{3a^3 + 2ab + b^2}{a^3 + ab + b^2} \times \frac{1}{4}h = 16.041375$  be the distance of the centre of gravity of  $DE$  from the point  $H$ . Again, put  $AB = c = 3$ , and  $GH = l = 3$ ; then will  $\frac{3c^3 + 2cb + b^2}{c^3 + cb + b^2} \times \frac{1}{4}l = 1.4523$  be the distance of the centre of gravity of  $DB$  from the point  $H$ . Now the solidity of the part  $DE$  is 600.21 inches and its weight 17.367lb; and the solidity of  $DB$  is 29.79 inches and its weight .86197lb. Put now  $d = 16.04137$ ,  $r = 1.4523$ ,  $w = 17.367$ , and  $x = .8197$ , also  $x$  = the weight sought; then  $rw + lx = dw$ , and hence  $x = (dw - rw) \div l = 92.447$ lb. the answer.



## V. QUESTION 867, by Mr. John Aspland, of Soham.

A smith, unskilled in mechanics, undertook to make a steelyard, to weigh hay and other large weights ; and having first made his beam hang in equilibrio by a fixed weight behind the centre of motion (so that it may be considered as without weight) he began his divisions from the said centre, and graduated the beam into 59 equal divisions, each division, with a weight of 118lb. weighing, as he supposes, 112lb ; at the 29th division he adds 125lb. to his former weight, which he then finds upon trial to weigh exactly 60 cwt. ; and with this weight he supposes each division now weighs 2 cwt. that is, at the 30th division 62 cwt. at the 31st 64 cwt. and so on to 90 cwt. which consequently he supposes to fall upon the 44th division. It is required to show how much the steelyard errs from the truth at every division, both with the small and large weight.

*Answered by Mr. Matt. Terry, Land Surveyor.*

Since the steelyard accurately weighs 60 cwt. or 6720lb. when 243lb. is suspended on the 29th division, therefore  $6720 : 243 :: 29 : 1\frac{109}{243}$  =  $d$  the length of the short arm of the steelyard ; and  $d : 1 :: 118 : 112 \cdot 5245$ , instead of 112. Again,  $d : 1 :: 243 : 231 \cdot 7242$ lb. instead of 2 cwt. or 224lb. Hence, in weighing with the small weight, or 118lb. the error on the first division is  $\cdot 5245$ , on the 2d twice as much, on the 3d thrice as much, and so on to the 29th division, where the error is  $15 \cdot 21$ lb. And in weighing with the greater weight, or 243lb. from the 29th to the 59th division, the error on the 30th division is  $7 \cdot 7242$ lb. on the 31st twice as much, on the 32d thrice as much, and so on to the last or 59th division, where the error is 30 times  $7 \cdot 7242$ , or  $231 \cdot 726$ lb. or 2 cwt. 7lb.

## VI. QUESTION 868, by Mr. Todd, of Darlington.

To determine the least semi-parabola that can circumscribe a given circle.

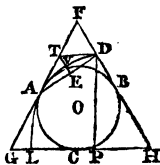
N.B. This question has been proposed elsewhere ; but it is here re-proposed on account of a dispute which has been held for some time concerning it.

There are two ways of considering this problem, viz. either as the semiparabola is a right one, or an oblique one. According to the common acceptation, it is taken as, and usually understood to be, a right parabola. But some of our correspondents have taken it as oblique, which makes the conclusion different, though each solution be right in its own way. We shall insert an example or two of each.

## 1. For the Oblique Parabola, by Amicus.

Circumscribe the given circle with an equilateral triangle  $FGH$ , touch,

ing it in A, B, C; bisect AF in T, and to FH apply  $TD = AT = TF$ ; join AD, and draw TE parallel to FH cutting AD in E; bisect TE in V; then with the vertex V, axis VE, and ordinate rightly applied AD, describe a parabola DVAL cutting GH in L, so is DALHD the semiparabola required.

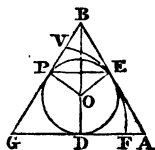


For since  $TV = VE$ , and TA and TD are tangents to the parabola: and since  $TD = TF = FD$ , TD is parallel to GH; consequently the ordinate LH is parallel to the tangent TD, and the abscissa DH to the axis VE; wherefore MDAL is a semiparabola; and because the tangent GF is bisected in A, it is manifestly that required.

By sim. triangles, as  $BH : DH :: 2 : 3 :: BO : AD = \frac{2}{3}BO = \frac{1}{3}BH\sqrt{3}$ , or  $BH = BF = BO\sqrt{3} = AB$ ; and  $DB : DH :: AB^2 : LH^2$ , or  $LH = 3BO$ ; and  $DF$  (perpendicular to GH)  $= \frac{1}{2}FC = \frac{1}{2}BO$ ; and the area of the semiparabola  $= \frac{9}{2}BO^2$ .

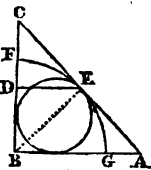
## 2. For the Oblique Parabola, by Mr. T. Todd, of Darlington.

It is proved in art. 20, Simpson's Fluxions, that the equilateral triangle GBA is the least that can circumscribe the given circle DEF, where BA, perpendicular to the radius OE  $= r$ , is a tangent to both curves in the point E; FE parallel to GA; and BD perpendicular to GA. Then  $OB = 2r$ , and  $GB = 2r\sqrt{3}$ . Also since  $BE = BA$ , and, by the parabola  $PV = VB = \frac{1}{4}GB$ , it appears by Simpson's Geom. page 201, that the required semiparabola GVEF is now given, being the greatest that can be inscribed in the said triangle, and the least that can circumscribe the given circle; hence then  $VF : VG :: 1 : 3 :: FE^2 : GF^2 = 9r^2$ , or  $GF = 3r = BD$ , and the area of the semiparabola GVEF is  $\frac{3}{2}GV \times GF \times \sin \angle G = \frac{1}{2}\sqrt{3} \times \frac{2}{3} \times \frac{1}{2}r\sqrt{3} \times 3r = \frac{9}{2}r^2$  required.



## 3. For the Right Parabola, by Mr. I. Saul, Holland, near Wigan.

By theorem 8, Simp. Max. and Min. the least right-angled triangle circumscribing the circle, is when the legs BC, BA are equal. Also the greatest parallelogram that can be inscribed in any curve is when the sub-tangent  $CD = DE = \frac{1}{2}CB$ . Therefore circumscribe the given circle with the right-angled triangle ABC, making  $AB = BC$ ; bisect BC in D, and DC in F; then to the vertex F, axis FD, and ordinate DE, describe the parabola FEG required. Then, putting  $a$  for the radius of the circle,  $BE = a + a\sqrt{2}$ , and  $BC = a\sqrt{6 + 4\sqrt{2}} = 2a + a\sqrt{21}$ , and  $BF = \frac{1}{2}BC$ , and  $DE = 2DF = \frac{1}{2}BC$ ; hence  $FD : FB :: DE^2 : BG^2$ , or  $1 : 3 :: \frac{1}{4}BC^2 : BG^2 = \frac{1}{4}BC^2$ , and  $BG = \frac{1}{2}BC\sqrt{3}$ . Consequently the area of the semiparabola, or  $\frac{3}{2}BF \times BG$ , is  $\frac{3}{2} \times \frac{1}{2}BC \times \frac{1}{2}BC\sqrt{3} = \frac{1}{4}BC^2\sqrt{3} = a^2 \times (\frac{3}{2}\sqrt{3} + \sqrt{6}) = 5.047a^2$ , as required.

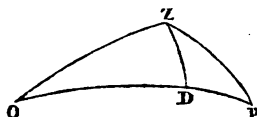


VII. QUESTION 869, by Mr. J. Williams, of Plymouth Dock.

On a certain day in the autumnal quarter, at 10 o'clock in the forenoon, in the latitude of  $50^\circ$  north, the sum of the sun's altitude and declination was  $34^\circ 40'$ . Required both the altitude and declination, or day the observation was made?

Answered by Mr. J. Howard, Teacher of the Mathematics, Carlisle.

Let  $z$  be the zenith,  $p$  the pole, and  $o$  the sun's place; then we have given  $zp = 40^\circ$  the co-latitude, also  $zo$  being the co-altitude, and  $po$  the co-declination, the altitude is  $90 - zo$ , and the declination  $po - 90$ , the sum of which is  $po - zo = 34^\circ 40' = pd$ , by taking  $od = oz$ . Then in the triangle  $zdp$ , are given  $pz$ ,  $pd$ , and  $\angle p = 30^\circ$ , to find the side  $dz$  and  $\angle pdz$ , the supplement of which is the  $\angle odz = \angle ozd$ . Then in the isosceles triangle  $ozd$  are given the base  $zd$ , and the angles at the base, to find  $oz$  or  $od = 67^\circ 31'$  the co-altitude, consequently  $po = pd + do = 102^\circ 11'$ , and then  $po - 90 = 12^\circ 11'$ , the declination south, answering to Oct. 25th.



The same by Mr. Richard Waugh, Bushblades, Durham.

Let  $a$  and  $b$  be sine and cosine of  $50^\circ$ ,  $c$  and  $d$  the sine and cosine of  $34^\circ 40'$ ,  $m$  = cosine of  $30^\circ$  the  $\angle p$ , and  $x$  and  $y$  the sine and cosine of the declination. Then  $cy - dx = \text{cosine } zo$ ; and, by a well-known theorem in spherics,  $bmy - ax = cy - dx$ ; hence  $\frac{x}{y} = \frac{c - bm}{d - a} = \text{tangent of } 12^\circ 8'$  the declination, answering to the 25th of October. Also the altitude  $= 22^\circ 32'$ .

VIII. QUESTION 870, by Jacobus de Viredi Sylva.

Suppose I throw a stone into a well, and that I observe a pendulum of 12 inches long make 20 vibrations from the moment of dropping the stone to the return of the sound from the bottom to my ear. Required the depth of the well?

Answered by the Rev. L. Evans, of Hungerford.

The times in which pendulums make an equal number of vibrations being as the roots of their lengths, we have  $\sqrt{39\frac{1}{2}} : \sqrt{12} :: 20 : 11.06$  the time in which a pendulum of 12 inches makes 20 vibrations. Now let  $x = \text{depth of the well}$ ,  $a = 16\frac{1}{2}$  feet the space through which a heavy body falls in the first second of time,  $b = 1142$  feet the space through which sound passes in the same time,  $c = 11''.06$  the time given from the first descent of the stone to the hearing of the

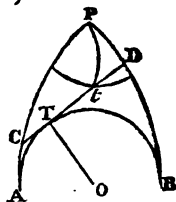
sound. Then, by the law of falling bodies  $a : x :: 1' : x \div a$ , therefore  $\sqrt{(x \div a)}$  = the time of the stone's descent ; and  $b : x :: 1'' : x \div b$  = time of the sound's ascent ; wherefore  $\frac{x}{b} + \sqrt{\frac{x}{a}} = c$ ; and hence  $x = (b^2 + 2abc - b\sqrt{(b^2 + 4abc)}) \div 2a = 1521 \cdot 341$  feet, the depth.

IX. QUESTION 871, by Mr. Isaac Dalby.

In a spherical triangle there are given the vertical angle, the perpendicular, and the perimeter or sum of the degrees in all the sides : to determine the triangle by stereographic projection.

*Answered by Amicus.*

If three tangents be drawn to the same given circle, their intersections will form a triangle, and if the intersection farthest from the centre be called the vertical angle, the tangent between it and the point of contact will be equal to half the perimeter of the triangle, if the triangle *does not* circumscribe the circle ; and to half the difference between the sum of the sides and base if it *does*. Which property holds equally in spherical triangles as in plane ones, and is too evident to need a particular demonstration here. This premised,



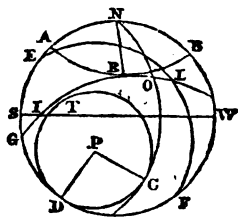
Let the great circles PA, PB, forming the given vertical angle at P, be drawn, so that AP = BP = half the given perimeter, and the less circle ATB touching them in A and B ; and round the pole P at a given distance therefrom = the given perpendicular of the triangle. draw the representation of another lesser circle ; then the representation of a great circle CD being drawn to touch those of the two lesser ones, will, from what is premised, form the representation CPD of the triangle required. Now the distances of the poles of the great circle CTD from each of the points of contact T and t must =  $90^\circ$  or a quadrant, the distance then from P = a quadrant — Pt, and from O the pole of the other lesser circle = a quadrant + To ; whence, by common spherical geometry, the representation of this pole, and consequently of the great circle CTD, are readily drawn. And that whether the projection be stereographic or orthographic.

*The same by Mr. Isaac Dalby.*

**Lemma.** If two great circles touch a lesser circle, the segments of the great circles between the points of contact and points of intersection on the same side of the lesser circle are equal. This is too evident to need a demonstration.

**Projection.** Make the  $\angle DNC$  = the given vertical angle ; take NC, ND each =  $\frac{1}{2}$  the perimeter, describe the arcs CP, DP perpendicular to NC, ND ; about P as a pole describe a lesser circle at the distance of PD

or PC, which, it is evident, will touch ND, NC in D and C : also about N as a pole describe a lesser circle ARB at a distance = the given perpendicular, and describe the great circles SW, ELF parallel to the lesser circles ARB, DTC respectively; then by prob. 21, Emerson's Stereog. Proj. describe a great circle GIOL to cut the great circle SW so that the  $\angle LIW = \text{comp. of NB or NA to } 90^\circ$ , and the great circle ELF so that the  $\angle GLE = \text{comp. of PD or PC to } 90^\circ$ ; and GNO will be the triangle required. For it is well known, that if a great circle be described to touch two lesser ones, it will cut their parallel great circles in angles = the complements of the lesser circle's distances from their poles, therefore the great circle GIOL will touch the lesser circles in T and R; hence by the foregoing lemma, the arc TG = GD, and TO = OC, therefore  $GT + TO + GN + NO = ND + NC = \text{the perimeter}$ , and describing the arc NR to the point of contact R, it will be perpendicular to the base GO, and = the given perpendicular by construction.

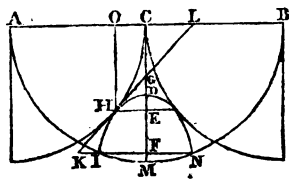


X. QUESTION 872, by Mr. Wm. Gooch, of Harlston School.

Suppose that on the diameter of a semicircle there be formed two quadrants, having their radii each equal to that of the semicircle, and their centres on the extremities of the said diameter, so that the arcs may meet each other in the centre of the semicircle. It is required to find the dimensions of the greatest parabola that can be inscribed in the curvilinear space formed by the arcs of the semicircle and the two quadrants?

Answered by Mr. James Young, of Pruddoe.

The figure being drawn as per question, &c. and LGHK a tangent to the curves at H; put the radius CA or CM =  $r$ , the ordinate HE = CO =  $y$ ; then is CE = OH =  $\sqrt{(2ry - y^2)}$ ; AO : AC :: OC : CL =  $ry \div (r - y)$ ; OL : CL :: OH : CG =  $CL \cdot CE \div OL$ ; hence GE = CE - CG =  $CO \cdot CE \div OL = EF$  when the parabola is a maximum by Simp. Max. and Min. hence CF = CE + EF =  $CE + CO \cdot CE \div OL = CE \times (2CO + CL) \div (CO + CL) = (3ry - 2y^2) \div \sqrt{(2ry - y^2)}$ ; and  $FI^2 = CM^2 - CF^2 = r^2 - (3ry - 2y^2) \div \sqrt{(2ry - y^2)}$ . But, by the nature of the parabola, DE = DG =  $\frac{1}{2}EF = \frac{1}{2}DF$  from above, therefore  $1 : 3 :: EH^2 : FI^2 = 3y^2$ . Consequently  $3y^2 = r^2 - (3ry - 2y^2) \div \sqrt{(2ry - y^2)}$ ; which equation reduced gives  $y^3 - 6ry^2 + 10r^2y - 2r^3 = 0$ ; the root of which is  $y = .79921r$  the base of the parabola, and its absciss DF =  $\frac{2}{3}GE = \frac{2}{3}(ry - y^2) \div \sqrt{(2ry - y^2)}$  is equal .416693r.



## XI. QUESTION 873, by Mr. George Sanderson, of London.

The shortest method, that I know of, for reducing the observed distance of the moon and sun, or moon and fixed star, to the true, by log. sines and tangents only ; is by the following rules :

*Rule 1.* To the apparent distance of the moon and sun, or moon and star, add the difference of their apparent altitudes, and take half the sum : also from the apparent distance subtract the difference of the apparent altitudes, and take half the remainder.

2. Add together the log. sines, of this half sum, of this half remainder, of the true zenith distances, and the arithmetical complements of those of the apparent zenith distances (or their log. cosecants) ; and take half their sum.

3. From this half sum of the six logs. subtract the log. sine of half the difference of the true zenith distances, and the remainder is the log. tangent of an arc ; the log. sine of which arc subtracted from the said half sum of the six logs. leaves the log. sine of half the true distance.

Required the investigation ?

*Answered by Mr. Geo. Sanderson, of London.*

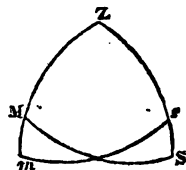
Let  $m$  and  $s$  represent the true places of the moon and star or sun, and  $m$  and  $s$  the apparent places of the same ; also  $z$  the zenith of the place. Put  $A$  = sine of the half sum, and  $R$  = sine of half the remainder, in rule 1 last Diary ;  $x$  = sine of half the true distance  $ms$  ;  $d$  = sine of half the diff. of the true zenith distances  $zm, zs$  ; and  $v$  = vers.  $\angle z$ . Then

$$v = \frac{A \times a}{\sin. zm \times \sin. zs} = \frac{\sin.(x+d) \times \sin.(x-d)}{\sin. zm \times \sin. zs}$$

by Sim. Trig. prop. 27, cor. 1.

Therefore  $\frac{A \times a \times \sin. zm \times \sin. zs}{\sin. zm \times \sin. zs} = (x+d) \times (x-d) = x^2 - d^2$ .

Put  $n$  = half sum log. sines of the factors in the numerator added to the arith. comp. log. sines of the factors in the denominator, from which subtract the log. of  $d$  ; then  $n - d = \log. \sqrt{(x^2 \div d^2 - 1)} = \log. \text{tang. of an arc whose secant is } x \div d \text{ to rad. 1, the log sine of which is the log. of } \sqrt{((x^2 - d^2) \div x^2)}$ , which put =  $l$  ; then  $n - l = \log. (\sqrt{(x^2 - d^2)} \div \sqrt{((x^2 - d^2) \div x^2)}) = \log. \text{ of } x$ , or log. sine of half the true distance, as was to be investigated.



## XII. QUESTION 874, by Mr. John Gould, of Spalding.

It is proposed to exhibit the fluent of  $(y\dot{x} - x\dot{y}) \times \dot{y} = ax^i$  in finite terms.

*Answered by Mr. Abel Whitehouse, Wolverhampton.*

Take the fluxion of the given equation, making  $\dot{y}$  constant, and by reduction it is  $2ax = y\dot{y}$ ; the fluents of which give  $4ax = y^2$ ; an equation to the parabola, the parameter being  $4a$ .

XIII. QUESTION 875, by Mr. Alex. Rowe, of Reginnis.

It is required to find the diameter of a circular parachute, by means of which a man of 200lb. weight may descend, from a balloon at a great height, with the uniform velocity of only 10 feet in a second of time. The parachute being supposed to be made of such materials, and thickness, that a circle of it of 50 feet diameter, weighs only 150lb.

*Answered by Mr. John Dalton, Kendal.*

If a falling body move with a uniform velocity, it must necessarily meet with a resistance, in the medium it is moving in, equal to its weight. Now it has been proved (Emerson's Mechan. prop. 108, cor. 3,) that the resistance to a cylinder, moving in a fluid in the direction of its axis, is equal to the weight of a cylinder of that fluid, of the same base, and its length equal to the height a body falls in vacuo to acquire its velocity. Put now  $g = 32\frac{1}{2}$  feet,  $v = 10$  the velocity, then  $g^2 : v^2 :: \frac{1}{2}g : v \div 2g$  the altitude fallen to acquire the given velocity  $v$ , which altitude call  $a$ ; put also  $p = .7854$ ,  $b = .075$ lb. the weight of a cubic foot of air,  $m = 200$ lb. the man's weight, also  $x =$  the diameter of the parachute. Then  $50^2 : x^2 :: 150 : \frac{3}{5}x^2$  the weight of the same, which added to 200 or  $m$ , must be equal to the resistance, namely  $abpx^2$  that is  $\frac{3}{5}x^2 + m = abpx^2$ ; and hence  $x =$

$$\sqrt{\frac{m}{abp - \frac{3}{5}}} = \sqrt{\frac{m}{bpv^2 \div 2g - \frac{3}{5}}} = \sqrt{\frac{10000}{375p \div 2g - 3}} = 100 \sqrt{\frac{2g}{375p - 6g}} = 80\frac{1}{2} \text{ feet nearly, the diameter sought.}$$

XIV. QUESTION 876, by Amicus.

If from three given points, not in the same right line, three lines be drawn to terminate in the same point, so, that the rectangle under two of them may be equal to the square of the third; to find how many, and what different species of curves, may be the loci of this terminating point; and under what particular variation of the positions of the given ones.

*Answered by Mr. John Farey, London.*

Let ABC be the three given points; join AB, which bisect in  $\kappa$ , and draw  $\kappa c$ , parallel to which draw  $\kappa B$ ,  $\Delta L$ ; also through  $\kappa$  draw  $\kappa I$  per-

pendicular to EC. Let P be a point in the curve, and join AP, BP, CP; also draw PL perpendicular to AL meeting EC in G.

Put  $EC = m$ ,  $KB = AI = n$ ,  $KE = EI = GL = r$ , the abscissa  $EG = z$ , and ordinate  $PG = v$ . Then  $(n + z)^2 + (v + r)^2 = AP^2$ ,  $(n - z)^2 + (v - r)^2 = BP^2$ , and  $(m - z)^2 + v^2 = CP^2$ . Hence by the question  $AP \times BP = CP^2$ , or  $\sqrt{((n + z)^2 + (v + r)^2)} \times \sqrt{((n - z)^2 + (v - r)^2)} = (m - z)^2 + v^2$ .

To reduce this equation to the Newtonian form, make  $z = x + \frac{m^2 + r^2 - n^2}{2m}$ ,

and  $v = y + \frac{rn}{m}$ ; then after proper reduction,

$$xy^2 + \frac{rn^3 - r^3n - m^2rn}{m^3} \times y = -x^3 + \frac{2n^3 - 2r^3}{m} \times x^2 + \frac{2m^3n^3 - 2m^3r^3 - m^4 - 5r^4 - 5n^4 + 14r^3n^2}{4m^3} \times x + \frac{m^4n^3 - m^4r^3 - 2m^3n^4 - 2m^3r^4 + 4m^2n^3r^2 - 7n^4r^2 + 7n^2r^4 + n^6 - r^6}{4m^3};$$

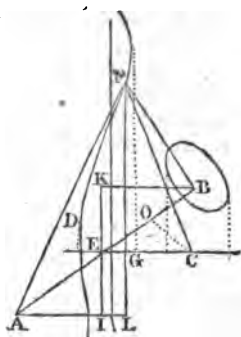
which equation reduced gives  $y = \frac{m^3nr + n^3r^2 - n^3r \pm \sqrt{c}}{2m^3x}$ , where  $c$  is the biquadratic equation under the vinculum. When  $c=0$ , the four roots or values of  $x$  are  $\frac{-r^2}{m}$ ,  $\frac{n^2 - r^2 \pm m\sqrt{(2n^2 - m^2 - 2r^2)}}{2m}$ , and

$\frac{n^2}{m}$ . From which it is evident that if  $\frac{1}{2}m^2 + r^2$  be less than  $n^2$ , the curve will be Sir I. Newton's 33d species, as in the figure; if  $\frac{1}{2}m^2 + r^2 = n^2$ , it will be the 34th species; and if  $\frac{1}{2}m^2 + r^2$  be greater than  $n^2$ , it will be the 37th species. Also if  $r = n$ , it will be the 38th species; if  $n^2 = m^2 + r^2$ , it will be the 40th species; and lastly if  $n = 0$ , it will be the 45th species.

*Corollary.* When  $r = 0$ , (or the three points are in the same right line, as in quest. 696), and if  $m^2$  be less than  $2n^2$ , the curve will be the 39th species; if  $m^2 = 2n^2$ , it will be the 41st species; if  $m = n$ , it will be a right line with a conjugate point; if  $m^2$  be greater than  $2n^2$ , it will be the 45th species; and lastly if  $m = 0$ , it becomes the conic equilateral hyperbola.

*Mr. George Sanderson, after giving the Solution nearly as above, makes the following observations:*

If D be the centre of a circle passing through the given points, and co perpendicular to AB (c being always considered as the point from which the mean proportional is drawn); then if co be less than a third



proportional to  $2AD$  and  $EB$ , the curve will be Newton's 33d species ; if  $co$  be equal to the 3d proportional, it will be the 34th species ; and if  $co$  be greater than the same, it will be the 37th species. If  $co = eo$ , it will be the 38th species ; and if  $eo = 0$ , or  $ac = cb$ , the 45th species ; the asymptote being parallel to  $AB$ , and its distance from  $e$  equal to  $AD$ .

If  $co = 0$ , or the three points in the same right line ( $AB$ ), the asymptote is perpendicular to  $AB$  ; and if  $ec^2$  be greater than  $2EB^2$ , the locus of the point  $P$  is still the 45th species ; if  $ec^2 = 2EB^2$ , the 41st species ; and if it be less, the 39th. If  $ec = EB$ , the asymptote passes through  $e$ , and is the locus of the point  $P$ , and  $B$  is a conjugate point. Lastly, if  $c$  fall in the point  $E$ , or  $ac = cb$ , it becomes a conic equilateral hyperbola whose foci are  $A$  and  $B$ .

#### PRIZE QUESTION, by Plus Minus.

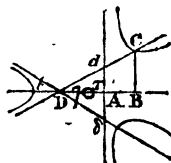
In the year 1717 Mr. Stirling published an illustration of Sir I. Newton's *Enumeratio Linearum tertii Ordinis*, and in it added four new curves to the catalogue, viz. 1. A redundant hyperbola with one diameter, and consisting of an inscribed and two ambiguous hyperbolas with an oval. 2. The same with a conjugate point. 3. A redundant hyperbola with three diameters and an oval. 4. The same with a conjugate point. After this Mr. E. Stone published his *Mathematical Dictionary* ; and in it, under the article *curves*, he says that he himself has discovered two more curves of this order, omitted by Newton, amongst the deficient hyperbolas denoted by the equation  $xy^2 = bx^3 + cx + d$ , viz. 1. When the equation  $bx^2 + cx + d = 0$  has two unequal negative roots. 2. When it has two equal negative roots. As no notice has been taken of any of these discoveries in the future editions of the *Enumeratio*, it is required to determine whether they have any existence or not, and if they have, to give an example of an equation for each.

*Answered by Mr. George Sanderson, London.*

*Lemma.* If  $-m$ ,  $-n$ ,  $-r$  be the three roots or values of  $x$  in the equation  $ax^3 + bx^2 + cx + d = 0$ ; then 1st, if  $r = m + n \pm 2\sqrt{mn}$ , then is  $b^2 = 4ac$ ; 2d, if  $r$  lie without  $m + n + 2\sqrt{mn}$  and  $m + n - 2\sqrt{mn}$ , that is greater than the former or less than the latter, then  $b^2$  is greater than  $4ac$ ; 3d, if  $r$  lie between  $m + n + 2\sqrt{mn}$  and  $m + n - 2\sqrt{mn}$ , then  $b^2$  is less than  $4ac$ ; 4th, in the first case, where  $r = m + n \pm 2\sqrt{mn}$ , if  $m = 4n$ , the two least roots,  $n$  and  $r$ , are equal. 5th, if the two least roots be equal, and the third greater than 4 times the least, (or equal root), then  $b^2$  is greater than  $4ac$ . Lastly, in either the first or second cases, namely when  $r$  does not fall between  $m + n \pm 2\sqrt{mn}$ , then half the sum of the roots is less than the greatest of the three. All the cases of this lemma will be evident to every one who

considers, that in the equation  $x^3 + px^2 + qx + r = 0$ ,  $p$  is equal to the sum of the roots, their signs being changed,  $q$  the sum of the products of every two, &c.

*Solution.* Figure 21 to Sir I. Newton's 15th species, consists of two ambigenous hyperbolas at  $d$  and  $\delta$ , and one inscribed at  $\nu$ , as in the annexed fig. but without oval or conjugate point. Here it appears that Sir Isaac considered the two least



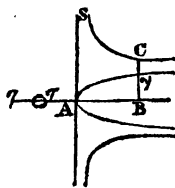
roots ( $Ar$ ,  $At$ ) as impossible when  $\frac{4ad}{b^2 - 4ac} = AB$

$= x$  (the corresponding abscissa to the ordinate  $BC$  at the point  $c$  where the asymptote cuts the curve) is affirmative, or when  $b^2$  is greater than  $4ac$ . But whoever considers the lemma, will find that examples are numerous in which the equation  $ax^3 + bx^2 + cx + d = 0$  has three real and unequal negative roots, (or three real negative roots with the two least equal), and  $b^2$  greater than  $4ac$ , or  $\frac{4ad}{b^2 - 4ac}$  a positive quantity, also  $AD$  less than the greatest of the three roots; and consequently the curve or locus of the equation  $y = \pm$

$\sqrt{\frac{ax^3 + bx^2 + cx + d}{x}}$  consists of an inscribed hyperbola at  $\nu$ , and two ambigenous ones at  $d$ ,  $\delta$ , with an oval in the triangle  $\nu d\delta$ , or the same with a conjugate point. Here the two greater roots cannot become equal, as in the 11th species. Whence it is manifest that these two species should follow Newton's 14th, so as to make the 15th and 16th, and not the 11th and 15th as Mr. Stirling has placed them at pages 99 and 100.

By the first case of the lemma examples may be found in which the equation  $ax^3 + bx^2 + cx + d = 0$  has three real unequal roots (or three real roots with the two least ones equal), and  $b^2 = 4ac$ , and  $AD$  less than the greatest root  $At$ . Consequently  $AB = x = \frac{4ad}{b^2 - 4ac}$  will be infinite; or the asymptote cut the curve at an infinite distance; and the figure consists of three inscribed hyperbolas with three diameters, and an oval in the triangle  $\nu d\delta$ , or the same with a conjugate point. If the two least roots be impossible, the curve becomes Newton's 22d species; and therefore these two should make the 24th and 25th species, as Stirling has placed them at page 102.

As to Mr. Stone's discoveries; I find that Sir I. Newton's 59th figure to his 55th species is the locus of the equation  $xy^2 = bx^2 - cx + d$ ; where the two roots or values of  $x$ , when  $y = 0$ , or  $bx^2 - cx + d = 0$ , are both positive. But as no notice is there taken of the locus of the equation  $xy^2 = bx^2 + cx + d$ , let us describe it here; taking for an example the numeral equation



$xy' = 4x' + 56x + 160$  or  $y = \pm \sqrt{\frac{4x' + 56x + 160}{x}}$ , where the two roots or values of  $x$  when  $y = 0$ , or  $4x' + 56x + 160 = 0$ , are  $-4$ , and  $-10$ .

Draw two indefinite right lines  $As$  and  $As$  cutting each other at right angles in  $A$ , (fig. 2). Let  $A$  be the beginning of the abscissa  $As$ , and  $As$  the first ordinate; then if  $As$  be represented by  $+x$ , and  $ac$  its corresponding ordinate by  $y$ ; on  $As$ , but on the contrary side of  $A$ , take  $Ar = -4$ , and  $A\ell = -10$ . If  $x = 0$ , then  $\pm ac$  is infinite, and the curve runs on infinitely towards  $s$ ; therefore  $As$  is an asymptote to two hyperbolic legs equally distant from  $As$ . If  $+x$  be infinite, then  $ac$  or  $y = \pm \sqrt{\frac{4x' + 56x + 160}{x}}$  becomes  $y = 2\sqrt{x}$ , or  $y^2 = 4x$ , an equation to the conic parabola, whose vertex is  $A$ , and parameter  $4$ : whence the curve has two parabolic legs joined to two hyperbolic ones, meeting the parabola at an infinite distance towards  $s$ . If  $-x$  be taken  $= -5$ , then  $y = \pm 2$ : if  $-x$  be taken  $= -4$  or  $-10$ , then  $y = 0$ , and the curve passes through the points  $r$  and  $\ell$ : if  $-x$  be taken less than  $-4$ , or greater than  $-10$ , then  $y = \pm \sqrt{\frac{4x' - 56x + 160}{-x}}$  it is impossible, because the quantity under the vinculum is negative, and no part of the curve can fall between  $As$  and  $r$ , or beyond  $\ell$ ; therefore the part of the locus lying on that side of  $As$  is an oval, and the curve consists of two hyperbolic-parabolic figures on one side of the asymptote  $As$ , with an oval on the contrary side.

If the two roots be equal, the oval becomes a conjugate point; which is Stone's second species. If the two roots be impossible, the curve becomes Newton's 57th figure to his 53d species; and therefore Mr. Stone's two should be the 57th and 58th species in the enumeration, the catalogue being deficient without them; that is, they should be the 53d and 54th in Newton's, and the 57th and 58th species in Stirling's enumeration.

*The same answered by Amicus.*

$As$ , in fig. 2d of Sir Isaac Newton's 1st species, where  $AD = -b \div 2a$  is greater than the greatest root of the equation for the limits, when the conic hyperbola that bisects the ordinates of the curve, coincide with its asymptotes, and the terms  $cy$  of consequence is wanting,  $r$  coincides with  $\omega$ ,  $\ell$  with  $\pi$ , and  $t$  with  $p$ , and that fig. 2d becomes the same with fig. 17th, or the 10th species, and when the oval vanishes, or the two less limits are equal, it becomes fig. 20, and species 13th: So, in fig. 1st of the 1st species, where  $AD$  is less than the greatest root  $Ap$ , when the conic hyperbola coincides with its asymptotes, it is evident without farther illustration, that the hyperbola whose vertex is  $t$  must be wholly within the asymptotes  $Dd$ ,  $dd$ , whilst the other two cut them (as in fig. 21st) and the oval still remains within the triangle  $dd\delta$

(as in the 10th species) bisected by the diameter  $AD$ ; and this is Mr. Stirling's 11th species: and when, the two less limits being equal, the oval vanishes into a conjugate point, it is his 16th, and the same in appearance with Sir Isaac's fig. 21st, 15th species. And when, in the limiting equation  $ax^3 + bx^2 + cx + d = 0$ ,  $4ac \pm b^2$ , or the curve has three diameters, it is manifest that the only difference this can make, will be, that the hyperbolas  $c$  and  $r$  will be wholly within the angles at  $d$  and  $\delta$ , as  $t$  is within  $D$ ; the oval must remain so long as the two less roots of this equation are real and unequal, which is Mr. Stirling's 24th species, when they are equal the oval becomes a conjugate point, and this is his 25th species, and the appearance as in Sir Isaac's fig. 28th, species 22d.

As to the curves expressed by the equation  $xy^2 = bx^2 + cx + d$ , when it takes the form  $xy^2 = c \times (x^2 + (b + d) \cdot x + bd)$ ; the first discovery of them is not due to Mr. Stone, for there is not one word about them in the first edition of his Dictionary, but to Mr. Nic. Bernoulli, who died in 1726, the year in which that edition was published. They consist of two hyperbolo-parabolic curves like those in Sir Isaac's 57th fig. with an oval on the contrary side of the asymptote, if  $b$  and  $d$  be unequal; but with a conjugate point only, if they be equal. For, in the equation  $xy^2 = c \times (x + b) \times (x + d)$ ,  $y$  is evidently equal 0, both when  $x = -b$ , and when  $x = -d$ .

*Questions proposed in 1788, and answered in 1789.*

I. QUESTION 878, *by* Mr. George Beswick, *Coalshaw Green.*

A beautiful couple had lately been ty'd,  
The groom was right lusty, and lovely the bride;  
A simple equation will easily shew  
The age of this couple, as noted below.

$2x^3 - x^4 - x^2 + 2x^2y - y^2 = \sqrt{x^2y} - \sqrt{x^3y} = 2xy$ , where  
 $x + y$  and  $y$  denote their ages in years.

*Answered by* Mr. George Beswick.

In the given equation  $2x^3 - x^4 - x^2 + 2x^2y - y^2 = 2xy$ , transpose all the terms to one side, and its square root will be  $x^3 - x - y = 0$ , or  $x^3 - x = y$ ; by which divide the equation  $\sqrt{x^2y} - \sqrt{x^3y} = 2xy$ , or  $(x^3 - x) \times \sqrt{xy} = 2xy$ , and the quotients give  $\sqrt{xy} = 2x$ , or  $y = 4x$ ; therefore the two values of  $y$ , viz.  $x^3 - x = 4x$ ; hence  $x - 1 = 4$ , or  $x = 5$ ; and consequently  $y$  or  $4x = 20$ . Therefore the ages are 25 and 20 years.

*The same by* Mr. George Stevenson.

The third quantity transposed to the same side with the first, gives

$(x^2 - x)^2 - 2y \times (x^2 - x) + y^2 = 0$ , which is evidently a square, and its root is  $x^2 - x - y$ , therefore  $(x - 1) \cdot x = y$ . And the latter of the given equations gives  $(x - 1) \times \sqrt{xy} = 2y$ ; this squared, and divided by  $y$ , gives  $(x - 1)^2 \cdot x = 4y$ ; therefore by substitution  $(x - 1)^2 \cdot x = (x - 1) \cdot 4x$ ; hence  $x - 1 = 4$ ; and  $x = 5$ . And the ages 20 and 25.

*The same by Mr. John Craggs, of Hylton.*

By transposing the 3d quantity, and extracting the root, it is  $y = x^2 - x$ . The second given equation squared, and divided by  $xy$ , gives  $x^4 - 2x^3 + x^2 = 4xy$ ; which added to the 1st and 3d quantities, gives  $y = 2x^2 - 6x$ . Consequently  $2x^2 - 6x = x^2 - x$ ; hence  $2x - 6 = x - 1$ , or  $x = 5$ ; and therefore  $y = 20$ . And the ages 20 and 25.

II. QUESTION 879, by Mr. T. Nield, *Writing Master, Hawarden.*

Measuring a small inclosure of a rectangular form, I observed that if 2 poles were added to the breadth and 5 to the length, the area would be increased by 430; but if 5 were added to the breadth and two to the length, it would be increased by 445. It is required from hence to find the length and breadth of the inclosure.

*Answered by Mr. Matt. Fleck, of Stella.*

Put  $x =$  the length, and  $y$  the breadth of the rectangular field; then is  $xy$  its area. Hence, by the question,  $(x + 5) \times (y + 2)$  or  $xy + 2x + 5y + 10 = xy + 430$ , and  $(x + 2) \times (y + 5)$  or  $xy + 5x + 2y + 10 = xy + 445$ . Or  $2x + 5y = 420$ , } The difference of these is  $3x - 3y = 15$ , and  $5x + 2y = 435$ . } or  $x - y = 5$ , and  $x = y + 5$ . This value taken for  $x$  in the equation  $2x + 5y = 420$ , gives  $7y + 25 = 435$ , and  $y = 58\frac{1}{2}$ ; hence  $x = 63\frac{1}{2}$ .

*The same, by Mr. Henry Tilney, junior.*

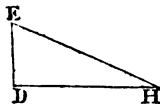
Let  $x$  and  $y$  be put for the length and breadth of the rectangle. Then per quest.  $(x + 5) \times (y + 2) = xy + 430$ , and  $(x + 2) \times (y + 5) = xy + 445$ . By comparing these two together, it appears that  $x = y + 5$ . Hence by substituting this value of  $x$  in the first equation, and reducing it we have  $y = 58\frac{1}{2}$ . Whence  $x = 63\frac{1}{2}$  poles.

III. QUESTION 880, by Mr. Jonathan Hornby, *Westerdale School.*

Observed in the spring quarter, in latitude 22 degrees north, when the sun was due east, the difference between his altitude and hour from 6, in degrees, a maximum. Query the time of observation?

*Answered by Amicus.*

In the right-angled spherical triangle EDH, DH is an arch of the equinoctial, measuring the hour from 6 when the sun is due east at E; EH the altitude, DE the declination, and DHE the latitude  $22^\circ$ . By Crackelt's translation of Mauduit's Trigonometry, page 68, prob. 3, as  $\cot. 11^\circ : \tan. 11^\circ :: s. (EH + DH) : s. (EH - DH)$ , therefore the sines of the sum and diff.



of EH and DH are in a given ratio; conseq. the greater the sine of the sum, the greater that of the diff. and of necessity the greater must the diff. itself be: but when the sum is a quadrant, its sine is the greatest possible; and therefore as  $\cot. 11^\circ : \tan. 11^\circ :: s. 90^\circ : s. (EH - DH)$  when a max. =  $2^\circ 10'$  *ferè*. Hence  $DH = 43^\circ 55'$ ,  $EH = 46^\circ 5'$ , and  $DE = 15^\circ 39'$  north declination, May 2, at  $55'$  past 8 A. M.

*The same by Mr. David Kinnebrook, junior.*

Let E be the sun's place at the time of observation, EH his altitude, DH part of the equator, DE the declination, then is the  $\angle H = 22^\circ$  the latitude of the place, whose cosine let =  $c$ , also the  $\tan. EH = x$ ; then per spherics,  $1 : c :: x : cx$  the  $\tan.$  of DH, whence the fluxion of the arc DH is  $cx \div (1 + c^2x^2)$ , and that of the arc EH is  $\dot{x} \div (1 + x^2)$ ; but the diff. of the said two arcs is, by the question, a maximum, consequently  $\dot{x} \div (1 + x^2) - cx \div (1 + c^2x^2) = 0$ ; hence  $x = \sqrt{(1 \div c)}$  the tangent of  $EH = 46^\circ 4' 58''$  the sun's altitude; and  $cx = \sqrt{c}$  the tangent of  $DH = 43^\circ 55' 2''$  the measure of the hour from 6; from whence by spherics the declination is found to be  $15^\circ 39' 18''$ , answering to May 2d, 8h. 55m. 40s. in the morning.

*The same by Mr. William Simpson, junior.*

Let E be the sun when due east,  $\angle H$  the latitude, DE the sun's declination, and DH the time from 6. By table 1, page 280, Simpson's Fluxions,  $EH : DH :: \cos. DE : \sin. \angle E$ . But when EH — DH is a max. then  $\dot{EH} = \dot{DH}$ , consequently  $\sin. \angle E = \cos. DE$ . By spherics, radius :  $\cos. DE :: s. \angle E = \cos. DE : \cos. \angle H$ , or radius  $\times \cos. \angle H = \cos^2. DE$ . Therefore when radius = 1,  $\cos. DE = \sqrt{\cos. H} = .9629040$  the  $\cos.$  of  $15^\circ 39' 18''$  the sun's declination answering to May 2. Also radius :  $\cot. \angle H :: \tan. DE : \sin. DH = 43^\circ 55' 2''$ . Hence the observation was made at 8h. 55m.  $4'' 8'''$ .

#### IV. QUESTION 881, by Mr. Timothy Simpson, Papplewick.

A certain gamester is willing to take the odds to a guinea, that he, with 9 halfpence, brings up 3 heads precisely, 4 times in 5 throws: what ought the odds to be?

*Answered by Mr. Alex. Rowe, of Reginnis.*

The probability that 3 heads precisely out of 9 halfpence at one throw, or, which is the same, that one halfpenny comes up a head precisely 3 times in 9 throws, by prob. 5, Simpson's Laws of Chance, is  $\frac{9 \cdot 4 \cdot 7}{3 \cdot 2^9} = \frac{21}{128}$ , and therefore that of the contrary is  $\frac{107}{128}$ . And, by

the same problem, the probability that it happens just 4 times in 5 trials, is  $5 \times \frac{4}{2} \times \frac{3}{3} \times \frac{2}{4} \times \frac{107 \cdot 21}{128^3} = \frac{104047335}{34359738368}$ . So that the odds are as 34255691033 to 104047335, or nearly as 329 guineas and  $\frac{1}{4}$  to one.

*The same, by Mr. James Ashton, of Harrington.*

The 9th power of 2, or  $2^9 = 512$  are all the chances; and it appears by the binomial theorem that there are  $\frac{84}{512}$  and  $\frac{428}{512}$ , or  $\frac{21}{128}$  and  $\frac{107}{128}$  chances respectively for and against 3 heads precisely at one throw. Put  $a = 21$ ,  $b = 107$ ,  $n = 5$ , and  $t = 4$ ; then, in the series  $a^n + 5a^4b$ , &c.  $5a^4b$  is the term in which the index of  $a$  is  $t$ ; therefore  $\frac{5a^4b}{(a+b)^n} = \frac{104047335}{34359738368}$  is the probability of happening precisely 4 times in 5 throws. Therefore the odds against the gamester are 34255691033 to 104047335, or 329 guineas, 4s. 10 $\frac{1}{2}$ d. to 1 guinea.

V. QUESTION 882, by Mr. Matthew Terry, *Land Surveyor.*

What must be the length of a pendulum vibrating seconds at the distance of 4 radii from the earth's centre?

*Answered by Mr. John Dalton.*

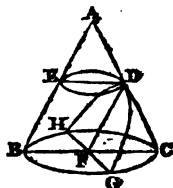
The lengths of pendulums are as the forces of gravity, and the squares of the times of their vibration. If, therefore, the times be constant, the lengths will be simply as the forces. And as gravity decreases in the inverse ratio of the square of the distances from the earth's centre; therefore its force at the distance of 4 radii, will be  $\frac{1}{16}$  of that at the surface, and consequently the length of the pendulum  $\frac{1}{16}$  of its length at the surface  $= \frac{1}{16}$  of  $39\frac{1}{8} = 2.445$  inches, or  $2\frac{1}{2}$  nearly.

VI. QUESTION 883, by Mr. Thos. Truswell, of Nuneaton.

If a given cone, whose altitude is 10, and base 8, be cut by two planes, the one of which is parallel to the side, and the other to the base; required the point in the side where the two planes meet, when the area of their sections are equal to each other.

*Answered by Mr. George Brown, of Newcastle.*

Let  $ABC$  be the cone,  $DE$  the diameter of the circular section, parallel to  $BC$ , and  $DFG$  half the parabolic section parallel to the side  $AB$ . Now  $BC$  being 8, and the perpendicular altitude 10, therefore  $AB^2$  or  $AC^2$  is  $\approx 116$ . By the nature of the circle,  $\sqrt{BF \cdot FC} = FG$  half the base of the parabola; and, by similar triangles  $BC : BA :: FC : FD = FC \cdot AB \div BC$  its altitude; therefore  $\frac{2}{3}FG \cdot FD$  or  $\frac{2}{3}\sqrt{BF \cdot FC} \cdot FC \cdot AB \div BC$  is the area of the parabolic section. And  $p \cdot DE^2$  or  $p \cdot BF^2$  is the area of the circular section, where  $p = \cdot 7854$ . Therefore by the question  $\frac{2}{3}\sqrt{BF \cdot FC} \cdot FC \cdot AB \div BC = p \cdot BF^2$ . Hence by squaring, &c. it is  $16AB^2 \cdot FC^2 = 9p^2 \cdot BC^2 \cdot BF^2$ , or  $BF^2 : FC^2 :: (4AB)^2 : (p \cdot 3BC)^2$ ; and hence  $BF : FC$  or  $AD : DC :: \sqrt[3]{16AB^2} : \sqrt[3]{p^2 \cdot 9BC^2} :: \sqrt[3]{29} : \sqrt[3]{9p^2} :: 1 \cdot 7351 : 1$ . Hence then  $BE = 5 \cdot 075$ ; and  $FC = 2 \cdot 925$ .



*The same answered by Mr. Joseph Peace.*

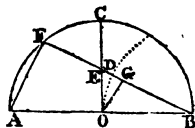
Put  $p =$  the perpendicular,  $b =$  the base  $BC$ ,  $s = AB$  the slant side of the cone,  $a = \cdot 7854$ , and  $x = DE$  or  $BF$  the diameter of the circular section. Then  $2\sqrt{(b-x) \cdot x} = 2\sqrt{(b-x) \cdot x} = EH$  the base of the parabola; and by similar triangles  $b : s :: b - x : (b-x) \cdot s \div b = DH$  its altitude. Therefore by the question  $2\sqrt{(b-x) \cdot x} \times (b-x) \times \frac{2}{3}s \div b = ax^2$ . Hence by reduction is found  $x = b \div (1 + \sqrt[3]{(9a^2b^2 \div 16s^2)}) = 5 \cdot 075$  nearly.

VII. QUESTION 884, by Mr. John Dalton, of Kendal.

In the semicircle  $ACB$ , whose diameter is  $AB$ , and  $OC$  perpendicular to it from the centre, from  $B$  there is drawn a chord  $BF$  to cut  $OC$  in  $E$ , and on the same chord there is taken  $BD$  equal to the radius of the semicircle; it is required to determine the rectangle  $DE \cdot EF$ , a maximum?

*Answered by Mr. John Cullyer, of Hingham.*

Let the figure be drawn as per question, and join  $AF$ . Assume the radius  $BO$  or  $BD = 1$ , and put  $x =$  sine of  $\angle BEO$  or  $BAF$ ; then is  $BF = 2x$ , and as  $x : 1 :: BO$  or  $1 : 1 \div x = BE$ . Hence  $EF = 2x - 1 \div x$ , and  $ED = 1 \div x - 1$ ; and consequently  $EF \cdot ED$  or  $(2x - 1 \div x) \times (1 \div x - 1)$  must be a maximum. This being put into fluxions, and reduced, there arises this cubic equation  $x^3 + \frac{1}{2}x = 1$ , the root of which is  $x = \cdot 835122$ , the sine of  $56^\circ 38'$  the  $\angle BEO$  or  $BAF$ ; and therefore  $BE = BO \div \cdot 835122 = 1 \cdot 197480$ .





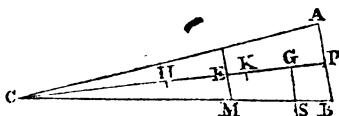
Put  $m = \frac{3b^2 + 2ab + a^2}{b^2 + ab + a^2}$ ; then  $\frac{ac + dx}{\sqrt{x^2 - c^2}} = \frac{m\sqrt{x^2 - c^2}}{4}$

which equation reduced, and the root found, it gives  $x = \frac{2d}{m} +$

$\sqrt{\left(c + \frac{4ac}{m} + \frac{4d^2}{m^2}\right)} = 22.5658 = \text{AH, and hence } \sqrt{(x^2 - c^2)}$   
 $= 22.415738 = \text{BG, the length of the frustum required.}$

*The same answered by Mr. John Dalton.*

It is evident that the block can only rest in equilibrio when its centre of gravity is supported; that is, when the needle produced would pass through the said centre; and that its under surface must also be parallel to the horizon.



Now to find the centre of gravity, put the perpendicular height of the whole pyramid  $= x$ ; then from the data will be the frustum's length  $= \frac{1}{3}x$ , and the solidities or weights of the frustum and remaining part, will be as 19 to 8. Then if on CP there be taken CH  $= \frac{1}{2}CE = \frac{1}{2}x$ , and CK  $= \frac{1}{4}x$ , the points H and K will be the centres of gravity of the upper part and whole frustum respectively; and then, by mechanics, 19 : 8 :: HK or  $\frac{1}{4}x$  : KG  $= \frac{19}{8}x$  the distance of the centre of gravity of the frustum from that of the whole pyramid; whence CK + KG = CG  $= \frac{47}{8}x$ . Again,  $PE^2 = 9^2 - \frac{1}{4} \text{ of } 9^2 = 60\frac{1}{4}$ ; therefore the slant side CA  $= \sqrt{(60\frac{1}{4} + x^2)}$ , and CS the distance of the needle from the vertex  $= CM + MS = CS = \frac{2}{3}\sqrt{(60\frac{1}{4} + x^2)} + 12$ ; and hence, by similar triangles, CP : CA :: CS : CG  $= (60\frac{1}{4} + x^2 + 18\sqrt{(60\frac{1}{4} + x^2)}) \div \frac{3}{2}x$ . This value being equated with that found above, and reduced, gives  $x$  in a quadratic; and when solved  $x = 67.2$ . Whence the length of the block  $= 22.4$  inches.

IX. QUESTION 886, by Mr. Thomas Todd, of Darlington, late of West Smithfield, London.

If the debts  $a, b, c, d$ , &c. in pounds, payable at the end of  $n, n', n'', n''',$  &c.  $t$  years, &c. Then I say the equated time from the first term, by compound interest, will be equal to the difference of the logarithm, of the sum of the debts, and logarithm of the sum of the present worths divided by the logarithm of one pound and its interest for one year; whether it be computed by the old method, by Kersey's, or by Malcolm's; all giving the same answer. Query the investigation by each method?

*Answered by Amicus.*

Let  $t =$  the time wherein a sum  $P =$  all the present worths would

amount to  $s$  = the sum of all the debts, and  $r$  = the amount of 1 pound in one year; then by the nature of compound interest  $rx^t = s$ , and consequently by the nature of logarithms  $t = (1.s - 1.r) \div 1.r$ , whence the whole is manifest.

*The same answered by Mr. Thomas Todd.*

*(Inserted verbatim, as the author desired).*

If  $x$  = time from the first term to the equated time, then  $t - x$  = time from the equated time to the last term,  $s$  = sum of debts and  $r$  = sum of all the present worths, and the rest of the notation as given in the question; then by the old method, we have  $(a + b + c \&c. + m) \times r^{t-x} = ar^{t-n} + br^{t-n'} + cr^{t-n''} + \&c. + m$ , the whole divided by  $r^{t-x}$ , transforms it into Mr. Kersey's method,  $a + b + c + \&c. + m = r^x \times ar^{-n} + br^{-n'} + cr^{-n''} + \&c. + mr^{-t}$ , or  $rx^x = s$ , therefore  $x = \frac{\log. \text{ of } s - \log. \text{ of } r}{\log. \text{ of } r}$  the time sought. And lastly, by Malcolm's method, we have the sum of interests  $ar^{x-n} - a + br^{x-n'} - b + \&c. = \text{sum of discounts } m - \frac{m}{r^{t-x}} (m - mr^{x-t}) + \&c. \text{ which}$  by transposition, gives  $ar^{x-n} + br^{x-n'} + cr^{x-n''} + \&c. + mr^{x-t} = a + b + c + \&c. + m$ , or  $rx^x = s$ , therefore  $r^x = \frac{s}{r}$  therefore  $x = \frac{\log. \text{ of } s - \log. \text{ of } r}{\log. \text{ of } r}$  the very same as given above.

*The same answered by Mr. John Dalton. (From the Supplement).*

The rule in this question is only Kersey's in a form applicable to compound interest. For let  $n' - n = m$ ,  $n'' - n = p$ ,  $n''' - n = q$ ; also  $r$  = amount of one pound for one year, and  $t$  = the equated time from  $n$ ; then, by Kersey's rule, we get this equation,

$$\left(a + \frac{b}{r^m} + \frac{c}{r^p} + \frac{d}{r^q}\right) \times r = a + b + c + d; \text{ whence } r^t = (a + b + c + d) \div \left(a + \frac{b}{r^m} + \frac{c}{r^p} + \frac{d}{r^q}\right) \text{ wherefore their logarithms will be equal also; but } \log. \text{ of } r^t = t \cdot \log. r, \text{ and } \log. (a + b + c + d) \div \left(a + \frac{b}{r^m} + \frac{c}{r^p} + \frac{d}{r^q}\right) = \log. (a + b + c + d) - \log. \left(a + \frac{b}{r^m} + \frac{c}{r^p} + \frac{d}{r^q}\right), \text{ consequently } t = ((\log. a + b + c + d) -$$

$\log. \left( a + \frac{b}{r^m} + \frac{c}{r^p} + \frac{d}{r^q} \right) \div \log. r$ , which is the rule in the question.

The truth of Malcolm's rule, which determines the equated time, from the supposition, that the interest on the debts foreborn, should be equal to the discount of the debts not due, is too evident to be here insisted on; and that Kersey's rule will, in this case, agree with this supposition, may be thus proved: Let the term of  $t$ , as found by this rule, fall between the terms of  $m$  and  $p$ , then will  $a + \frac{b}{r^m} + \frac{c}{r^p} + \frac{d}{r^q}$  + their interest for the time  $t = a + b + c + d$ , but the amount of  $\frac{b}{r^m}$  for the time  $t = b$  + its interest for the time  $t - m$ , and the

amounts of  $\frac{c}{r^p}$  and  $\frac{d}{r^q}$  at the term of  $t =$  the then present worths of

$c$  and  $d = c + d$  — their discount from the term of  $t$ ; wherefore, as  $a + b$  + their interest at the term of  $t + c + d$  — their discounts from that term  $= a + b + c + d$ , it follows that the interests of the two former are equal to the discounts of the two latter, which agrees with Malcolm's supposition. The common method will also give the same result on Moreland's principle, but not on Hatton's or Cocker's: for, putting  $v$  for the distance of the equated term, from that when the debt last payable becomes due, we have, on his supposition,  $(a + b + c + d) \cdot r^v = ar^q + br^{q-m} + cr^{q-p} + d$ , and dividing both sides by  $r^v$ ,  $a + b + c + d =$

$$\frac{ar^q + br^{q-m} + cr^{q-p} + d}{r^v} = \left( a + \frac{b}{r^m} + \frac{c}{r^p} + \frac{d}{r^q} \right) \cdot r^{q-v(t)}, \text{ an}$$

expression the same as that deduced from Kersey's rule above.

*Mr. Todd afterwards repropoed this question in the Scientific Repository; to which he gave the following solution.*

If  $x =$  years from the first term the debts are owing from to the equatement,  $t - x =$  years from the equatement to the last debt,  $s =$  sum of the debts,  $p =$  sum of the present worths, and the rest of the notation as given in the question; then by the true principle of the old method (not those of Cocker's nor Hatton's) we have the amount  $(a + b + c + d + \&c. + m) \times r^{t-x} = ar^{t-x} + br^{t-x} + cr^{t-x} + dr^{t-x} + \&c. + m$ , this equation divided by  $r^{t-x}$  transforms it to Kersey's method, viz.  $a + b + c + d + \&c. + m = r^x \times (ar^{-x} + br^{-x} + cr^{-x} + dr^{-x} + \&c. + mr^{-x})$ ; whence  $pr^x = s$ , and therefore  $r^x = s \div p$ , therefore  $x = (\log. \text{ of } s - \log. \text{ of } p) \div \log. \text{ of } r$ , the time (or rule given in the question).

And lastly by Malcolm's method, the sum of all the interests  $ar^{x-n}$   
 $- a + br^{x-n'} - b \&c. = m - \frac{m}{r^{t-x}}$ , or  $m - mr^{x-t} +$  all the rest  
 of the discounts; now by transposing all the discounts to the contrary  
 side of the equation, they will be subtracted from all the interests, and  
 the equation will be equal to 0, by Malcolm's method, viz.  $ar^{x-n}$   
 $- a + br^{x-n'} - b + cr^{x-n''} - c + dr^{x-n'''} - d + \&c. + mr^{x-t}$   
 $- m = 0$ , and therefore  $r^x \times (ar^{-n} + br^{-n'} + cr^{-n''} + dr^{-n'''} +$   
 $\&c. mr^{-t}) = a + b + c + d + \&c. + m$  the same equation as above.

Hence the same rule (as that given in the question) is derived by all  
 the three methods in compound interest, where  $p = ar^{-n} + br^{-n'} +$   
 $cr^{-n''} + dr^{-n'''} + \&c. + mr^{-t}$ , and if  $n = 0$ , and the rest of the  
 times as before; then  $r^0 = 1$ , and  $\frac{a}{r^0} = a$ , therefore  $p = a + br^{-n'}$   
 $+ cr^{-n''} + dr^{-n'''} + \&c. + mr^{-t}$ , the sum of the present worths,  
 $s = a + b + c + d + \&c. + m$ , the sum of the debts.

In compound interest, in the solution it is not necessary to know  
 (as in simple interest) between which two debts the equatement will  
 fall, that will be given by calculation by the rule, the debts and times  
 being given.

*Scholium.* These three methods by simple interest gives different  
 answers owing to the inequitable nature of that interest; for each  
 method, is founded on equity. But equity in this respect will not take  
 place in simple interest. Malcolm's method by nearly all the late  
 writers of Arithmetic is said to be the only true method in simple in-  
 terest; but they seem not to know the reason why they differ. In  
 simple interest the creditor may always make more of his money in the  
 whole time, than he could do by receiving his debts as they come due,  
 which destroys the justness of the equatement, wholly owing to the  
 injustice of simple interest. As for instance, if A owes B 4000*l.* due  
 directly, and 3000*l.* more due in fifty years, allowing .5 per cent.  
 per annum simple interest, then Malcolm's equated time is ten years  
 from the first term, by which the creditor B, may gain 4000*l.* more  
 than if he had received his debts, as they came due. For, the interest  
 of 4000*l.* at 5 per cent. for ten years ( $= 4000 \times .05 \times 10$ ) is  
 equal to the discount of 3000*l.* payable forty years hence

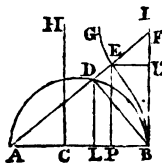
$\left( = \frac{3000 \times .05 \times 40}{1 + .05 \times 40} \right) = 2000*l.*$  which proves the equated time to  
 be right found. And the whole interest B could make by receiving  
 his debts when due, would only be  $(4000 \times .05 \times 50) = 10000*l.*$   
 but the interest by Malcolm's time would be  $(7000 \times .05 \times 40) =$   
 14000*l.* Just 4000*l.* more than the former.

## X. QUESTION 887, by Mr. John Farey, London.

Let  $GB$  be a defective hyperbola of Sir Isaac Newton's 44th species,  $A$  its conjugate point,  $B$  the vertex, and  $CH$  its asymptote; draw  $BI$  a tangent to the vertex parallel to the asymptote, on  $AB$  describe a semi-circle, and draw any line  $AF$ , cutting the circle in  $D$ , the curve in  $E$ , and the tangent  $BI$  in  $F$ . Then I say  $AC : CB :: DE : EF$ . Required a demonstration?

*Answered by Amicus.*

Draw  $EF$  and  $DL$  parallel to  $BI$ . Then since, by the question,  $AC : CB :: DE : EF$ , by similar triangles,  $AE : CB :: LP : PB$ , and  $AB : CB :: LB : PB$ , or  $AB : LB :: CB : PB$ , or  $LB : AL :: PB : CP$ . But  $LB : AL :: DB^2 = LB \cdot AB : AD^2 = AL \cdot AB :: EF^2 : AP^2 :: PB : CP$ . That is,  $EF^2 \cdot CP = PB \cdot AP^2 = (CB - CP) \cdot (AC + CP)^2$ , which is Sir Isaac Newton's equation of the 44th species.



*The same by Mr. John Farey, the Proposer.*

The lines being drawn as in the question, from  $E$  let fall the perpendiculars  $EF$  and  $EL$ , and draw  $DB$ . Put  $AC = n$ ,  $CB = m$ , the absciss  $CP = x$ , and ordinate  $EF = y$ . Then  $\sqrt{((n+x)^2 + y^2)} = AE$ , and  $AP : PB :: AE : ((m-x) \div (n+x)) \sqrt{((n+x)^2 + y^2)} = EF$ , also  $AE : AP :: AB : (n+x) \cdot (n+m) \div \sqrt{((n+x)^2 + y^2)} = AD$ , and hence  $AE \cdot AD = DE = ((n+x)^2 + y^2 - (n+x) \cdot (n+m)) \div \sqrt{((n+x)^2 + y^2)}$ . Then take, as per question,  $AC : CB :: DE : EF$ , or  $AC \cdot EF = CB \cdot DE$ , that is,  $n \cdot \frac{m-x}{n+x} \sqrt{((n+x)^2 + y^2)} = m \cdot \frac{(n+x)^2 + y^2 - (n+x) \cdot (n+m)}{\sqrt{((n+x)^2 + y^2)}}$  which equation reduces to  $xy^2 = (m-x) \cdot (n+x)^2$ , or  $xy^2 = -x^3 + (m-2n)x^2 + (2mn - nn)x + mn^2$ , an equation belonging to the 44th species of Sir Isaac Newton's curves.

*Answered by Lieut. Wm. Mudge, of the Royal Artillery. (Suppl.)*

Sir Isaac Newton's equation to the 44th species, or the defective hyperbola, is  $CP \cdot PE^2 = BP \cdot AP^2$ , that is,  $CP : PB :: AP^2 : PE^2$ . But, by similar triangles  $AP^2 : PE^2 :: AD^2 : DB^2 :: AL : LB$  by the nature of the circle; therefore, by equality,  $AL : LB :: CP : PE$ ; and, by composition,  $AB : LB :: CB : PB$ , or  $AB : CB :: LB : PB ::$  (by similar triangles)  $DF : EF$ ; lastly, by division,  $AC : CB :: DE : EF$ .

XI. QUESTION 888, by Mr. John Cullyer, Assistant at Mr. M<sup>r</sup> Kain's School, Bungay, Suffolk.

Being in a thunder storm, and having a cylindrical stick, 5 feet

long; I held one end of it in my hand, and caused it to turn round in a conical motion, in which the other end described circles, parallel to the horizon, 6 feet in diameter; and the stick made 7 revolutions from the instant of my seeing the lightning till I heard the thunder. From which I desire to know the distance of the thunder-cloud from me.

*Answered by Mr. Isaac Saul.*

The length of the stick, or slant side of the cone, being 5, and the radius of its base 3 feet, therefore  $\sqrt{5^2 - 3^2} = \sqrt{16} = 4$  is the altitude of the whole cone described by the stick. But the centre of oscillation is at  $\frac{2}{3}$  of the length of the stick, and therefore  $\frac{2}{3}$  of 4, or  $2\frac{2}{3}$  is the altitude of the cone described by the part to the centre of oscillation, which call  $a$ . Then, by page 243 of Simpson's Fluxions;  $3 \cdot 14159 \times \sqrt{(2a \div 16\frac{1}{3})} = 1'' \cdot 809066$  is the time of one revolution of the stick; consequently  $1 \cdot 809066 \times 7 \times 1142 = 14461 \cdot 7$  feet, or 2 miles and  $1300\frac{1}{2}$  yards, is the direct distance of the cloud as required.

*The same answered by Allensis.*

The slant side of the whole cone being 5, and the radius of its base 3 feet, therefore  $\sqrt{25 - 9} = \sqrt{16} = 4$  is its altitude; and because the centre of oscillation is at  $\frac{2}{3}$  of the length of the stick; therefore  $\frac{2}{3}$  of 4, or  $\frac{8}{3}$  is the altitude of the cone above the centre of oscillation, which call  $a$ ; also  $n = 3 \cdot 1416$ , and  $p = 16\frac{1}{3}$ . Then, by problem 9, Emerson's Centripetal Forces, we have  $n\sqrt{(2a \div p)} =$  the periodic time of one revolution. And as sound flies at the rate of 1142 feet in one second, and the stick made 7 revolutions from the instant of seeing the lightning, till the report of the thunder, we have  $n\sqrt{(2a \div p)} \times 7 \times 1142 = 14461 \cdot 67$  feet  $= 2 \cdot 739$  miles, the distance of the thunder cloud required.

XII. QUESTION 889, by the Rev Mr. John Hellins.

If there be four numbers  $A, B, C, D$ , in arithmetical progression, whose common difference is 1, that is,  $A - 1 = B, B - 1 = C$ , and  $C - 1 = D$ ; and if there be put  $c = a, \frac{a+2}{2a+1} = p$ , and the modulus of Briggs's logarithms  $= m$ ; and if  $a$  be not less than 100, then shall  $\frac{m}{pa^2 - \frac{1}{4}}$  express the third difference of the logarithms of those numbers, true to 18 places of figures. Query the demonstration?

*Answered by Mr. Alexander Rowe, of Reginnis.*

Since the fluxion of the logarithm of any quantity is equal to the fluxion of that quantity divided by the same quantity; if the quantity be  $x + c$ , where  $c$  is a small given number, the fluxion of it is  $\frac{x}{x+c}$ , and

the fluxion of its logarithm is  $\frac{x}{x+c}$ , which, by dividing the numerator by the denominator, is

$$\frac{x}{x+c} = \frac{x}{x} \pm \frac{cx}{x^2} + \frac{c^2x}{x^3} \pm \frac{c^3x}{x^4} + \frac{c^4x}{x^5} \&c.$$

then taking the fluent of every term, we have the

$$\log. \text{ of } x \mp c = 1. x \mp \frac{c}{x} - \frac{c^2}{2x^2} \mp \frac{c^3}{3x^3} - \frac{c^4}{4x^4} \&c.$$

Now if we take the four numbers mentioned in the question to be  $x - \frac{3}{2}$ ,  $x - \frac{1}{2}$ ,  $x + \frac{1}{2}$ ,  $x + \frac{3}{2}$ , which have the common difference 1; then making  $c$  successively equal to  $-\frac{3}{2}$ ,  $-\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{3}{2}$ , the above theorem for the log. of  $x \mp c$  will give these four logs. viz.

$$1. (x - \frac{3}{2}) = 1. x + m \times (-\frac{3}{2x} - \frac{9}{4.2x^2} - \frac{27}{8.3x^3} - \frac{81}{16.4x^4} \&c.$$

$$1. (x - \frac{1}{2}) = 1. x + m \times (-\frac{1}{2x} - \frac{1}{4.2x^2} - \frac{1}{8.3x^3} - \frac{1}{16.4x^4} \&c.$$

$$1. (x + \frac{1}{2}) = 1. x + m \times (+\frac{1}{2x} - \frac{1}{4.2x^2} + \frac{1}{8.3x^3} - \frac{1}{16.4x^4} \&c.$$

$$1. (x + \frac{3}{2}) = 1. x + m \times (+\frac{3}{2x} - \frac{9}{4.2x^2} + \frac{27}{8.3x^3} - \frac{81}{16.4x^4} \&c.$$

where  $m$  is the modulus of the system of logarithms. Then, taking the successive differences of these logarithms, the third difference

$$\text{is } 6m \times (\frac{3^3-1}{3.2^3x^3} + \frac{3^4-1}{5.2^4x^3} + \frac{3^5-1}{7.2^5x^3} + \frac{3^6-1}{9.2^6x^3} \&c.)$$

$$\text{or } 2m \times (\frac{1}{x^3} + \frac{3}{2x^3} + \frac{39}{16x^3} + \frac{205}{48x^3} \&c.).$$

$$\text{Now, by the question, } a = x - \frac{1}{2}, \text{ and } p \text{ or } \frac{a+2}{2a+1} = \frac{2x+3}{4x};$$

$$\text{therefore } pa^3 - \frac{1}{2} = \frac{16x^4 - 24x^3 - 3}{32x}, \text{ and } \frac{m}{2pa^3 - \frac{1}{2}} =$$

$$\frac{32mx}{16x^4 - 24x^3 - 3} \text{ is } = 2m \left( \frac{1}{x^3} + \frac{3}{2x^3} + \frac{39}{16x^3} + \frac{189}{48x^3} \&c. \right),$$

which agrees with the series for the 3d diff. except in the last term, in

which it differs from it by only  $\frac{2}{3x^3}$  which, when  $x = 100$ , will have

cyphers in the first 18 places of decimals, and a 6 in the 19th place, to multiply by  $m$  the modulus.

*The same answered by Amicus. (Suppt.)*

It is now pretty well known, that if  $a =$  the least but one of four numbers whose common difference is unity, their logarithms will be expressed by these four following series :

$$l. (a - 1) = l. a - m \times : \frac{1}{a} + \frac{1}{2a^2} + \frac{1}{3a^3} + \frac{1}{4a^4} \&c.$$

$$l. (a) = l. a$$

$$l. (a + 1) = l. a + m \times : \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} - \frac{1}{4a^4} \&c.$$

$$l. (a + 2) = l. a + m \times : \frac{2}{a} - \frac{4}{2a^2} + \frac{8}{3a^3} - \frac{16}{4a^4} \&c.$$

And the 3d difference of these four quantities

$$\text{is } m \times : \frac{8-2}{3a^3} - \frac{16-4}{4a^4} + \frac{32-2}{5a^5} - \frac{64-4}{6a^6} + \frac{128-2}{7a^7} + \frac{256-4}{8a^8} \&c.$$

$$\text{or } m \times : \frac{2}{a^3} - \frac{3}{a^4} + \frac{6}{a^5} - \frac{10}{a^6} + \frac{18}{a^7} - \frac{63}{2a^8} + \frac{170}{3a^9} - \frac{102}{a^{10}} \&c.$$

And if we suppose this to be a recurrent series, and by the known method of substitution and comparison, try to find such a fraction as will produce it, it will be found that  $\frac{4a+2}{2a^4+4a^3-2a-1}$  will by ex-

pansion, produce the first six terms of the series, and that the 7th and 8th terms of this real recurrent series will be  $+\frac{56}{a^7} - \frac{99}{a^{10}}$  instead of

$$\frac{56\frac{2}{3}}{a^7} - \frac{102}{a^{10}}, \text{ differing from them by } \frac{2}{3a^9} - \frac{3}{a^{10}}, \text{ which, when } a=100,$$

$$\text{is } \frac{63\frac{2}{3}}{100^9} = .0000000000000000063\frac{2}{3}; \text{ the fraction therefore gives}$$

the value of the series true to 18 places of decimals, and is  $= \frac{1}{pa^3 - \frac{1}{2}}$ , as per question.

*The same by Lieut. Wm. Mudge, of the Royal Artillery. (Suppl.)*

If A, B, C, D, be any four numbers; then, because the difference of the logs. of any two quantities is equal to the log. of their quotient, by dividing these numbers by each other, the quotient will be the series of the first differences, and dividing the terms of the first differences will give the second differences, and these divided again, give the 3d differences, and so on: hence the several orders of differences of the logs. of A, B, C, D, will be

1st Dif.	2d Dif.	3d Dif.	So that the 3d diff. of the logs. is equal to the log. of $\frac{AC^3}{DB^3}$ . Now, according to the question, putting $a=c$ , then is $A=a+2$ , $B=a+1$ , $C=a$ , $D=a-1$ ; and consequently $\frac{AC^3}{DB^3} = \frac{a+2}{a-1} \times \frac{a^3}{(a+1)^3} = (a+2) \times$
$l. \frac{A}{B}$	$l. \frac{B^2}{AC}$	$l. \frac{AC^3}{DB^3}$	
$l. \frac{B}{C}$	$l. \frac{C^2}{BD}$		
$l. \frac{C}{D}$			

$\frac{a^3}{a^4 + 2a^3 - 2a - 1} \approx \frac{pa^3}{pa^3 - 1}$ ; and the log.  $\frac{AC^3}{DB^3}$  or leg.  $\frac{pa^3}{pa^3 - 1}$   
 is  $\frac{1}{pa^3} + \frac{1}{2p^2a^6} + \frac{1}{3p^3a^9}$  &c. the true 3d dif. of the logs. But this  
 last series, it is evident, is nearly  $= \frac{1}{pa^3 - \frac{1}{2}}$ , for by division  $\frac{1}{pa^3 - \frac{1}{2}}$   
 $= \frac{1}{pa^3} + \frac{1}{2p^2a^6} + \frac{1}{4p^3a^9}$  &c. which differs from the former series  
 only in the 3d term, having 4 in the denominator of that term instead  
 of 3, so that the dif. between the two is only  $\frac{1}{12p^3a^9}$ . And as  $p = \frac{1}{2}$   
 nearly, therefore  $p^3 = \frac{1}{8}$ , and  $12p^3 = \frac{3}{2}$  nearly; and so  $\frac{1}{12p^3a^9} =$   
 $\frac{2}{3a^9}$  nearly; which, when  $a = 100$ , will have 18 places of cyphers,  
 and a 6 in the 19th place of decimals. Consequently  $\frac{m}{pa^3 - \frac{1}{2}}$  gives  
 the third dif. true to 18 places of decimals.

And the same method may be used for five or more numbers in arithmetical progression.

XIII. QUESTION 890, by *Lieut. W. Mudge, of the Royal Artillery.*

If a string, with the weights  $w, w$ , one at each end, be hung on a pulley, and the greater weight  $w$  touch the pulley, so that the less may have the whole length of the thread to vibrate by; and, if, at the instant when it has completed one vibration, and is about to describe another, the weight  $w$  be suffered to descend; query the time of vibration, and the nature of the curve.

*Answered by Amicus.*

In this question, if the angle of vibration be of any considerable magnitude, the final equation will involve second fluxions squared when freed from surds, and be so complex, as to render the separability of the unknown quantities in a manner hopeless. But if they be exceedingly small, let  $b$  = the versed sine of the arc of vibration to the constant radius  $= a$  = the length of the string,  $x$  = the part of that versed sine answerable to the vertical descent in the vibration of the less body during the time  $t$ , gravity  $= 32\frac{1}{2} = 2s$ ,  $u$  = the distance of that body from the pulley at the end of that time, and  $v$  = the velocity in the direction of the string; putting  $m^2 = \frac{w + w}{w - w}$ , then by the nature of

motion, and the question  $v\dot{v} = -\frac{2s}{m^2}u$  nearly, and  $v = 2\sqrt{\frac{a-u}{m^2}}$ .

But the vibrating velocity of the body perpendicular to the string  $= 2\sqrt{sx}$ , and the space described with that velocity  $= \frac{\dot{x}\sqrt{u}}{\sqrt{(2b-2x)}}$ ,

hence  $\dot{t} = -\frac{\dot{u}}{v} = -\frac{m\dot{u}}{2\sqrt{s \cdot (a-u)}} = \frac{\dot{x}\sqrt{u}}{2\sqrt{2s} \sqrt{(bx-x^2)}}$ ,  $t = \frac{m\sqrt{(a-u)}}{\sqrt{s}}$ , and  $-\frac{m\dot{u}\sqrt{2}}{\sqrt{(au-u')}} = \frac{\dot{x}}{\sqrt{(bx-x^2)}}$ ; let  $\Lambda =$  the arc

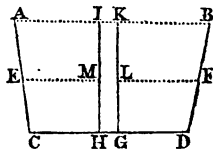
whose versed sine  $= 2x \div b$  to radius 1,  $\mathbf{B} =$  that to versed sine  $2u \div a$ , and  $p = 3.14159$ , then the equation of the correct fluents is  $(p - \mathbf{B}) \cdot m\sqrt{2} = \Lambda$ , from which equation the value of  $x$  and  $\sqrt{(bx-x^2)}$ , answering to any value of  $u$ , becomes known, shewing the nature of the track of the weight. And when the string becomes vertical,  $\Lambda = p$ , at which time therefore the arc  $\mathbf{B}$  becomes given and  $= p \times (m\sqrt{2} - 1) \div m\sqrt{2}$ , whose versed sine  $2u \div a$  gives  $u$  the length of the vibrating string when vertical; let this value of  $u = e$ , then in the same manner as before it will be found that at the end of one whole vibration, the arc whose versed sine is  $2u \div e$ , to radius 1, is  $= p \times (m\sqrt{2} - 1) \div m\sqrt{2}$ , consequently  $2u \div e = 2e \div a$ , and  $u = e^2 \div a =$  the distance of the less weight from the pulley at the end of one vibration, and the time of one whole vibration  $= \sqrt{(a + \sqrt{e})} \times m\sqrt{((a - e) \div sa)}$ , as required.

XIV. QUESTION 891, *by Major E. Williams, of the Royal Artillery.*

It is required to assign the time of exhausting the ditches of a fortress of water, to within one inch of the bottom, by means of a rectangular cut or notch in the side from top to bottom, of 2 feet wide; the depth of the ditch or water being 9 feet, the breadth at top 30 feet, at bottom 32 feet, and the whole length of the ditches one mile.

*Answered by Lieut. Wm. Mudge, of the Royal Artillery.*

I apprehend the numbers expressing the breadth of the ditch at top and bottom, have been interchanged; I shall therefore change the dimensions, and take the ditch as narrowest at the bottom. Let therefore  $ABDC$  be the end of the ditch, and  $GHIK$  the cut; put  $x = MM$  any variable altitude of the water within, and  $g = 16\frac{1}{2}$  feet; by the data  $HI : HM :: AB - CD : EE - CD =$



$\frac{2}{3}x$ , therefore  $EF = CD + \frac{2}{3}x = 30 + \frac{2}{3}x$ , and hence  $(30 + \frac{2}{3}x) \times 1$  mile  $= (30 + \frac{2}{3}x) \times 5280$  the area of the surface of the water when it is at  $EF$ ; and the quantity running through the cut  $GM$  is equal to  $\frac{2}{3}$  of what would run through an equal aperture with the greatest velocity, or that at  $GH$ , which velocity is equal to that of a heavy body

falling through  $MH$  or  $x$ , namely  $2\sqrt{gx}$ , that is, the quantity per second running through  $MG$  is  $\frac{2}{3}MG \times 2\sqrt{gx}$  or  $\frac{4}{3}x\sqrt{gx}$ ; and hence, dividing this quantity by the surface of the water at  $EF$ , the quotient  $\frac{x\sqrt{gx}}{135+x}$  will be the velocity  $v$  per second with which the surface of the water descends; therefore, by uniform motions,  $v : -x :: 1'' : t$   
 $= \frac{-x}{v} = \frac{-440x}{\sqrt{g}} \times \frac{135+x}{x^{\frac{3}{2}}}$  the fluxion of the time of exhaust-

ing. And the correct fluent of this, it being nothing when  $x = 9$ , is  $\frac{880}{\sqrt{g}} \times \left( \frac{135-x}{\sqrt{x}} - 42 \right) = t$  the time of exhausting till the depth is  $x$ . And when  $x = 1$  inch, or  $\frac{1}{12}$  foot, this expression gives  $t = (880 \div \sqrt{193}) \times (1619 - 84\sqrt{3}) = 93338$  seconds  $= 25$  hrs. 55 min. 38 sec. the time required.

Had the dimensions been as in the question, or the ditch narrowest above, by a similar process the time of exhausting to one inch deep would be 98330 seconds, or 27 hrs. 18 min. 50 sec. And the time of a complete exhaustion, in both cases, is infinite.

*Note.* This solution is on the supposition that the velocity of issuing water is equal to that acquired by a body in falling through the whole height of the surface above the orifice.

*The same answered by Amicus. (Suppt.)*

Let  $ACED$  represent the end of the ditches,  $TB$  the chink, whose breadth  $LX = 24$  inches  $= c$ ,  $TB = 108 = a$ ,  $AC = 384 = b$ ,  $DE = 360 = c$ ,  $f = 63360 = 1$  mile,  $g = 386 =$  gravity. Then, if the water has no obstruction in running off, its fluxion per second with the first velocity (nearly) at  $x = \frac{1}{12}$ ,  $\frac{1}{12} \times \frac{1}{12} \times c\sqrt{g}$ , and through the whole length  $TB$ ,  $\frac{2}{3}TB^{\frac{3}{2}} \times c\sqrt{g}$ , and when its depth is  $BX = y$ ,  $\frac{2}{3}cg^{\frac{1}{2}}y^{\frac{3}{2}}$ . And as  $DF = a : 2AF = b - c :: DI = a - y : 2GI$ , and  $(2GI + DE) \times f =$  the area of the surface at  $x = bf - nfy$ , putting  $n = (b - c) \div a$ ; and the velocity of the descending surface  $gx = \frac{c\sqrt{g}}{3f} \times \frac{y\sqrt{y}}{b - ny}$ , which applied to  $-y$ , gives the elementary time, whose correct fluent is  $\frac{3f}{2\sqrt{g}} \times \left( \frac{b}{\sqrt{y}} - \frac{b}{\sqrt{a}} + n\sqrt{y} - n\sqrt{a} \right) =$  the time; which, when  $y = 1$ , gives  $139060'' = 38^h 37^m 40^s$  the time required.

The above solution is given on the supposition that the velocity of issuing water, is equal to that acquired by a body in falling through only half the height of the surface above the orifice.

## PRIZE QUESTION, by Mr. George Sanderson, London.

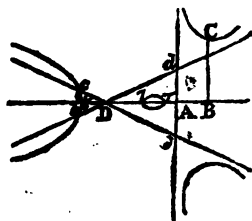
In considering the prize question for last year, I find that figure 20 of Newton's catalogue consists of two inscribed hyperbolas at  $d$  and  $\delta$ , and one containing its asymptotes within its own space at  $\nu$ ; and that figure 21 consists of two ambigenous hyperbolas at  $d$  and  $\delta$ , and one inscribed at  $\nu$ , without oval or conjugate point. But between these two there are five more curves, essentially different from either; two of which have been described by Mr. Stirling at pages 99 and 100. It is therefore required to determine the other three, with an example of a numeral equation for each?

*Answered by Amicus.*

The general equation of the redundant hyperbolas having one diameter only is  $xy^2 = c \cdot (x+e) \cdot (b+x)(d+x) = cx^3 + c \cdot (e+b+d)x^2 + c \cdot (bd + e \cdot (b+d))x + cebd$ , where  $AB = x$ ,  $CB = y$ ,  $A\delta = -x$ ,  $Af = e$ ,  $A\eta = b$ ,  $A\tau = d$ ,  $DA = \frac{1}{2}(e+b+d)$ ,  $\frac{Ad}{AD} = \sqrt{c}$ ,

$\frac{CB}{DB} = \frac{y}{x+DA}$ , which at the point where

the curve cuts the asymptote  $= \frac{Ad}{AD} = \sqrt{c}$



$= \frac{y}{x+DA}$ ; this equation reduced is  $x = \frac{4ebd}{(e-b-d)^2 - 4bd}$ , for

the value of the abscissa at the point where the curve cuts the asymptote. Hence it appears, that so long as  $e-b-d$  is greater than  $2\sqrt{bd}$ , this value of  $x$  will be affirmative, and the two hyperbolas adjacent to  $\delta$  and  $d$  will be ambigenous; and since then also  $e$  is greater than  $b+d$ , therefore  $e$  must be greater than  $DA$  or  $\frac{1}{2}e + \frac{1}{2}(b+d)$ ; consequently whilst these two hyperbolas are ambigenous,  $Af$  must also be greater than  $AD$ , and that adjacent to  $\nu$  an inscribed one, and the curve of one of the two species discovered by Mr. Stirling. But if  $2\sqrt{bd}$  be greater than  $e-b-d$ , the above value of  $x$  is negative, and the point of intersection on the contrary side of  $Ad$ ; and here  $DA$  or  $\frac{1}{2}(e+b+d)$  may be either less or greater than  $Af$  or  $e$  according as  $b+d$  is less or greater than  $e$ ; if  $b+d$  be greater than  $e$ , then  $e-b-d$  is necessarily less than  $2\sqrt{bd}$ , and the curves are those drawn in Sir Isaac's 17th, 18th, 19th and 20th figures. But if  $e$  be greater than  $b+d$ , and  $e-b-d$  less than  $2\sqrt{bd}$ , then must  $DA$  be less than  $Af$ , and the above value of  $x$  negative: here then the curve is not as described in those figures, but as in that here annexed, the hyperbola adjacent to  $\nu$  cutting its asymptotes, and then including them within itself, whilst the other two are inscribed ones. If  $A\eta$  and

$ax$  are unequal, the curve has an oval ; if equal, a conjugate point. But if they be impossible, or the equation of the curve be  $xy^2 = cx^3 + c \cdot (e + b) \cdot x^2 + c \cdot (a^2 + be) \cdot x + cea^2 = (x + e) \times (x^2 + bx + a^2)$ , where  $a$  is greater than  $\frac{1}{2}b$ , then  $-x = \Delta b = \frac{4ea^2}{4a^2 - (e-b)^2}$  at the intersection of the curve and asymptote : and if  $2a$  be greater than  $e - b$ , the curve will still be as in the annexed figure, but without oval or conjugate point.

*Ex. gr.* 1. Let  $e = 10$ ,  $b = 5$ ,  $d = 4$  ; then  $\Delta b = 10 \cdot 126582$ , and  $7r = 1 =$  the diameter of the oval.

2. Let  $e = 10$ , and  $b = d = 4 \cdot 5$  ; then  $\Delta b = 10 \cdot 125$ , and the oval becomes a point.

3. Let  $e = 10$ ,  $b = 9$ , and  $a = 5$  ; then  $\Delta b = 9 \cdot 5$ , and  $\Delta b = 10 \cdot 1010$  &c ;  $\Delta l$  and  $\Delta r$  being impossible.

*Scholium.* Though these three curves thus differ in figure from those drawn by Sir Isaac for the 10th, 13th, and 14th species, they cannot, with propriety, be said to constitute new ones ; for they are all included in his descriptions of those species ; which descriptions will equally hold for these, word for word, and letter for letter. Moreover, the two species discovered by Mr. Stirling ought not to follow Sir Isaac's 14th, but the first of them ought immediately either to precede or follow Sir Isaac's 10th, and the second his 13th species.

*The same answered by Mr. J. Farey, of London. (Suppl).*

The figure designed by the equation  $xy^2 = ax^3 + bx^2 + cx + d$  will have only one diameter if  $b^2$  be either greater or less than  $4ac$ , which Mr. Stirling has shewn in his *Lineæ Tertii Ordinis Newtonianæ*, prop. 16, also that  $\Delta D = -b \div 2a$ ,  $\Delta d = \Delta d = b \div 2\sqrt{a}$ , and  $\Delta B = 4ad \div (b^2 - 4ac)$  ; from the nature of equations it is evident that  $-b \div 2a$  or  $\Delta D$  is equal to half the sum of the three roots ; and by Mr. Sanderson's Lemma, in the last Diary, if  $-m, -n, -r$  be the three roots, and  $r$  be greater than  $m + n - 2\sqrt{mn}$ , but less than  $m + n + 2\sqrt{mn}$ , then  $b^2$  will be less than  $4ac$ , and half the sum of the roots may be either less, equal to, or greater, than the greatest. This being premised, we may observe that, in the 17, 18, 19 and 20th figures of Sir Isaac Newton's Enumeration,  $b^2$  is less than  $4ac$ , and  $\Delta l$ , the greatest of the roots, less than  $\Delta D$  ; it is evident also, from the above, that  $\Delta l$  may be equal to, or greater than  $\Delta D$  ; in which case the figure will be different from any of Newton's, as follows :

*Case 1.* When, in the equation  $ax^3 + bx^2 + cx + d = 0$ ,  $b^2$  is less than  $4ac$ , and the three roots real, unequal, and negative ; and the greatest  $\Delta l$  greater than  $\Delta D$  ; then the hyperbola at  $D$ , having its vertex in  $t$ , will cut the asymptote in  $c$ , and contain the part beyond within its own space, as in the figure ; and there will be an oval within the triangle. For example, let  $xy^2 = \frac{1}{4}x^3 + 4x^2 + 17x + 20$

(where  $a$ , which may be any positive number, is assumed equal  $\frac{1}{2}$ , in order to have  $\Delta d = \frac{1}{2}\Delta b$ ), and the three roots are  $\Delta r = -2$ ,  $\Delta l = -4$ , and  $\Delta t = -10$ ; also we have  $\Delta b = -b \div 2a = -8$ , and  $\Delta b = 4ad \div (b^2 - 4ac) = -20$ .

*Case 2.* When the two least roots are equal, the oval becomes a conjugate point. For example, if  $xy^2 = \frac{1}{4}x^3 + 4x^2 + 17\frac{1}{4}x + 22\frac{1}{2}$ , the roots are  $-3$ ,  $-3$ , and  $-10$ ; also  $\Delta b = -8$ , and  $\Delta b = -18$ .

*Case 3.* When the two least roots are impossible, the oval vanishes. For example, if  $xy^2 = \frac{1}{4}x^3 + 4x^2 + 18x + 30$ , the roots are  $-3 - \sqrt{-3}$ ,  $-3 + \sqrt{-3}$ , and  $-10$ ; also  $\Delta b = -8$ , and  $\Delta b = -15$ .

The two greatest roots cannot here become equal, as in Newton's 18th figure; nor the three roots equal, as in the 19th figure. The same will also hold if  $\Delta t$ , the greatest root, be equal to  $\Delta b$ , or when the vertex of the hyperbola is in D. These three cases should be inserted between the 16th and 17 figures of Newton's Enumeration.



*Questions proposed in 1789, and answered in 1790.*

**I. QUESTION 893, by Mr. James Ashton, of Harrington.**

In what time will an annuity of 83*l.* 10*s.* discharge a debt of 900*l.* allowing interest on each at  $4\frac{1}{2}$  per cent.?

*Answered.*

This question is answered by our Correspondents, on three different principles; 1st by making the whole amount of the annuity equal to that of the original sum, and its interest for the time sought, both computed at simple interest; 2dly, by making the same amounts equal, when computed at compound interest; and 3dly, subtracting the annuity every year from the principal sum or debt, and its interest, till the whole shall be extinguished. We shall therefore insert a solution upon each of these three different principles. And first by simple interest.

*1st. By Mr. B. Benson, of Croston.*

Put  $p = 900$  the principal,  $a = 83.5$  the annuity, and  $r = .045$  the rate, also  $t$  the time sought. Then  $p + ptr$  is the amount of the debt in  $t$  years, and  $at + \frac{1}{2}at^2r - \frac{1}{2}atr$  is the amount of the annuities and the interests; therefore  $p + ptr = at + \frac{1}{2}at^2r - \frac{1}{2}atr$ ; which reduces to this  $t^2 + \left(\frac{2}{r} - \frac{2p}{a} - 1\right) \times t (st) = \frac{2p}{ar}$ ; and hence  $t = \sqrt{\left(\frac{s^2}{4} + \frac{2p}{ar}\right) - \frac{s}{2}} = 13.5268$  years, the time sought.



to nothing, that is  $pr^t - \frac{r^t - 1}{r - 1} \times a = 0$ , or  $pr^t = \frac{r^t - 1}{r - 1} \times a$ , consequently,  $r^t = \frac{a}{p + a - pr} = \frac{a}{a - pr}$ , putting  $r = r - 1 = .045$ , and  $t = \frac{\log. a - \log. (a - pr)}{\log. r} = 14.174$  years, the time sought in this case.

### II. QUESTION 894, by Philalethes Cleasbyensis.

q of Amsterdam sends to r of Paris 2000 crowns at 91d. Flemish per crown, at double usance, or 2 months, and pays  $\frac{2}{5}$  per cent. brokerage; with orders to remit him again the value at 93d. per crown, allowing at the same time  $\frac{1}{3}$  per cent. for commission: What is gained per cent. per annum by a remittance thus managed?

*Answered by A. Whitehouse, of Wolverhampton.*

$100 : 100\frac{3}{5} :: 2000 \times 91d. : 182273$  q's cost of 2000 crowns sent r to Paris, and  $100 : 99\frac{2}{3} :: 2000 \times 93d. : 185380d.$  r's remittance sent q, their difference is 3107d. = q's 2 months gain; this multiplied by 6 = 18642d. = his year's gain at that rate; therefore 182273d. : 18642d. ::  $100 : 10\frac{4147}{182273}$  = q's gain per cent. per annum, the same as given by Mr. Clare.

N. B. This question was first proposed by Mr. Clare, in his Introduction to Trade and Business, from whence it has been taken by Messrs. Birks and Vyse, and placed in their Arithmetics, but both of them have given false answers to it.

### III. QUESTION 895, by Mr. N. Hoskins.

A merchant began trade with a certain sum of money, which amounted at the end of 7 years to 62500*l.* and had accumulated in the following manner, viz. at the end of the 3d year he had just doubled the first sum. The next year he gained the square root of that doubled sum, and 10*l.* more. And the last 3 years he squared the whole. Query the first sum?

*Answered by Mr. Matthew Fleck, of Stella.*

Put  $\frac{1}{2}x^3$  for the sum the merchant began with. Then, by the quest,  $x^3$  is the amount at the end of the 3d year, and  $x + 10$  is the gain the 4th year, therefore  $x^3 + x + 10$  is the amount at the end of the 4th year, consequently  $(x^3 + x + 10)^3 = 62500$  the amount at the end of 7 years. Hence  $x^3 + x + 10 = \sqrt[3]{62500} = 250$ , and consequently  $x = 15$ . Then  $\frac{1}{2}x^3 = 112\frac{1}{2}$  or 112*l.* 10*s.* the sum at beginning.

### IV. QUESTION 896, by Mr. John Birch, of Moulton.

Having a conical vessel full of liquor, standing upon its less end,

the radius of which is 20 inches, into which I immersed a cone of equal base and altitude, the convex superficies of which is  $2827\cdot44$ , and is equal to the area of the top of the vessel. Required its content, and the quantity of liquor in ale gallons that overflowed by so doing?

*Answered by Mancuniensis.*

Let  $ABED$  be the conical vessel, and  $DCE$  the immersed cone, of the same height and base. Then  $\sqrt{(2827\cdot44 \div \cdot7854)} = 60 = AB$ ; and, per page 182, Dr. Hutton's Mensuration, 2d edit.  $2827\cdot44 \times 2 \div 3 \cdot 1416 \times DE = 45 = DG$ ; also  $\sqrt{(DC^2 - DG^2)} = CF = 40\cdot31129$  nearly; but, by page 185 of the same,  $(AB^2 - DE^2) \times \cdot7854 \times CF \div (AB - DE) \times 282 \times 3 = 284\cdot42045$  the content of the vessel in ale gallons; and  $\cdot7854 \times CF \times DE^2 \div 282 \times 3 = 59\cdot877989$  ale gallons, the content of the cone  $DCE$ , or the quantity overflowed by immersing it.



*The same by Mr. Wm. Slatter, at Adderbury School.*

First, the surface of the cone divided by half the circumference of the base, will give the slant height, that is  $2827\cdot44 \div 3\cdot1416 \times 20 = 45 = DG$ ; and hence  $\sqrt{(DC^2 - DG^2)} = \sqrt{(45^2 - 20^2)} = 40\cdot311288 = CF$  the height of the cone, or of the vessel. Then, by page 188 of Hutton's Mensuration, new edition,  $(2827 \times \cdot44 \times 40^3 \times \cdot7854 \times \sqrt{(2827\cdot44 \times 40^3 \times \cdot7854)}) \times 40\cdot311288 \div 3 = 80206\cdot5635$ , the content of the vessel in inches, which divided by 282, gives  $284\cdot42$  ale gallons.

Also  $40^3 \times \cdot7854 \times 40\cdot311288 \div 3 = 16885\cdot59231744$  the content of the cone in inches, which divided by 282, gives  $59\cdot878$  ale gallons; and so much of the liquor will overflow by the immersion of the cone.

V. QUESTION 897, by Mr. Terry, Land-Surveyor, Askrigg.

To determine the ratio of two elastic balls A and B, so that A, by striking B at rest, shall lose one fifth of its motion.

*Answered by Mr. S. Woolcott, of South Moulton.*

Put  $v$  for the velocity of A before the stroke. Then will  $((A - B) \div (A + B)) v$  be its velocity after the stroke, which by the question is  $\frac{4}{5}v$ ; therefore  $(A - B) \div (A + B) = \frac{4}{5}$ , or  $A + B : A - B :: 5 : 4$ , and by adding and subtracting  $2A : 2B :: 5 + 4 : 5 - 4$ , or  $A : B :: 9 : 1$ , the ratio required.

*Remark.* Had the bodies been non-elastic, it would have been  $Av \div (A + B) = \frac{4}{5}v$ , or  $A + B : A :: 5 : 4$ , or  $A : B :: 4 : 1$ , the ratio in this case.

*The same by Mr. John Dalton, of Kendal.*

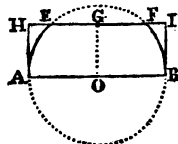
The bodies being supposed perfectly elastic, they will recede from each other after the impulse with the same velocity as they approached before it, that is with the velocity of the body A. Let this velocity be called  $a$ ; then the motion of A before the impulse is  $aA$ ; and its motion after the same, by the question, is  $\frac{3}{4}aA$ ; therefore the motion of the body B after the impulse is  $\frac{1}{4}aA$ ; because action and reaction are equal; but the velocities, by Mechanics, is equal to the motions divided by the masses or bodies; that is  $\frac{3}{4}a$  for the body A, and  $aA \div 5B$  for the body B; and the difference of these velocities must be equal to the first velocity of A, that is  $(aA \div 5B) - \frac{3}{4}a = a$ ; and hence  $\frac{1}{4}a - \frac{3}{4}a = B$ , or  $A = 9B$ : so that A is to B as 9 to 1, as required.

VI. QUESTION 898, *by Mr. Isaac Saul.*

Given the curve superficies of the frustum of a sphere, equal to 1600; and the difference between its solid content, and that of a cylinder of the same diameter and altitude, equal to 1800; to determine both the diameter and altitude.

*Answered by Mr. James Hannah, of Broughton.*

Let AEFB be the zone or frustum, and AHHI its circumscribing cylinder. Put  $x = AO$  the radius of the sphere,  $y = AH = GO$  the common height of the frustum and cylinder,  $s = 1600$  the curve surface, and  $d = 1800$  the difference between the solid contents, also  $a = 3 \cdot 1416$ . Now  $x^2 - y^2 = EG^2$  the square of the radius of the top; hence, by prob. 17, page 209, Hutton's Mensuration, 2d edit.  $(x^2 + x^2 - y^2 + \frac{1}{3}y^2) \times \frac{1}{2}ay$  or  $ax^2y - \frac{1}{3}ay^3$  is the solidity of the frustum or zone, and  $ax^2y$  that of the cylinder; therefore their dif. or  $\frac{1}{3}ay^3 = d$  the dif. of the solidities; and hence  $y^3 = 3d \div a$ , and  $y = \sqrt[3]{(3d \div a)} = \sqrt[3]{(5400 \div 3 \cdot 1416)} = 11 \cdot 9788$  the height of the solids.



Again, by page 197 Mensuration,  $2axy = s$  the surface, therefore  $x = s \div 2ay = 1600 \div 75 \cdot 2652 = 21 \cdot 258162$  the radius, and its double or  $42 \cdot 516324$  is the diameter sought.

VII. QUESTION 899, *by Mr. Alexander Rowe.*

A bets B 5 guineas to 10 shillings, that in throwing up 5 halfpence, they shall not come up either all heads, or all tails, once in 4 throws: whether has the advantage, and how much?

*Answered by Mr. Alexander Rowe, of Reginnis.*

The number of different ways that 5 things, with 2 faces each, can

come up in one throw, is  $2^4$ , or 32 ways, out of which there are 2 ways for all heads or all tails, and 30 for the contrary; therefore the probability of throwing all heads or all tails, in one throw, is  $\frac{2}{32}$  or  $\frac{1}{16}$ , and the probability of the contrary, or missing them both, is  $\frac{30}{32}$  or  $\frac{15}{16}$ ; and consequently the probability of missing them 4 times

running is  $\frac{15^4}{16^4}$  or  $\frac{50625}{65536}$ ; and taking this from 1, the remainder  $\frac{14911}{65536}$

is the probability that the event will happen at least once in 4 throws. Whence the former probability is to the latter, as 50625 to 14911, or nearly as 3.3952 to 1, instead of  $10\frac{1}{2}$  to 1, as staked by A. And to

find the sum or value of his disadvantage, take the  $\frac{50625}{65536}$  part of the

whole stake 5*l.* 15*s.* which is 4*l.* 8*s.* 10*d.* the value of A's chance of the stake, being 16*s.* 2*d.* less than he deposits, and therefore his disadvantage is 16*s.* 2*d.* on every stake.

*The same by Mr. John Griffith, of Sandbach.*

The number of combinations with 5 halfpence, is  $2^5$  or 32; the number of chances for bringing 5 heads, or 5 tails, is 2; and the number, for failing, 30. Whence the probability of not bringing 5 heads, or

5 tails, at least once in 4 throws, is  $\frac{30^4}{32^4}$  or  $\frac{810000}{1048576}$ . Therefore as

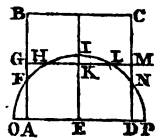
238576 to 810000 :: 10*s.* : 34*s.* nearly, instead of 10 to 105 which A deposited.

VIII. QUESTION 900, *by Mr. William Hardy, of Cottingham.*

There is a geometrical square, whose side is 12 inches, required the radius of a circle, whose centre shall be in the middle of one of its sides, that shall cut the said square into equal parts.

*Answered by Mr. Thomas Leybourn, of North Shields.*

Let ABCD be the given square, and AGMD the half of it, or AGKE one-fourth of it. Now it is easy to construct the figure nearly by the eye, for it is only to make the small segment HIK equal to the small external trilineal FGH; for then the mixed area AFHIE will be equal to the small square AGKE. Now by the first measurement it is found that the radius EI is equal to 7 very nearly. Then to compute the small segment HIK or its equal AOF; the versed sine IK = 1, to the diameter 14; then by the large table of circular segments at the end of Hut-



ton's Mensuration, the semi-segment  $\Delta OF$  is easily found to be  $2.44028$ , which taken from the area of the quadrant  $EOI$  or  $38.4846$ , leaves  $36.0443$  nearly for the area  $\Delta FHI E$ , which ought to be just  $36$ , therefore the error is  $.0443$  too much.

Take again, therefore, the radius a little less; as suppose  $EO$  or  $FI$   $= 6.9$ ; then to this radius, and the versed sine  $\Delta O$  or  $IK$   $= 0.9$ , by the same table, the semi-seg.  $\Delta OF$  is  $2.0736$ , which taken from the quadrant  $EOI$  or  $37.3929$ , leaves  $35.3193$ , which ought to be  $36$ , therefore the error is  $.6807$  too small. Hence, by the rule of Double Position, the radius is found to be nearly  $6.9939$ , the answer.

*The same by Mr. Wm. Pearson, of North Shields.*

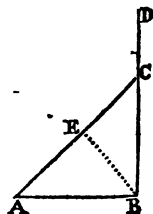
The square and circle being drawn as above, put the side  $\Delta B = 12 = a$ ,  $.7854 = c$ , and the versed sine  $\Delta O$  or  $IK = v$ . Then  $a + 2v$  is the diameter  $OP$ , and  $(a + 2v)^2 \times \frac{1}{2}c$  the area of the semicircle  $OIP$ , which is equal to half the square plus the segment  $HIL$ . Then, by Hutton's Mensuration, (rule 5, page 105, first edit. or rule 6, page 140, 2d edit.) the segment  $HIL = \frac{1}{3}v\sqrt{(av - \frac{1}{3}v^2)}$ ; and hence arises this equation  $\frac{1}{3}v\sqrt{(av - \frac{1}{3}v^2)} + a^2 = (a + 2v)^2 \times c$ ; from which, by Trial and Error, or otherwise, is found  $v = .9943$ ; and hence  $\frac{1}{2}a + v = 6.9943$  is the radius of the circle nearly.

IX. QUESTION 901, *by Mr. John Cullyer, of Hingham.*

A carpenter having nearly felled a tree 60 feet high, wishes to be informed at what height he must fix a rope to it, of 70 feet long, that when standing on the ground he may draw the tree down with the most ease.

*Answered by Mr. James Ashton, of Harrington School.*

By the principles of Mechanics, the effect of the power acting at  $c$ , to break the tree at  $B$ , is directly as the length of the lever  $BC$ ; and, when acting in the direction  $AC$ , it is also directly as the sine of the angle of direction  $c$ , or as the distance  $AB$ , since the hypotenuse  $AC$  is given  $= 70$ ; hence the effect will be as the rectangle of the two  $AB \times BC$ , which in this case must be a maximum, and will therefore happen when the two are equal, that is  $AB = BC$ , and consequently the angles  $A$  and  $c$  each half a right angle. Hence  $BC^2 = \frac{1}{2}AC^2$ , and  $BC = AC\sqrt{\frac{1}{2}} = 49\frac{1}{2}$  feet nearly.



*The same by Mr. A. Buchanan, jun. of Sedgefield.*

Let  $BD$  represent the tree, and  $AC$  the rope in its required position. Put  $BD = 60 = a$ ,  $AC = 70 = c$ , and the height required  $BC = x$ ;

then will  $AB$  be  $= \sqrt{(c^2 - x^2)}$ , and sine of  $\angle C = \sqrt{(c^2 - x^2)} \div c$ . But the effect of the force or rope  $AC$  is as  $BC \times \text{sine } \angle C$ , therefore  $x\sqrt{(c^2 - x^2)}$  must be a maximum, or  $c^2x^2 - x^4$  a maximum; then its fluxion  $2c^2xx' - 4x^3x' = 0$ , or  $c^2 = 2x^2$ , and  $x = c\sqrt{\frac{1}{2}} = 49 \cdot 497$  feet nearly, the height sought. Hence also the distance  $AB$  is  $=$  the height  $BC$ .

*The same by Mr. John Haycock, of Ware.*

Let  $AC$  be the rope fixed to the tree  $BD$  at  $c$ , and  $BE$  perpendicular to  $AC$ . Then, by mechanics, the effect of the force in  $AC$ , to break the tree at  $B$ , is as  $BC \times \text{sine } \angle C$ , that is, since  $AB$  is as the sine of the angle  $c$  when the hypotenuse  $AC$  is given, the effect is as  $AB \times BC$ , or as the area of the triangle  $ABC$ , which is also as  $AC \times BE$  which is equal to the double area; but  $AC$  is given, and therefore the effect is simply as the perpendicular  $BE$  of the right-angled triangle  $ABC$  having the hypotenuse  $AC$  given; which effect being a maximum, the perpendicular  $BE$  must be a maximum, and consequently must stand upon the middle of the hypotenuse, making the triangles on both sides of it equal in all respects. Consequently the  $\angle A = \angle c =$  half a right angle, and the height  $CB =$  the distance  $AB = AC \sqrt{\frac{1}{2}} = 49\frac{1}{2}$  nearly.

X. QUESTION 902, by Mr. John Farey, of London.

There is a cubical block of marble, whose side in inches is expressed by two digits; the superficies of the block is equal to 864 times the sum of the said digits, and its solidity is equal to 576 times the square of the sum of the said digits: required the dimensions?

*Answered by Mr. J. Hartley, of Fleet-street.*

Make  $a = 864$ ,  $b = 576$ ,  $x =$  sum of the two digits, and  $y =$  side of the block. Then  $6y^2$  is the superficies, and  $y^3$  the solidity of the block. Hence, by the question  $y^3 = bx^2$ , and  $6y^2 = ax$ , or  $36y^4 = a^2x^2$ . Hence,  $a^2bx^2 = 36by^4 = a^2y^3$ , or  $36by = a^2$ , and  $y = a^2 \div 36b = 36$ , the side of the cube sought.

*The same by Mr. Isaac Saul, of Holland, near Wigan.*

Put  $x$  and  $y$  for the two digits. Then  $x + y$  is their sum, and  $10x + y$  the side of the cube. Hence, by the question,  $(10x + y)^2 \times 6 = (x + y) \times 864$ , or  $(10x + y)^2 = (x + y) \times 144$ , and  $(10x + y)^4 = (x + y)^3 \times 576$ . From the 1st equation  $(x + y)^2 = (10x + y)^4 \div 144^2 = (10 + x)^3 \div 24^2$  from the 2d; hence  $10x + y = (144 \div 24)^2 = 6^2 = 36$  the required side of the cube.

*The same by Mr. Wm. Weatherill, of York.*

Let the two digits be denoted by  $x$  and  $y$ ; so will  $10x + y$  be the

side of the block. Then, per question  $(10x + y)^2 \times 6 = (x + y) \times 864$ , and  $(10x + y)^2 = (x + y)^2 \times 576$ . Divide 6 times the latter equation by the former, then  $10x + y = 4x + 4y$ , and hence  $y = 2x$ ; this substituted in the first makes it  $144x^2 = 3x \times 144$ , which divided by  $144x$ , gives  $x = 3$ . Consequently  $y = 6$ , and the side of the block is 36.

XI. QUESTION 903, by the Rev. Mr. John Hellins.

How many cubical feet of water will freely flow through a circular hole, of one foot diameter, in a board fixed perpendicular to the horizon, in one hour; the surface of the water being kept always level with the top of the hole?

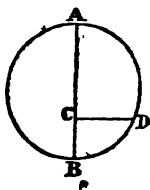
*Answered by the Rev. Mr. J. Hellins.*

Put  $32\frac{1}{6} = f$ , and  $x =$  any variable height above the bottom of the hole; then will the velocity, per second, of the issuing water, at that depth, be  $\sqrt{(2f \cdot (1 - x))}$ ; at which the width of the orifice will be  $2\sqrt{(x - xx)}$ . Therefore  $2x\sqrt{(x - xx)} \cdot \sqrt{(2f \cdot (1 - x))}$  or  $\sqrt{8f} \cdot (x^{\frac{1}{2}}x - x^{\frac{3}{2}}x)$  is the fluxion of the quantity of water that flows out of the lower segment of the aperture in the first second; the fluent of which is  $\sqrt{8f} \cdot (\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}})$  or  $\frac{1}{15}(10x - 6xx) \div \sqrt{8fx}$ ; which, when  $x = 1$ , becomes  $\frac{4}{15}\sqrt{fx} = \frac{8}{15}\sqrt{64\frac{1}{6}} = 4 \cdot 2777634$  cubical feet; and this number multiplied by 3600, the seconds in an hour, gives 15400 cubical feet nearly, the quantity sought.

N. B. This solution is on the supposition that the velocity of the issuing water, at any depth, is equal to that acquired by a heavy body in falling through that space; and if the number here given be diminished in the ratio of  $\sqrt{2}$  to 1, the result will give the quantity that would run out in one hour, if the velocity be every where equal to that which is due to half the height.

*The same by Mr. John Dalton, of Kendal.*

Let AB represent the diameter of the hole, perpendicular to the horizon, = 1 foot, and CD an ordinate to it; put 3600 seconds =  $t$ ,  $s = 32\frac{1}{6}$ , and  $AC = x$ , and suppose CD to flow from B towards A. Then, by Hydrostatics,  $\sqrt{(s \cdot AC \cdot 2AC \cdot CB)}$  drawn into the fluxion of AC = the fluxion of the water =  $\sqrt{s} \cdot 2\sqrt{(1 - x)} \cdot x\dot{x}$ ; and its correct fluent multiplied by  $t$ , is  $\frac{8}{15}t\sqrt{s} \times (1 - (1 - x)^{\frac{3}{2}} \times (1 + \frac{2}{3}x))$ . Wherefore when  $x = 0$ , the fluent becomes  $\frac{8}{15}t\sqrt{s} = 10889 \cdot 4$  cubical feet.



*Note.* This solution is on the supposition that the velocity is equal to that generated by gravity through half the height.

*Corollary.* Hence it appears that the water flowing out of a circular hole, is to that flowing out of a square one, whose side is equal to the circle's diameter, in this circumstance, as 4 to 5.

*The same answered by Wirksworthiensis.*

Put  $a = 12$  inches the diameter of the orifice,  $b = 32\frac{1}{2}$  feet  $= 386$  inches the force of gravity per second, and  $x =$  any variable part of the diameter from the top. Now admitting the velocity of the water at  $x$  depth to be equal to the velocity of a heavy body acquired in falling through the same space; then  $\sqrt{\frac{1}{2}b} : \sqrt{x} :: b : \sqrt{2bx} =$  velocity per second of any part of the ordinate corresponding to the depth  $x$ ; and the quantity flowing out, as the velocity and aperture, or  $2x \times \sqrt{2bx} \cdot \sqrt{(ax - x^2)} = 2\sqrt{2b} \cdot xx\sqrt{(a - x)}$ ; the correct fluent of which is  $\frac{8}{15}\sqrt{(2a'b)} - \sqrt{(8b)} \times (a - x)^{\frac{3}{2}} \times \frac{1}{15} \cdot (a + \frac{1}{2}x)$ , which when  $x$  is equal to the diameter of the circle becomes  $\frac{8}{15}\sqrt{(2a'b)} = 7391 \cdot 975$  cubic inches in one second; whence 15400 cubic feet of water will flow through the whole in one hour.

If the velocity of water issuing through an orifice be supposed equal to that acquired by a heavy body in descending through half the distance from the surface, as per prop. 2, book 2, of Sir Isaac Newton's Principia, then proceeding as before, 10889 cubic feet will be found to issue through in an hour.

## XII. QUESTION 904, by Mr. John Bonnycastle.

It is asserted by Mr. Castillioneus, in his Commentary upon Sir Isaac Newton's Arithmetick, that any rational cubic equation of the irreducible case, (as  $x^3 - 15x = 3$ ), will have at least one rational root: it is required to shew the truth or falsity of this assertion?

*Answered by Amicus.*

In the equation  $x^3 - 15x = 3$ , or generally, in  $x^3 - qx = r$ , where  $q$  and  $r$  are integers, let  $x = r \div p$ , then  $r^3 \div p^3 - qr \div p = r$ , and  $r^3 \div p^3 - p = q$ , where if  $r$  and  $q$  be integers, and  $r \div p$  rational,  $\pm p$  if rational must also be an integer. For, suppose the contrary, and that  $p$  is a fraction  $= \pm m \div n$  in its lowest terms, or,  $m$  and  $n$  prime to each other, then  $r^3 n^3 \div m^3 \mp m \div n = q$ , and  $r^3 n^3 \mp mm^3 \div n = m^2 q$  a rational integer; but  $n$ , being prime to  $m$ , must by Euclid 7, 27, be also prime to  $m^2$ , therefore  $mm^3 \div n$  must be a fraction, therefore  $r^3 n^3$  cannot be an integer, but it is necessarily an integer, consequently  $\pm m \div n = p$  must be an integer. And in the case in hand  $r^3 = 9$ ,  $q = 15$ , and  $9 \div p^3 - p = 15$ , and  $p$  an integer, also  $p^3$  a divisor of 9 and  $p$  of 3, consequently  $p$  must either be equal to  $\pm 1$  or  $\pm 3$ ; but none of the four will answer, therefore  $x^3 - 15x = 3$  has no rational root; and the assertion of *Castillioneus* is not true.

*Scholium.* By the above operation, simple as it is, may always be known, whether the cube root of any binomial surd, possible or impossible, can be extracted or not, and the trouble of such extraction avoided when it is only to find the roots of an equation; such, for example, as  $x^3 - 15x = 4$ , where, by *Cardan's* rule  $x = \sqrt[3]{(2 + 11\sqrt{-1})} + \sqrt[3]{(2 - 11\sqrt{-1})}$ , but  $16 \div p^3 - p = 15$ , consequently  $p = 1$  and  $x = 4$ . It moreover hence appears, that the objections made by Dr. Saunderson, M. de Moivre, and many others, to Dr. Wallis's rule for extracting the root of an impossible binomial are without foundation, since that rule will always find the root when there is such an one, and that may always be known from what is done above. The Doctor is also right in asserting, that, strictly speaking, there is no such thing as an *irreducible case*, or one that *Cardan's* rule will not reach; for that rule is general for all, and when any root is rational it may always be found by the method above, and when none can be found thus, it is also above demonstrated that the equation has none; and then in all cases equally the root may be approximated by extracting the cube roots of the binomials in series, as is largely shewn by Dr. Hutton in the Philosophical Transactions. And the method which I have given above is general for all, not only when  $q$  and  $r$  are integers, but also when they are rational fractions, for such equation may always be transformed to one wherein they are integers.

*The same answered by the Proposer, Mr. Bonnycastle. (Suppl.)*

This question will be best answered, by shewing that no equation whatever, above a simple one, which has its exponents, coefficients, and the absolute term whole numbers, can have a finite fractional root. And for this purpose little more is necessary to be proved, than that if two numbers be prime to each other; any powers of them will also be prime to each other; a truth which is demonstrated by Euclid in the 7th book of the Elements, and which is, indeed, nearly self-evident. For, as any powers of prime numbers, are compounded *only* of factors which are the same as the numbers themselves, it is plain that none of these factors can be common to each product, because, by the hypothesis, they are prime to each other; whence the products or powers themselves, must also be prime.

To apply this to the question, let us take the general equation  $x^n \pm ax^{n-1} \pm bx^{n-2} \pm cx^{n-3} \dots \pm qx = p$ ; where the exponents, coefficients, and absolute term  $p$  are all whole numbers. And in order to try whether  $x$  can be equal to any vulgar fraction, let it be assumed equal to  $\frac{y}{z}$ ,  $y$  and  $z$  being whole prime numbers.

Then by substituting  $\frac{y}{z}$  for its equal  $x$ , we shall have

$$\frac{y^n}{z^n} \pm a \times \frac{y^{n-1}}{z^{n-1}} \pm b \times \frac{y^{n-2}}{z^{n-2}} \pm c \times \frac{y^{n-3}}{z^{n-3}} \&c. \dots \pm q \times \frac{y}{z} = p;$$

and by multiplying by  $z^{n-1}$ , it is

$$\frac{y^n}{z} \pm ay^{n-1} \pm by^{n-2}z \pm cy^{n-3}z^2 \&c. \dots \pm qyz^{n-2} = pz^{n-1};$$

$$\text{or } \frac{y^n}{z} = pz^{n-1} \mp ay^{n-1} \mp by^{n-2}z \mp cy^{n-3}z^2 \&c. \dots \mp qyz^{n-2}$$

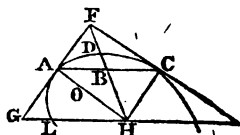
But  $\frac{y^n}{z}$  is an irreducible fraction; and all the terms on the right hand side of the equation are whole numbers, by the premises. The former, therefore, cannot be equal to the latter; and of consequence no root of the equation can be a finite fraction; for the same reasoning will hold whether  $\frac{y}{z}$  be positive or negative. If some of the terms in the equation be wanting, it is only to multiply by a different power of  $z$ , and the proof will be the same. Thus, in the equation proposed,  $x^3 - 15x = 3$ , let  $x = \frac{y}{z}$  as before; then  $\frac{y^3}{z^3} - \frac{15y}{z} = 3$ ; or  $\frac{y^3}{z^3} - 15y = 3z$ ; or  $\frac{y^3}{z^3} = 3z + 15y$ ; that is an irreducible fraction equal to a whole number, which is absurd.

### XIII. QUESTION 905, by Amicus.

What are the transverse and conjugate axes of the least ellipsis, such that a circle whose radius is unity may be the greatest that can be inscribed in any one quadrant thereof?

*Answered by the Proposer, Amicus.*

Let  $FGH$  be the equilateral triangle circumscribing the given circle whose radius is  $AO = 1$ ; let fall the perpendicular  $HA$ , draw  $AB$  parallel to  $GH$ , produce it till  $BC = AB$ , draw  $FC$  and  $HC$ . Then, from what is done for the parabola, at page 37, *Diary 1788*, the ellipsis must touch  $GF$  at  $A$ , and  $FC$  at  $C$  (because  $AB = BC$ ), and since  $AO = 1$ ,  $FO = 2$ ,  $AH = 3$ , and  $AF = CH = \sqrt{3}$ . But, because  $AB = BC = BF$ , the angle  $AFC$  is right; consequently  $ADCH$  is a quadrant of the ellipsis,  $CH = \sqrt{3}$  is the semiconjugate, and  $AH = 3$  is the semitransverse axis. And because the radius of curvature of the ellipsis at  $A$  is a third proportional to  $AH$ ,  $CH$ , and  $= 1 = AO$  the radius of the given circle, that circle may be cut from the ellipsis. Had the radius of curvature been less, the circle could not have been cut from such a curve, because in that case it must have fallen without it.

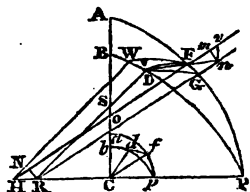


## XIV. QUESTION 906, by Mr. Isaac Dalby.

Suppose the earth an ellipsoid, having the equatorial and polar diameters 6993480 and 6954420 fathoms, respectively : now if a flagstaff be placed perpendicular to the horizon in latitude  $50^\circ$  north, longitude 0 ; and a theodolite in latitude  $49^\circ 40'$  north, longitude  $30'$  east ; what will be the observed horizontal angle, taken with the theodolite, between an object placed in its meridian, and the flagstaff ; supposing the flagstaff is long enough to be seen through the telescope when it is horizontal, and its axis 10 feet above the surface of the earth ?

*Answered by the Proposer, Mr. Isaac Dalby, of London.*

Suppose  $c$  to be the centre ;  $CA$ ,  $CP$  the equatorial and polar semi-diameters of the earth ;  $PA$ ,  $PB$  two meridians on the spheroid ; and  $pa$ ,  $pb$  two corresponding ones on a sphere having the same centre  $c$ . Let the points  $D$ ,  $F$ ,  $d$ ,  $f$  have the same latitudes and difference of longitude on both figures ; and draw the verticals  $DR$ ,  $FR$ . Then since the angles  $DSB$ ,  $FOA$  in the spheroid are  $=$  the latitudes of  $D$  and  $F$  respectively, and  $=$  the angles  $dcB$ ,  $fca$  in the sphere, therefore the verticals  $DR$ ,  $FR$  are  $\parallel$  to the radii  $dc$ ,  $fc$ . Now suppose the latitude of  $F$  or  $f$  to be greater than that of  $D$  or  $d$ , and let it be required to make the horizontal  $\angle FDG$  on the spheroid  $=$  the horizontal  $\angle pdf$  on the sphere. Because the horizontal  $\angle pdf$  is measured by the inclination of the planes  $fdc$ ,  $pdc$ , and  $cd$ ,  $rd$  are parallel, and in the same plane, therefore where the horizontal  $\angle FDG$  is  $=$   $pdf$ , the plane  $GDG$  must be parallel to the plane  $fdc$  ; hence, if  $RG$  be drawn  $\parallel$   $cf$  it will give the point  $G$  in the meridian  $FR$  making the  $\angle FDG$  on the spheroid  $=$   $pdf$  on the sphere. And for the same reason, if  $HW$  is drawn  $\parallel$   $RD$  or  $cd$ , the plane  $FWH$  will be  $\parallel$  to the plane  $GDG$  or  $fdc$ , and the  $\angle FRW = \angle pfd$ . Hence it is evident, if  $D$  be the place of observation, and  $F$  an object on another meridian, that the horizontal  $\angle FDR$  between the north part of the meridian and  $F$ , will be greater (by the  $\angle FDG$ ) than it would be on a sphere, (the latitudes and longitudes being the same on both) as long as the latitude of  $F$  is greater than that of  $D$  ; for when the latitudes are the same, the planes  $DGR$ ,  $FWH$  will coincide, and the angles be the same as on a sphere : but if the latitude of the place of observation ( $F$ ) is the greatest, the observed  $\angle FRD$  will be less by the  $\angle WFD$ , which, because the planes  $HWF$ ,  $RDG$  are parallel, is  $=$  the  $\angle GDF$  the excess on the other side. Hence, when the species of spheroid is given, the method of determining the  $\angle FDG$  may be thus : In the spherical  $\triangle dpf$ , with the co-latitudes  $pd$ ,  $pf$ , and included  $\angle$ , or dif. of long. find the  $\angle$ s at  $f$  and  $d$ , and the side  $df$ , or  $\angle dcf$  ; then from the nature



of the spheroid, find the length of the vertical  $DR$ , and also  $RH$  the distance between the points where the verticals meet the axis; on  $RH$  let fall the perpendicular  $RN$ , which will also be  $\perp RG$ , and because it is in the plane of the meridian  $FR$ , it will evidently be  $\equiv$  the arc  $GR$  extremely near: now the  $\angle FHR$  being  $\equiv$  the co-latitude of  $F$ , we have, radius:  $HR :: \cosine \text{ latitude} : RN$ ; and because the arc  $DR$  (considered as an arc of a circle) or  $\angle DRG$  is the same as the arc  $df$  or  $\angle dcf$ , if  $GR$  be taken as an arc of a circle to the same radius ( $DR$ ), the sides  $DR$ ,  $GR$ , and included  $\angle DGR (= \text{comp. of } \angle dfp)$  will give the  $\angle DRG$ , or dif. of the horizontal angles on the sphere and spheroid, when the telescope is pointed to the surface at  $F$ .

This is general for any spheroid. When the figure is the ellipsoid in question, we get  $DR = 3508112$ , and  $RH = 148.3$  fath. hence  $RN (GF) = 95.3$  fath. and the spherical  $\triangle pfd$  gives  $\angle pfd = 135^\circ 45' 16''.2$ ,  $\angle pfd = 43^\circ 51' 18''.2$ , and the arc  $df = 27' 49''.7 = \angle DRG$ ; this will give the  $\angle DRG = 8' 5''$  nearly for the dif. between the horizontal angles on the sphere and spheroid.

But when the telescope is horizontal, the dif. will evidently be something less: Let  $m$  and  $n$  be the points where the vertical (or flag-staff)  $hm$ , and its parallel  $rn$ , cut the plane of the horizon of  $D$ ; then  $mn$  (in the plane of the meridian  $FR$ ) will be what subtends the true difference, which may be determined as follows: Seeing  $DN$  (which is a perpendicular to the vertical  $DR$ , or the tangent to the  $\angle DRN$ ,  $DR$  being the radius) and the tangent to the meridian at  $D$  are both in the plane of the horizon, conceive the tangent of the co-latitude of  $D$  to be drawn to meet the axis  $CP$  (produced) then radius:  $DR :: \text{tangent co-latitude } (40^\circ 20' = \angle DRP) : 2978606 \text{ fath.} = \text{the tangent};$  and radius:  $DR :: \text{tangent } DRG (27' 49''.7) : 28398.5 \text{ fath.} = DR$ , this, and 2978606 the other tangent and the included angle  $43^\circ 51' 48''.2$  ( $\angle GDR$  or  $\angle DRP$ ) as a plane triangle, gives the  $\angle$  at  $n = 135^\circ 45' 19''.6$ , and its comp.  $44^\circ 14' 40''.4$  is  $\equiv$  the  $\angle mnd$ . Now suppose  $r$  to be the vertex,  $rn$  the axis, and  $nd$  the radius of the base of a cone whose base is in the plane of the horizon; then if  $a = \text{sine of } 44^\circ 14' 40''.4$  ( $mnd$ ), and  $s = \text{sine } 67^\circ 52' 39''.8$  (half its comp.)  $l = DR$ , and  $m = dn$  (the radius of the base) it will readily be seen that  $\sqrt{(l^2 + a^2 m^2 \div s^2)} = 3508177.8$  fath. is the length of a line drawn from  $r$  to meet  $mn$  (produced) in the circumference of the base; this line, the axis  $rn = 3508227.2$  fath. and 28398.5 (the radius of the base) being the sides of a plane triangle in the plane of the meridian  $AP$ , will give the  $\angle mnr = 89^\circ 40' 6''.3$ ; hence the  $\angle mnd$  ( $nd$  being  $\parallel RN$ )  $= 19^\circ 53''.7$ , and  $mn$  being  $= 95.3$  fath.  $= rn$ , we have  $nm = 95.35$  fath. nearly; this, with  $dn = 28398.5$ , and the included  $\angle mnd = 44^\circ 14' 40''.4$ , gives the  $\angle ndm = 8' 4''.4$  the difference of the horizontal angles; hence the observed  $\angle$  at  $D = 43^\circ 51' 48''.2 + 8' 4''.4 = 43^\circ 59' 52''.6$ .

PRIZE QUESTION, by *Lieut. W. Mudge, of the Royal Artillery.*

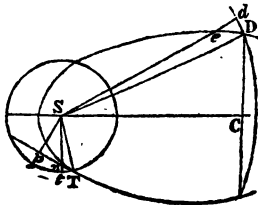
It is required to determine the quantity of heat received by the great comet, expected to appear in the beginning of the year 1789, during its passage from the aphelion to its perihelion, the quantity received in one second when at the mean distance of the earth being given equal to  $q$ ; and to compare the mean heat of the earth to the greatest heat of the comet when in its perihelion: the period of the comet being  $128\frac{1}{4}$  years, and its perihelion distance  $0.44851$ , the radius of the earth's orbit being 1.

*Answered by Amicus.*

According to the method made use of by *Sir I. Newton* for determining the heat of the comet of 1680, the heat at that in question will be to the heat of the summer sun here as  $4.9711$  to 1, or nearly as 5 to 1; and generally at the distance  $y$  from the sun, the heat will be as  $y^{-2}$ , and if  $z$  = the arc or angle to a given radius 1, described by the comet about the sun in the time  $t$ ,  $y^{-2}t$  will be as the heat received in the time  $t$ ; but  $t$  is as the elementary area described about the sun in that time, that is  $t$  is as  $\frac{1}{2}y^2z$ , consequently the heat in the time  $t$  will be as  $y^{-2}y^2z$  or as  $z$ , and in the finite time  $t$  as  $z$  the angle described about the sun in that time, let the distance be what it will. Which conclusion, though differently brought out, is agreeable to that of *Mr. Simpson* at Art. 473 of his *Fluxions*. Hence then whilst the comet in any part of its orbit describes an angle =  $180^\circ \div 64\frac{1}{8}$ , it receives at the point turned towards the sun a quantity of heat equal to that received at a like point of the earth during the course of a year. Because the earth must make  $128\frac{1}{4}$  revolutions to receive the quantity of heat which the comet does in one of its revolutions, therefore in one year the earth can only receive the  $1 \div 128\frac{1}{4}$  of this, or a quantity = that received by the comet during the time of describing an angle about the sun =  $360^\circ \div 128\frac{1}{4} = 2^\circ 48' 42''$ ; and the time which the comet in different parts of its orbit takes to describe this angle, is given by *Kepler's* problem.

*The same answered by Mr. John Dalton, of Kendal.*

**Lemma 1.** Suppose two like bodies to revolve round the sun in concentric circles; then the quantities of heat received by each body in one revolution will be inversely as the square root of their distances from the sun. For let  $\pi, h$  be the quantities received in any small given time, as one second;  $\tau, t$  their periodic times in seconds;  $\pi, r$  their distances from the sun. Then as the density, and consequently the heat of the



solar rays, is inversely as the square of the distances from the sun, it will be  $H : h :: r^2 : R^2$ , and it is well known that  $T : t :: R^{\frac{3}{2}} : r^{\frac{3}{2}}$ , therefore by multiplication  $TH : th :: \sqrt{r} : \sqrt{R}$ .

*Lemma 2.* Suppose two like bodies to revolve round the sun, the one in a circle and the other in an ellipse, whose transverse is equal to the diameter of the circle; then the quantities of heat received by each in one revolution will be inversely as the areas of their orbits. For let the orbits be as in the figure, and draw  $sd$ , and  $sed$  indefinitely near it; then since the velocities of the two bodies when at  $d$  are equal (Principia lib. i. prop. 16, cor. 4), the times of describing, and consequently the heat acquired in  $nd$ ,  $ne$ , will be as  $nd$  to  $ne$ , that is, from similar  $\triangle s$ , as  $dc : sd$ , or as the area of the ellipse to the area of the circle. But (Simpson's Fluxions art. 473) the whole quantities received in one revolution are as the quantities received in the  $\angle vsd$ ; therefore the truth of the lemma is manifest.

*Solution.* To determine the comet's orbit, from the data and the laws of centripetal forces, it will be  $1^3 : 1^3 :: (128\frac{1}{4})^3$ : the cube of the semitransverse of its orbit; which by extracting the root is had  $= 25.4314769 = a$ , and the eccentricity  $=$  semitransverse — perihelion distance  $= 24.9829669 = b$ ; from which (47 Eucl. 1.) the semiconjugate is had  $= 4.755142 = c$ . Then from the first lemma it will be  $\sqrt{a} : \sqrt{1} :: sq : sq \div \sqrt{a} =$  the heat that would be received by the comet in one revolution round the sun in a circle, whose diameter  $=$  the transverse of its orbit, where  $s$  denotes the number of seconds in 1 year. But by the 2d lemma,  $c : a :: sq \div \sqrt{a} : sq \sqrt{a} \div c =$  the heat received by the comet in one revolution in its proper orbit; half of which  $= 16733512q =$  the quantity of heat received in its passage from aphelion to perihelion, as required. Also, the heat of the comet in perihelion will be to the mean heat of the earth, as 1 to .44851<sup>2</sup>, or as 4.97113 to 1 nearly.

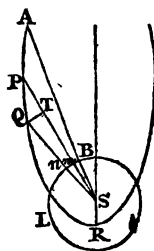
*Otherwise thus, without the Lemmas.*

The dimensions of the orbit being found as above, and the notation retained, half the latus rectum  $= c^2 \div a$ . Draw  $st$  to the intersection of the orbits of the earth and comet (which orbits may be supposed in the same plane, without error in this consideration); also draw a tangent to the point  $t$  of the ellipse, and  $sp$  perpendicular thereto; which last will be (Emerson's Conics, prop. 24)  $= c \div \sqrt{(2a - 1)}$ . Let  $tn$  be the small part of the comet's orbit run through in a second at that place, and draw  $snt$  to cross its orbit in  $n$  and meet the earth's orbit in  $t$ ; then the triangles  $spt$  and  $tn$  being similar,  $rt : tn :: sp (c \div \sqrt{(2a - 1)}) : ts (1)$ . But the velocity of the comet at the point  $t$  is to the velocity of a body revolving in a circle at that distance from the sun (Principia, lib. 1. prop. 16, cor. 9) as  $\sqrt{(st \times \frac{1}{2} \text{ latus rectum})}$  is to  $sp$ , that is as  $c \div \sqrt{a}$  to  $c \div \sqrt{(2a - 1)}$ , or as  $\sqrt{(2a - 1)}$  to  $\sqrt{a}$ ; and the times being as the spaces directly and veloci-

ties inversely, it will be, as time in  $\tau t$  : time in  $\tau n$  ::  $\frac{c}{\sqrt{(2a-1)} \cdot \sqrt{a}}$   
 ::  $\frac{1}{\sqrt{(2a-1)}} :: \frac{c}{\sqrt{a}} : 1$  :: the heat received in  $\tau t$  : the heat received in  $\tau n$  (the body being given); and as this ratio likewise obtains for the whole quantity of heat received by any given body in one revolution in each orbit (see Simpson above) and the whole quantity received in one revolution in the earth's orbit being =  $sq$ , it follows that  $c \div \sqrt{a} : 1 :: sq : sq\sqrt{a} \div c$  = the heat received by the comet in one revolution as above.

*The same answered by the Rev. Mr. William Sewell, A. M. of Northwalsham. (Suppl.)*

Let  $APR$  represent any portion of the comet's orbit;  $A$  the place from which we are to reckon the commencement of the time;  $P$  the place of the comet at any instant afterwards. With radius 1, representative of the earth's mean distance from the sun, or centre of force  $S$ , describe the circular arc  $BL$ : draw  $SP$ , and let  $sq$  be conceived indefinitely near it. Put  $A$  for the area described, by the ray  $SP$  in one complete revolution, and  $r$  for the correspondent period: then as the areas are descriptive of the times, it is evident  $\frac{P}{A} \times$



$\frac{PS \times QT}{2}$  will be the time of describing the incremental arc  $rq$ ; and as the heat received is as its intensity and time conjunctly, we have  $\frac{q}{PS^2} \times \frac{P}{A} \times \frac{PS \times QT}{2}$ , or  $\frac{QP}{2A} \times \frac{QT}{PS}$ , or its equal  $\frac{QP}{2A} \times mn$ , for the incremental quantity of heat received in the same time; and consequently the integral quantity itself, received by the comet in its passage thro'  $AP$ , will be truly defined by  $(qr \div 2A) \times$  circular arc  $bm$ . And therefore the heat received in half a revolution is  $(qr \div 2A) \times 3.14159$ , let the figure of the orbit be what it may. Now the quantity of heat received by any equal magnitude of the earth's disk in the same time, is evidently  $\frac{1}{2}qr$ ; therefore the heat of the earth will be to the heat of the comet in its perihelion ::  $A$  : area of a circle whose radius is 1, or as 121 to 1 nearly\*, supposing no heat to be lost, but to be uniformly dispersed throughout the substance of each body.

\* The quantity  $A$  may be easily determined from the given perihelion distance and periodic time, thus: by Newton, in case of elliptic orbits,  $r : 1 :: a^3 : 2t^3$  ( $a$  being the major axis), therefore  $a = 2r^{\frac{1}{3}} = 50.863$ . Put  $d = .44851$ , the given perihelion distance, then by Conics  $\sqrt{((a-d) \times d)}$  = semiconjugate axis = 4.7572; whence  $A = 120.93 \times 3.14159$ , &c.

*Scholium.* From this theorem it may readily be adduced, that the heat received by the great comet of 1680, during one of its periods, does not exceed above  $\frac{1}{16}$ th of the quantity which our earth receives in the same time, notwithstanding the density of the solar rays acting upon its surface, when in the perihelion, is 2000 times greater than sufficient to heat iron red hot; and in respect to the comet in question, is more than twice sufficient for that purpose. Whence we learn, that the heat, received by comets, upon the whole, falls far short, *cæteris paribus*, of that received by our earth.

It is to be noted, it is not the whole body or mass of the comet that can be heated to the degree above mentioned, but the matter near the surface only, and that more especially to which the sun is vertical, and even here not without taking into the account what probably is not possible, an unclouded atmosphere.

And hence the cometarians, if such inhabitants there are, of which I little doubt, may not suffer so great inconvenience from the perihelion heat as may at first be suspected: for soon after the comet is descended within the sphere of the earth's orbit, the solar influence upon the waters, or other fluids, would soon be sufficient to raise a thick and impenetrable fog into its atmosphere, so as to prevent the inhabitants even of the equatorial parts from perishing, so nicely does the adjusting hand of Providence regulate the laws of nature, and make them operate to the due preservation of life.

It may not be so easy, perhaps, to account for the subsistence of these inhabitants, whilst they are sluggishly carried thro' the cold regions of the aphelion, unless the electric fluid, which is esteemed, by experimentalists, the grand vehicle of animal and vegetable life, and of which comets are observed to superabound on the comparison with planets, which move in more regular orbits, may be supposed to contribute a principal share towards a solution of the difficulty.

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*Questions proposed in 1790, and answered in 1791.*

**I. QUESTION 908, by Mr. A. Buchanan, jun. of Sedgefield;**

From these equations here subjoin'd \*

In next year's Di, pray ladies shew

My age.—With ease you'll soon it find

In years and months from what's below.

\* Given  $x \cdot y + x^3 = 7581$ , } Where  $x$  is the year,  
and  $xy^3 + y^3 = 513$  } and  $y$  is the month.

*Answered by Mr. John Cavill, of Beighton,*

By the Question  $x$  and  $y$  must be whole numbers, and  $y$  less than

12. Now in the first equation  $y = (7581 - x^3) \div x^2$ ; hence  $x$  is less than 20, and (to have  $y$  within its limits) greater than  $\sqrt[3]{6}$ ; therefore trying the three intermediate numbers, 19 is the only one that will answer without a remainder. Hence, the gentleman's age was 19 years and 2 months.

*The same answered by Mr. John Dalton, of Kendal.*

To determine the limits of  $x$  in the first equation, it is obvious if  $y = 0$ ,  $x$  is then the greatest possible, and  $\sqrt[3]{7581} = 19$  with a remainder; also  $x$  is least when  $y = 12$ , and the equation then becomes  $12x^2 + x^3 = 7581$ , in which  $x$  is above 16; so that  $x$  must be either 17, 18, or 19; these numbers being severally substituted in the equation,  $y$  is determined to be a whole number only when  $x = 19$ , it being then  $= 2$ ; but these values substituted in the second equation do not succeed; from which it may be concluded there is some error in this equation.

*The same by Mr. Tho. Leybourn, of North-Shields.*

The second equation is wrong printed, and the two ought to be either  $\begin{cases} x^2y + x^3 = 7581 \\ xy^2 + x^2 = 513 \end{cases}$  or  $\begin{cases} x^2y + x^3 = 7581 \\ xy^2 + y^3 = 156 \end{cases}$ .

Put  $a = 7581$ ,  $b = 156$ , and  $c = 513$ . Now, by the nature of the question,  $x$  and  $y$  must both be whole numbers; and since  $y = \frac{a - x^3}{x^2}$   $(a - x^3) \div x^2$  from the first equation, it is evident that  $x$  must be a whole number below 20, and  $y$  a whole number not greater than 12; then by making trial of 19 for  $x$ ,  $y$  comes out 2, which two numbers answer the question exactly.

Or thus, by using the latter Equations. From the first  $y = (a - x^3) \div x^2$  which substituted in the second, it is  $x \times (a - x^3) \div x^3 + (a - x^3)^2 \div x^4 = b$ ; which gives  $x = 19$ , and  $y = 2$ . Or, by using the former,  $x \times ((a - x^3) \div x^2)^2 + x^3 = c$ ; which again gives  $x = 19$ , and  $y = 2$ .

II. QUESTION 909, by Mr. J. Fildes, Schoolmaster, Liverpool.

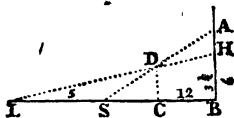
Being in a room opposite to the side of a window, the bottom of which was just the height of my eye, I observed that up the edge of the window I could see 42 courses of bricks in a wall on the opposite side of the street; but walking in a direct line towards the window 5 yards, I found that I could see 72 courses. Required the height of the window, supposing the breadth of the street to be 12 yards, and 4 courses of brick work to the foot in height?

*Answered by Miss Betty Claxton, Bentwell, near Newcastle upon Tyne.*

Let  $r$  and  $s$  denote the observer's eye at the first and second stations

respectively. Draw LH and SA: then will BH and BA represent the given portions of the wall seen from those respective stations through the window CD.

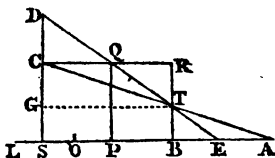
Now BA being = 6 yards, BH =  $3\frac{1}{2}$ , LS = 5, and the breadth of the street BC = 12 yards; by similar triangles it is, first  $sc + 12 : 6 :: sc : CD$ , and  $sc + 17 : 3\frac{1}{2} :: sc + 5 : CD$ ; then, by comparing these two together, it is  $sc + 12 : sc + 17 :: 6 \times sc : 3\frac{1}{2} \times (sc + 5)$ ; then multiplying extremes and means, and reducing, gives  $sc^2 + 17sc = 84$ ; the resolution of which quadratic equation gives  $sc = 4$  yards; and hence  $CD = 1\frac{1}{2}$  yards = 4 feet 6 inches, the height of the window sought.



*The same by Amicus.*

Let E and A be the places of the eye, TB the window, and DS the wall; then AE is given = 15, CS =  $10\frac{1}{2}$ , DS = 18, and SB = GT = CR = 36, to find TB or BE.

By similar  $\triangle s$  DC : CS :: CQ : PE, and AB : AS :: TB : BR :: BE : PE, also by equal. DC : CS :: AB  $\times$  CQ : AS  $\times$  BE = LE  $\times$  BE, taking LS = AE; but AB : SB = CS :: AE : CQ, therefore DC : CS :: SB  $\times$  AE = SB  $\times$  LS : LE  $\times$  BE, which is therefore given. Bisect LB in O; then  $LE \times BE = OE^2 - LO^2 = OE^2 - 51^2 \div 4 = 10\frac{1}{2} \times 15 \times 36 \div 7\frac{1}{2}$ , and  $OE = 37\frac{1}{2}$ ; hence BE = 12, SE = 48, and SE : BE :: DS : TB =  $4\frac{1}{2}$  the window's height required.



III. QUESTION 910, by Mr. Hepworth, of Northwalsham Academy.

Required the roots of the equation  $4x^4 + 8x^3 - 89x^2 + 28x + 49 = 0$ , by quadratics only?

*Answered by Mr. John Harrison, of Manchester.*

If  $121x^2$  be added to both sides of the given equation, both sides become complete squares, viz.  $4x^4 + 8x^3 + 32x^2 + 28x + 49 = 121x^2$ , the roots of which give  $2x^2 + 2x + 7 = 11x$ , or  $2x^2 - 9x + 7 = 0$ , the roots of which quadratic are 1 and  $3\frac{1}{2}$ .—Again, the original equation divided by this last, viz.  $2x^2 - 9x + 7 = 0$ , gives  $2x^2 + 13x + 7 = 0$ , the roots of which are  $\frac{-13 + \sqrt{113}}{4}$  and  $\frac{-13 - \sqrt{113}}{4}$ .

Consequently 1,  $\frac{7}{2}$ ,  $\frac{-13 + \sqrt{113}}{4}$ ,  $\frac{-13 - \sqrt{113}}{4}$  are the four roots.

*The same by Mr. John Ryley, Teacher of the Mathematics, at Leeds.*

By the well known method of finding trinomial divisors, it is discovered that either  $2x^3 - 9x + 7$ , or  $2x^3 + 13x + 7$ , will divide the given equation: also, if the said equation be divided by one of these expressions, the quotient will always be the other. And because the original equation  $4x^4 + 8x^3 - 89x^2 + 28x + 49 = 0$ , is universally equal to  $(2x^3 - 9x + 7) \times (2x^3 + 13x + 7) = 0$ ; of course  $2x^3 - 9x + 7 = 0$ , and  $2x^3 + 13x + 7 = 0$ : from these

equations, viz. from the former  $x = \frac{9 \pm 5}{4}$ , and from the latter  $x = \frac{-13 \pm \sqrt{113}}{4}$ . Therefore the four roots of the given equation are 1, and  $3\frac{1}{2}$ , and  $-5.92463547$  &c. and  $-5.907536453$  &c. as required.

IV. QUESTION 911, by Mr. John Biebford, *Westminster.*

It is required to find a point in a right line between the earth and moon, from which an equal quantity of the surface of these two bodies might be seen: the earth's diameter being 7964, the moon's 2192, and the distance of their centres 240000 miles?

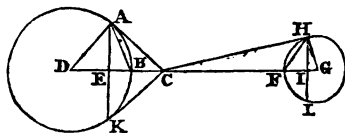
*Answered by Mr. John Haycock, of Ware.*

Let  $c$  be the point required, in the line joining the centres,  $D$ ,  $G$ , of the earth and moon;  $CA$ ,  $CH$ , tangents at  $A$ ,  $H$ ; and  $AK$ ,  $HL$ , perpendicular to  $DG$ . Then the surfaces of the two segments  $ABK$ ,  $HFL$ , are to be equal.

Put  $a = DG = 240000$ ,  $b = DA = 3982$ ,  $c = GH = 1096$ , and  $x = DC$ ; then  $DC : DB :: DB : DE = b^2 \div x$ , and  $BE = b - b^2 \div x$ ; in like manner  $FI = GF - GI = c - c^2 \div (a - x)$ . But, by Dr. Hutton's *Measurement*, page 199, 2nd edition, or *Compend. Measurer*, page 136, the surface of a segment is as the product of the radius of the sphere and height of the segment; therefore  $DB \times BE = GF \times FI$ , that is

$$b \times (b - \frac{b^2}{x}) = c \times (c - \frac{c^2}{a-x}), \text{ or } x^3 - (a + \frac{b^3 + c^3}{b^2 - c^2}) \times x +$$

$\frac{ab^3}{b - c^2} = 0$ ; hence  $x = 4306.74585$  miles, the distance from the earth's centre, or  $324\frac{1}{2}$  miles nearly from the earth's surface.

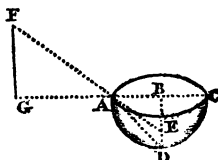


## V. QUESTION 912, by Mr. Matt. Terry, of Askrigg.

Suppose a semispherical vessel, whose diameter is 12 inches, be filled with water; at what distance from its edge must a person stand, the perpendicular height of whose eye above the top of the same is 4 feet, so as just to see the centre of its bottom?

*Answered by Mancuniensis.*

Let ABCD be the semispherical vessel, its altitude BD. Join AD, and take AE to AD as 396 to 529, or as 3 to 4 nearly, the ratio of the sine of refraction to the sine of incidence, out of water into air, according to Sir I. Newton, and continue EA to F so that FG be the given height 4 feet, of the eye; so shall DAF be the refracted ray, and AG the distance sought.



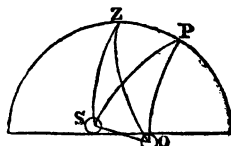
*Calculation.* As 4 : 3 :: AD = AB  $\sqrt{2}$  =  $\frac{1}{2}\sqrt{2}$  : AE =  $\frac{3}{8}\sqrt{2}$ ; hence BE =  $\sqrt{(AE^2 - AB^2)}$  =  $\frac{1}{8}\sqrt{2}$ ; and, by similar triangles BE : BA :: FG = 4 : AG =  $8\sqrt{2}$  = 11.3137 the distance sought.

## VI. QUESTION 913, by Mr. Thomas Crosby, of York.

Admit that on June the 21st the body of the sun be observed to rise out of the horizon, in 8 minutes time; what is the latitude of the place?

*Answered by Mr. James Adams, of Stonehouse, near Plymouth.*

The sun's semidiameter on the 21st of June is 15' 47'', refraction near the horizon 33', parallax 9'', and declination 23° 28'. Let P represent the pole of the world, z the zenith of the place of observation, o the place of the sun when his upper limb was observed in the horizon, and s his place when the under limb was observed there. The zenith distances being cleared of refraction and parallax, we obtain zo = 89° 42' 56'', and zs = 89° 11' 22''; also po = ps = 66° 32' the polar distance, the  $\angle ops = 2^\circ$ , which corresponds to 8<sup>m</sup> the given time. Hence, from the data in the triangle sro, are found os = 1° 50' 4'' 27''', and the  $\angle ros = 89^\circ 36' 6''\frac{1}{4}$ . In the triangle zos are given the three sides, to find the  $\angle zos = 73^\circ 19' 40''$ ; whence by subtraction is found the  $\angle roz = 1^\circ 16' 26''$ . Lastly, in the triangle roz are given two sides and the included angle, to find z the co-lat. = 27° 31' 6'' : and therefore the required latitude is 62° 28' 54''.



*The same by Mr. Da. Kinnebrook, jun. of Norwich.*

The figure being premised, as before, then, without considering the effect of refraction and parallax, in the isosceles triangle  $osp$  are given the two sides  $op$  and  $sp$  the complement of the declination =  $66^{\circ} 32'$ , and the included  $\angle ops = 2^{\circ}$ ; to find the 3d side,  $os = 1^{\circ} 50' 9''$ , and the  $\angle pos = 89^{\circ} 36' 4''$ . Again, in the triangle  $ozs$  are given all the sides  $oz = 90^{\circ} 15' 47''$ ,  $sz = 89^{\circ} 44' 13''$ , and  $os = 1^{\circ} 50' 9''$ ; to find the  $\angle zos = 73^{\circ} 20' 59''$ , whence  $\angle poz = \angle pos - \angle zos = 16^{\circ} 15' 5''$ . Then, in the triangle  $opz$  are given the two sides  $op$ ,  $oz$ , and the included  $\angle poz$ ; to find the 3d side  $zp = 28^{\circ} 30' 6''$ , the complement of the latitude; whence  $61^{\circ} 29' 54''$  is the latitude required.

VII. QUESTION 914, *by the Rev. Mr. L. Evans.*

The counterpoise of a steelyard being lost, it was observed, that there are on one edge of the arm 53 divisions, and on the other  $11\frac{1}{2}$ , each exhibiting in weighing one pound avordupois; moreover it was found that a weight of 1oz. 5dts. 12grs. placed on the last, or  $11\frac{1}{2}$  division of the less scale kept the balance in equilibrio. What must be the weight of a new counterpoise?

*Answered by Amicus.*

Put  $p = .0875$  lb. avordupois, the given weight which hung at A will keep the balance in equilibrio, and  $w =$  the required counterpoise. Then

$$\begin{aligned}
 p \times (AD + DC) \div DC &= w = CB, p \\
 \times AD \div DC &= w - p, DC = AD \times \\
 p \div (w - p), AD + DC + CB &= AB; OB = CB \times 11.5 \div 53, \\
 DO = AD \times 11 \div 53, AD + DO + OB &= AB = AD + DC + CB, \\
 DO + OB = DC + CB = 11 \times 11.5 \div 53 + w \times 11.5 \div 53 &= 11.5p \\
 \div w - p + w; \text{ reduced } w^2 - p + 11 \times 28 \div 83 \times w &= -.64 \\
 \times 23p \div 83 = w^2 - 2bw = -a^2, w = b \pm \sqrt{b^2 - a} &= 2.5193659 \\
 \text{or } .6163341, \text{ the counterpoise required.}
 \end{aligned}$$

VIII. QUESTION 915, *by Mancuniensis.*

Ye British Philomaths profound,  
Th' equation plac'd below expound;  
And you my age will quickly see,  
For three times  $x$  exact will be  
As many times as Terra's run  
Her annual course around the sun,  
Since first into this world of strife  
I came to drink the cup of life.

$$3x + \frac{201684}{x} \text{ a minimum.}$$

*Answered by Mr. Jacob Park, of Morpeth.*

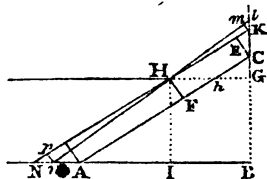
The given expression  $3x^4 + \frac{261684}{x}$  or  $3x^4 + \frac{a}{x}$  being a minimum, its fluxion  $12x^3 - \frac{ax}{xx} = 0$ ; therefore  $12x^3 = a$ , and  $x = \sqrt[3]{\frac{a}{12}} = 7$ ; whence  $3x = 21$ , the age required.

IX. QUESTION 916, *by Mr. John Haycock, of Ware.*

Given the breadth of a river, BG; and the length and breadth of a barge, AC, CE; to find the least breadth of a new cut, GH, at right angles, sufficient for the barge to turn into; supposing BG less than AC.

*Answered by Mr. John Dalton, of Kendal.*

Conceive the corners A and c of the barge to slide along the margins AB and GC; then will the point H, the intersection of the other side of the barge and margin of the river, recede gradually till a certain point, and then return again towards G, and the greatest distance GH will be when the line of the barge, prolonged to the margins NHK is divided in H so that BK : BN :: HN : HK. For, through H draw a right line nhl, making an indefinitely small angle with NK, to represent the position of the barge the next moment, and draw the two small perpendiculars km and pn; also let fall the perpendicular HI = GB: then it will be HN : HK :: pn : km = HK × pn ÷ HN = AB × pn ÷ BC; also AB : AC :: km = AB × pn ÷ BC : kl = AC × pn ÷ BC, and BC : AC :: pn : nn = AC × pn ÷ BC = kl; moreover it will be HN : HI :: HK : KG = HI × HK ÷ HN = AI × AB ÷ BC, and BC : AB :: HI : NI = HI × AB ÷ BC = KG; again, NI : IH :: KG : GK, therefore IN × GK = IH × HG; and as KG is to be a maximum, and IH is constant, therefore IN × GK must be a maximum, which is evidently the case, since the two legs are equal, and their respective fluxions nn and kl are also equal, with opposite signs.



*The same answered by Mr. Da. Kinnebrook, jun. of Norwich.*

Let  $a = AC$  the length, and  $b = CE$  the breadth of the barge;  $c = BG$  the breadth of the river, and  $x = AB$ : then  $BC = \sqrt{(a^2 - x^2)}$ , whence  $CG = \sqrt{(a^2 - x^2)} - c$ ; now by similar triangles  $AB : AC :: CE : CK = ab ÷ x$ , then  $KG = ab ÷ x + \sqrt{(a^2 - x^2)} - c$ ; again,  $BC : AB :: KG : GH = (ab - cx) ÷ \sqrt{(a^2 - x^2)} + x$  a maximum, which put into fluxions and reduced gives  $(a^2 - x^2)^3 = (ac - bx)^2 × a^2$ , from which equation the value of  $x$  may be found.

X. QUESTION 917, *by the Rev. Mr. Wm. Sewell, A. M.*

Query, are those picture frames we purchase of the oval frame turners, true Apollonian ellipses, or not?

*Answered by the Rev. L. Evans.*

There are several ways of turning those frames the ingenious proposer alludes to. One way is by what is called a trammel, which forms the frames truly elliptical, because its construction is similar to the elliptical compasses. Another way is, by a contrivance of putting the two puppets, which hold the collar and mandrel, to move backwards and forwards, by which means a true elliptical frame may be made, provided the model which regulates this motion be an ellipsis.—A third way is by a slider and hoop placed at any distance from the centre of the mandrel's motion, according to the eccentricity of the oval intended to be turned. And as the difference of the transverse and conjugate diameters of ovals turned this way, the hoop remaining in one particular state, is a constant invariable quantity, they are truly elliptical, as will be evident by attending to the nature of describing an ellipse by a continual motion. Vide Emerson's Conics, book 1, prob. 30.\* A true elliptical frame of small eccentricity might also be made in the same manner the *rose-work* is performed, viz. by putting the collar end only of the mandrel in reciprocal motion, while a plate of the intended elliptical form, not very eccentric as was before observed, fixed to the mandrel, is kept by a spring to act against a pulley, thereby giving a similar form to the frame turned.

\* The invariable quantity is the distance from the centre of the mandrel to the centre of the hoop.

The principal properties of the machine for turning ovals may be seen in a paper of Mr. Ludlam's, in the Philosophical Transactions, vol. 70, page 378.

*Observation by a Lunarian, taken from the Suppt. for 1792.*

“In answer to question 917, it may be observed, that neither of the methods given in the last Diary is generally practised by the picture-frame makers. They construct two equilateral triangles on the same base; then the angular points of this rhombus are the centres of 4 circular arcs which constitute the circumference of the oval. It is easy to perceive that these arcs, respectively, lie between the sides of the triangles produced. This is the old and common method (which still continues among the workmen) of making an oval. See Clavius's Geom. Also Guldini Centrobaryca, lib. 3, page 227.”

XI. QUESTION 918, *by Archimedes.*

To what constant height must the weight of a pile engine be raised, so as to have the greatest effect in a given time, the time of raising the weight being always as the height raised?

*Answered by the Proposer, Archimedes.*

Let  $y$  be the time in which the weight  $w$  is raised to the height  $x$ ,  $g$  = the given time, and  $a = 16\frac{1}{2}$  feet. Then  $\sqrt{x \div a}$  = the time of its falling from that height;  $\sqrt{x \div a} + y$  = time of one stroke; and  $g \div (\sqrt{x \div a} + y)$  the number of strokes in the given time  $g$ . Also  $\sqrt{a} : 2a :: \sqrt{x} : 2\sqrt{ax}$ , the velocity gained per second in falling through  $x$ ; hence, if the momentum or force is as the velocity drawn into the weight, we have  $2\sqrt{ax} \times gw \div (\sqrt{x \div a} + y)$  for the whole force in the time  $g$ . Next suppose the weight to be constantly drawn up to the height  $nx$  (instead of  $x$ ), then proceeding as above, we get  $g \div (\sqrt{nx \div a} + ny)$  for the number of strokes, and  $2\sqrt{anx} \times gw \div (\sqrt{nx \div a} + ny)$ , or its equal  $2\sqrt{ax} \times gw \div (\sqrt{x \div a} + y + y\sqrt{n})$ , for the whole momentum. Now it is evident that if  $n$  be less than unity, or, which is the same thing, if the weight falls from any height less than  $x$ , this last expression will be greater than the former, that is,  $2\sqrt{ax} \times gw \div (\sqrt{x \div a} + y\sqrt{n})$  greater than  $2\sqrt{ax} \times gw \div (\sqrt{x \div a} + y)$ ; therefore the less the height to which the weight is raised, the greater would be the effect (the effect being as the force) in a given time: but this is known to be directly contrary to experiment; and therefore in this case the force cannot be as the *simple* velocity into the quantity of matter.

If we suppose the momentum to be as the mass into the square of the velocity, and retain the notation as above; we get  $4ax \times gw \div (\sqrt{x \div a} + y)$  for the whole force when the weight falls from the height  $x$ , and  $4anx \times gw \div (\sqrt{nx \div a} + ny)$ , or its equal  $4ax \times gw \div (1 \div \sqrt{n}) \times (\sqrt{x \div a} + y)$  for the force when it falls from the height  $nx$ ; this expression, when  $n$  is greater than unity, will be greater than  $4ax \times gw \div (\sqrt{x \div a} + y)$ ; therefore on this supposition, the higher the weight is raised the more would be the effect in a given time.

Though it is evident from the above, that the force cannot be as the *simple* velocity into the mass, yet it does not follow that it is accurately as the *square* of the velocity into the mass. But Professor *Bugge* in his paper on the pile engine in the *Philosophical Transactions*, 1779, has adopted this latter hypothesis. And Mr. Robins's experiments at the end of his *Gunnery* seem to favour it. It is to be observed too that in this solution, the small time between descending and the next ascent is omitted, as well as the small time of the pile's sinking at each stroke.

XII. QUESTION 919, by the Rev. Mr. John Hellins.

Given  $y = \frac{y\dot{x}}{xx} + \frac{\dot{x}}{xx} + 3\dot{x} + 2x\dot{x} - \frac{4\dot{x}}{x}$ , to find the correct fluent of  $y\dot{x}$ , generated while  $y$  from 0 becomes = 44.

N. B. The reason for reproposing this question will appear in the solution.

*Answered by the Rev. Mr. J. Hellins.*

By assuming  $y =$  the ascending series  $A + Bx + cx^2 + dx^3 \&c.$  and substituting this value for it in the given equation, we obtain  $A = -1$ ,  $B = 4$ ,  $C = 1$ ,  $D$ , and all the succeeding coefficients,  $= 0$ . Therefore we have  $y = -1 + 4x + x^2$ ; which value is attainable also from the assumption  $y = rx^3 + qx^2 + B + sx^{-1} \&c.$  for  $r$ ,  $q$ , and  $B$ , in this assumption, will come out  $= C$ ,  $B$ , and  $A$ , respectively, in the former assumption; and  $s$ , and all the succeeding coefficients in this series, will vanish.

The value of  $y$  being now obtained, we have  $y\dot{x} = x^2\dot{x} + 4x\dot{x} - \dot{x}$ , the fluent of which is  $\frac{1}{3}x^3 + 2x^2 - x$ . But when  $y = 0$ ,  $xx + 4x - 1 = 0$ , and  $x = -2 + \sqrt{5}$ ; therefore the correct fluent is  $\frac{1}{3}x^3 + 2x^2 - x + \frac{1}{3}(10\sqrt{5} - 22)$ ; which, when  $y = 44$  ( $x$  being then  $= 5$ ), will be  $= 79 + \frac{1}{3}(1 + 10\sqrt{5}) = 86.78689$  nearly.

N. B. The equation given in this question may be found in Colson's translation of Sir Isaac Newton's Fluxions, page 40, and in Emerson's Fluxions, Prop. 10. Ex. 87; in both which places the value of  $y$ , given in an infinite descending series, is wrong.

XIII. QUESTION 920, by Lieut. W. Mudge, of the Royal Artillery.

If a heavy weight of iron, having a rod of 20 feet long fastened to the bottom of it, be suffered to descend from a height less than the length of the rod, the lower end of which slides freely along the ground as the weight descends; and if a small ring of metal be run on the rod, touching the weight when it first begins to descend: Query, how high should the weight fall from, so that when it arrives at the ground, the ring shall have run over one fourth of the rod.

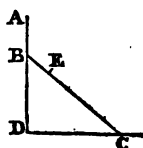
*Answered by the Proposer, Lieut. W. Mudge, of the Royal Artillery.*

Before attempting a solution to this question, the proposer observes that in wording it, he considered the heavy descending body as falling freely by the force of gravity without being at all resisted either by the rod, which is considered as without weight, or by the ring, which is considered as very small; on which account he omits any consideration about the centrifugal force on the ring, as foreign to the intent and meaning of the question.

Let  $BC$  be the rod, and  $AB$  the space the weight  $B$  has fallen at the end of any time  $t$ , at the same instant let the ring be at  $E$ , sliding freely down the rod  $BC$ ; and put  $AD$  the first height of the great weight from the ground  $= h$ ,  $BC = r$ ,  $AB = x$ ,  $BE = y$ , the velocity of the ring at  $E = v$ , and  $2g = 32\frac{1}{2}$ . Now  $2g \times BD \div BC = (h - x) \times 2g \div r$

is the force of gravity on the ring, and by the nature of forces ( $h-x$ )

$\times 2gt \div r = v$ ; but  $t = x \div 2\sqrt{gx}$ ; and hence  $v = (\sqrt{g} \div r) \times (hx \div \sqrt{x} - x\sqrt{x})$ , the fluents of which give  $v = (\sqrt{g} \div r) \times (2h\sqrt{x} - \frac{2}{3}x\sqrt{x})$ . Again,  $y \div v = x \div 2\sqrt{gx}$ , and substituting for  $v$ , then  $y = hx \div r - x^2 \div 3r$ , and the fluents give  $y = (6hx - x^2) \div 6r$ , the distance of the ring from the great weight. But by the question when  $x = h$ ,  $y = \frac{4}{3}r$ , therefore  $\frac{4}{3}r = 5h^2 \div 6r$ , and  $h = r\sqrt{(3 \div 10)}$ , the height required.



XIV. QUESTION 921, by Mr. John Bonnycastle.

If any equation, above a simple one, have its exponents, coefficients and absolute term whole numbers, neither of its roots can be expressed by means of any simple surd whatever, uncompounded with other quantities. Required the demonstration?

*Answered by Mr. John Bonnycastle, the Proposer.*

This question is liable to certain restrictions, which render it less general than was at first supposed; but as some useful conclusions may be derived, even from the cases in which it fails, it is not the less proper on that account, as whatever throws any light upon the nature of equations cannot be thought undeserving attention. That the rule holds in any equation  $x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \&c. = \pm p$ , where the signs of all the terms on one side are affirmative, may be

readily shewn as follows: Take  $y^{\frac{n}{r}}$  for any simple surd, and let it be substituted for  $x$  in the given equation, as equal, if possible, to one of

its roots. We shall then have  $(y^{\frac{n}{r}})^r + a.(y^{\frac{n}{r}})^{\frac{n-1}{r}} + b.(y^{\frac{n}{r}})^{\frac{n-2}{r}} +$

$c.(y^{\frac{n}{r}})^{\frac{n-3}{r}} \&c. = \pm p$ ; where it is manifest that since  $p$  is rational, every term of the equation must also be rational; the sum of any number of simple surds being always a surd. But in order that each

of the terms may be rational, the indices  $\frac{n}{r}, \frac{n-1}{r}, \frac{n-2}{r}, \frac{n-3}{r},$

$\&c.$  must be all whole numbers, which is impossible; for taking  $\frac{n}{r} = q$ , a whole number, we have  $n = rq$ , which substituted for  $n$  in the

second term, gives  $\frac{rq-1}{r} = q - \frac{1}{r}$ , a mixed number, the part  $\frac{1}{r}$  being evidently a fraction; so that, whether the rest of the terms are

wanting or not, it is plain that in this case, neither of the roots can be equal to a simple surd. And the same will be true in any change of the signs, except when one or more negative terms are exactly equal to one or more affirmative ones, in which case it sometimes happens that the irrational parts of the equation will cancel each other, and of course not affect the absolute term, which is a whole number. This is the case in the equation  $x^3 + ax^2 - bx = ab$ , where one of the roots is equal to  $\sqrt{b}$ , and several others might be formed of a like nature; in any of which instances, whenever they occur, the root may be easily discovered by comparing those terms together which can destroy each other; and from thence many useful rules may be derived, but which for want of room we must omit.

There is another exception to the universality of the rule, which takes place when the indices of the terms of the equation are composite numbers, as in the equation  $x^4 + 3x^2 = 18$ , where one of the roots is  $\sqrt{3}$ ; but these kind of equations are always reducible to those of lower dimensions, in which the rule will commonly hold.

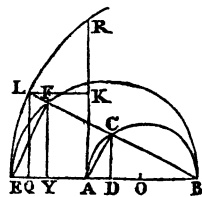
*This question was also answered by Amicus, Mr. John Dalton of Kendal, and Mr. T. White; who all agree in shewing that the proposition is not general, but that many such equations as mentioned in it may have a simple surd for one of their roots; of which Mr. Dalton gives this instance,  $x^3 + 5x^2 - 2x - 10 = 0$ , one root of which is the  $\sqrt{2}$ .*

**XY. QUESTION 922, by the Rev. Mr. Robert Bownas, of Bardsey.**

To determine geometrically (i. e.) without the help of Algebra or Fluxions, the arc of a circle such, that the excess of any multiple of its sine above its chord, may be a maximum.

*Answered by Amicus.*

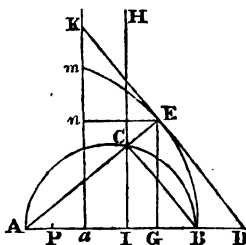
Produce the diameter BA of the given circle ABC till BE =  $n \times BA$ , then per fig.  $FY = n \times CD$ , and when  $FY - CA$  is a maximum  $AB \times FY - AB \times AC$  is a maximum, but this is  $= FB \times AC - AB \times AC = BQ \times AC - AB \times AC$ , the circle ELR being described with the radius BE, hence  $LQ = EF = n \times AC$ , therefore when  $(BQ - AB) \times AC = AQ \times AC$  is a maximum  $n \times AC \times AQ$  or  $LQ \times AQ$  is a maximum: and the problem is reduced to finding the greatest rectangle LQAK that can be inscribed in the given circular segment AELB, the method of doing which is well known.



*The same by Mr. Geo. Sanderson, of London.*

Let ACB be a given semicircle; CI the required sine,  $n$  its multiple, and CB its chord.

**Construction.** On the diameter  $AB$  take  $aa$  such, that  $AB$  may be the same multiple of  $aa$  as  $IH$  is of  $IC$ ; erect the indefinite perpendicular  $AK$ , and with  $A$  as a centre and  $AB$  radius describe the arch  $BEM$  cutting it in the point  $m$ ; bisect  $aa$  in  $P$ , and make, by Simpson's Geometry, *b*, 18, as  $AG : \frac{1}{2}AB (\frac{1}{2}AE) :: AB : GP$ ; erect the perpendicular  $GE$  meeting the arch  $BEM$  in  $E$ ; draw  $AE$  cutting the given semicircle in  $C$ ; and  $CB$  is the arch required.



**Demonstration.** Draw  $ne$  parallel to  $AB$  meeting  $AK$  in  $n$ . The triangles  $EAG$  and  $ABC$  having the angles  $EGA$  and  $BCA$  equal,  $EAB$  common, and  $AE=EB$ , have also  $EG=BE$ , and  $AG=AC$ ; whence  $AE (AB) : AG (AC) :: EG (CB) : CI$ ; but  $aa : AB (AE) :: CI : IH$ ; therefore  $aa : AG :: EG (CB) : IH$ ; and hence  $aa : AG - aa (AG=En) :: CB : IH - CB$ ; therefore  $aa \times (IH - CB) = \text{rectangle } aE$ , which is a maximum when  $IH - CB$  or  $IH - GE$  is a maximum.

Draw the tangent  $DEK$ : then since (by construction.)

$$AG : \frac{1}{2}AB (\frac{1}{2}AE) :: AB : GP = ag + ar;$$

$$AG : AE :: AE : 2GP :: (\text{by sim. } \Delta s) AE : AD;$$

therefore  $AD = 2GP = 2ga + 2pa = GD + ga + 2pa$ ; hence  $GD = ga$ ; and  $pa$  being bisected in  $G$ , the tangent  $DK$  must be bisected in  $E$ , and consequently the rectangle  $aE$  is a maximum by theorem 8 of Mr. Simpson on the Maxima and Minima, of Geometrical Quantities.

PRIZE QUESTION, by Amicus.

If from any three given points lines be drawn to meet in a fourth point, so that one of them shall be always an arithmetical mean to the other two: required the vertices, asymptote and species of the curve which is the locus of this fourth point.

Answered by Mr. John Farey, of London.

The figure and notation used in my solution of the 876 question will serve equally for this, by making  $AP + BP = 2CP$ , instead of  $AP \times BP = CP^2$ , and the equation of the curve will be  $\sqrt{((n+z)^2 + (v+r)^2)} + \sqrt{((n-z)^2 + (v-r)^2)} = 2\sqrt{((m-z)^2 + v^2)}$ ; then reduce

this to the Newtonian form by making  $z = x + \frac{m^2 - n^2}{2m}$ , and  $v = y +$

$\frac{nr}{2m}$  and extract the root, so shall  $y = \frac{m^2 nr - n^2 r}{4m^2 x} \pm \frac{1}{x} \sqrt{(-x^4 +$

$$\frac{2m^2 + 3n^2 - r^2}{2m} x^3) + \frac{2m^2 r^2 - m^4 - 3n^4 + 3n^2 r^2}{4m^4} x^5$$

$$+ \frac{m^4 n^2 - m^4 r^2 - 2m^2 n^4 - 2n^4 r^2 + n^6}{8m^3} x + \frac{m^4 n^2 r^2 - 2m^2 n^4 r^2 + n^6 r^2}{16m^4};$$

the four roots, or values of  $x$  which cause the equation under the vinculum to vanish are  $-\frac{r^2}{2m}$ ,  $\frac{n^2}{2m}$ , and  $\frac{(m \mp n)^2}{2m}$ ; whence it appears

that if neither  $m = 2n$ , nor  $m = n$ , nor any of the quantities  $m, n, r$  wanting, the curve will be Sir Isaac Newton's 33d species; if  $m = 2n$ , it will be the 34th species; if  $r = 0$ , the 29th species; if  $m = n$ , the 40th species; if  $r = 0$ , and  $m = 2n$ , it will be the 41st species; if  $n = 0$ , the 44th species; if  $r = 0$ , and  $m = n$ , the locus is a right line and circle; if  $r = 0$ , and  $n = 0$ , it becomes a right line and point; and lastly if  $m = 0$ , it is a right line, coincident with  $AB$ . The vertex of the curve is always the centre of a circle passing through the three given points, and the asymptote always perpendicular to  $EC$  at a distance from  $E$  equal to  $(m^2 - n^2) \div 2m$ .



*Questions proposed in 1791, and answered in 1792.*

1. QUESTION 924, by Mr. J. Hunt, of Stoney Stratford.

What's high in esteem,  
As the Ladies now deem,  
To declare, from below, condescend;  
In Diary next year  
Pray let it appear,  
'Twill oblige your poetical friend.

$$xyz = 160, x^2 + y^2 + z^2 = 465, x^3 + y^3 + z^3 = 8513.$$

*Answered by Mr. Joseph Garnett, from Mr. Rodham's Academy, Richmond, Yorkshire.*

By the question  $x, y$ , and  $z$ , are to be whole numbers, and from the 3d equation no one can be above 20, and some one must be more than 14. Now from the 1st equation the three roots must evidently be some of the divisors of 160, which are 1, 2, 4, 5, 8, 10, 16, and 20, among which there are only two above 14, viz. 16 and 20; therefore the numbers are either

$$16 \begin{cases} 10 \text{ and } 1, \\ 5 \text{ and } 2, \end{cases} \text{ or } 20 \begin{cases} 8 \text{ and } 1, \\ 4 \text{ and } 2; \end{cases}$$

of which 20, 8, and 1 are the only ones that will answer the conditions, and the word is HAT.

*The same answered by Mr. J. Holt, of Manchester.*

Because no word can be formed without a vowel, the value of one of the unknown quantities must answer to a vowel, and be such that the product of the other two in the first equation be a composite number; but 1 and 5 only have these properties. Now if 5 be substituted for  $x$  in the 2d equation; then  $y^2 + z^2 = 440$ ; but no two perfect squares whatever will make this number; therefore  $x = 1$ , consequently  $y^2 + z^2 = 464$ ,  $yz = 160$ ; from this latter equation  $z = 160 \div y$ , which value of  $z$  substituted in the former, it becomes  $y^2 + 25600 \div y^2 = 464$ , or  $y^4 - 464y^2 = -25600$ , the two roots of which quadratic equation are 8 and 20, which are the values of  $y$  and  $z$ . Hence the required word is HAT.

*The same by Mr. D. Kinnebrook, jun. of Norwich,*

Let  $x + y + z = s$ , and  $xy + xz + yz = r$ ; then will  $x^2 + y^2 + z^2 = s^2 - 2r$ ; also  $x^3 + y^3 + z^3 - 3xyz (480) = (s^3 - 2r) \times s - rs$ , and  $x^3 + y^3 + z^3 = s^3 - 3rs + 480$ , whence come these two equations  $s^2 - 2r = 465$ , and  $s^3 - 3rs + 480 = 8513$ ; from the first  $r = \frac{1}{2}(s^2 - 465)$ , which makes the latter become  $s^3 - \frac{1}{2}(3s^3 - 1395s) + 480 = 8513$ , or  $s^3 - 1395s = -16066$ , where  $s = 29 = x + y + z$ . Now, by the 1st and 2d original equations,  $xy = 160 \div z$  and  $x^2 + y^2 = 455 - z^2$ , therefore  $x + y = \sqrt{(455 - z^2 + 320 \div z)}$ , and  $x + y + z = \sqrt{(455 - z^2 + 320 \div z)} + z = 29$ ; hence  $z^3 - 29z^2 + 188z = 160$ , the roots of which equation are 1, 8, 20; which, as the unknown quantities are alike concerned, are the values of  $x, y, z$  answering to HAT.

## II. QUESTION 925, by Dubliniensis.

Given  $\sqrt[3]{(x^3 + y^3)} = \sqrt[3]{(x^3 - y^3)}$ ; to determine  $x$  and  $y$  in rational numbers.

*Answered by Mr. M. Mooney, Dublin.*

Put  $3z = y$ , and  $4z = x$ ; then by substitution, &c.  $5z = \sqrt[3]{7z^3}$ ; hence  $z = \sqrt[3]{\frac{7}{125}}$ ; consequently  $x = \frac{28}{125}$ , and  $y = \frac{91}{125}$ , the least values of  $x$  and  $y$ .

*Or, universally:* Put  $mz = x$ , and  $nz = y$ ; then, by substitution,  $\sqrt[3]{((m^3 + n^3)z^3)} = \sqrt[3]{((m^3 - n^3)z^3)}$ . Put  $m^3 + n^3 = a^3$ , and  $m^3 - n^3 = c$ ; then the equations become  $az = \sqrt[3]{cz^3}$ , or  $a^2z^2 = c^2$ , and hence  $z = c \div a^2 =$  (by restoring the values of  $a$  and  $c$ )

$$\frac{m^3 - n^3}{(m^3 + n^3)^{\frac{2}{3}}}; \text{ consequently } x = \frac{m^3 - n^3}{(m^3 + n^3)^{\frac{2}{3}}} \times m, \text{ and } y =$$

$\frac{m^2 - n^2}{(m^2 + n^2)^{\frac{1}{2}}} \times n$  where  $m$  and  $n$  may be taken any numbers at pleasure, so that  $m$  be greater than  $n$ , and the sum of their squares a square. If  $m = 4$ , and  $n = 3$ , then will  $x = \frac{28}{125}$ , and  $y = \frac{21}{125}$ , the same as before; but if  $m = 12$ , and  $n = 5$ , then will  $x = \frac{1428}{2197}$ , and  $y = \frac{505}{2197}$ ; and so of others.

*The same by Mr. John Dalton, of South Cave.*

Since  $x$  and  $y$ , in the given equation  $\sqrt{(x^2 + y^2)} = \sqrt{(x^2 - y^2)}$ , are to be rational numbers, it is plain they must denote the two legs of a right-angled triangle whose three sides are rational numbers. And because  $3^2 + 4^2 = 5^2$ , say as  $x : y :: 4 : 3$ , or  $y = \frac{3}{4}x$ ; this being substituted for  $y$  in the given equation, by reduction it becomes  $125x^3 = 28x^2$ , or  $x = \frac{28}{125}$ ; and consequently  $y = \frac{3}{4}x = \frac{21}{125}$ . This question admits of as many answers as there are right-angled triangles whose three sides are rational numbers.

*The same by a Bengal Officer.*

By raising each side of the given equation to the 3d power, it becomes  $(x^2 + y^2)^3 = (x^2 - y^2)^3$ . Let  $x = ny$ , and it will become  $y^3 \cdot (n^2 + 1)^3 = (n^2 - 1)^3$ , and  $y = (n^2 - 1) \div (n^2 + 1)^{\frac{3}{2}}$ . Here  $n$  must be greater than unity to be affirmative, and  $n^2 + 1$  a square number to be rational. This can only happen when  $n$  is taken equal the base of a right-angled triangle divided by the perpendicular, as  $\frac{4}{3}$ ,  $\frac{12}{5}$ ,  $\frac{20}{9}$ , &c.

Now if  $n = \frac{4}{3}$ ,  $y = \frac{21}{125}$ , and  $x = \frac{28}{125}$ .

But if  $n = \frac{12}{5}$ ,  $y = \frac{505}{2197}$ , and  $x = \frac{1428}{2197}$ , &c.

Had the given equation been  $\sqrt{(x^2 + y^2)} = \sqrt{(x^2 - y^2)}$ , then would  $n =$  the hypotenuse divided by either leg. See question 536.

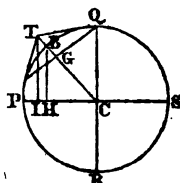
III. QUESTION 926, by Mr. T. Whiting, Schoolmaster, Lambeth.

From the top of a lofty mountain may be seen 48012260 acres of land on the spherical surface of the earth, exclusive of the mountain's base, which contains just 1000 acres; it is also known that the summit of the mountain is carried by the earth's rotation on its axis 6 miles in 24 hours more than the centre of its base: Required the height of the mountain, and latitude of the place, supposing the earth's radius 3982 miles?

*Answered by Mr. George Barnes, of Wigton.*

First, 48012260 acres = 75019 square miles. Now the radius being 3982, the circumference is 25019 miles; hence (by Dr. Hutton's Mensuration, 2d edit. pa. 197)  $75019 \div 25019 = 3$  miles nearly is = BG the versed sine of the segment; hence CG = 3979, and then CG : CB :: CB : CT = 3985 miles; hence BT = CT - CB = 3 miles, the height of the mountain.

Now, if QR be the equator, and PS the polar axis of the earth; also HB and IT are the radii in which the bottom and top of the mountain move, and the  $\angle HCB$  the co-latitude, the sine of which angle put =  $x$ ; then  $3 \cdot 1416 \times 2CB \times x$  = space gone over by B in 24 hours, and  $3 \cdot 1416 \times 2CT \times x$  = space gone over by T in the same time; therefore  $3 \cdot 1416 \times 2BT \times x$  is their difference = 6 miles; and hence  $x = 1 \div 3 \cdot 1416 = \cdot 31831$ , the natural cosine of  $71^\circ 27'$  the latitude of the mountain sought.

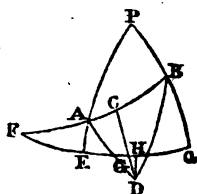


IV. QUESTION 927, by Mr. Samuel Hill, of Walworth.

Suppose two ports, one in the north latitude of  $30^\circ$ , and the other in north latitude of  $40^\circ$ , the difference of longitude between them being  $50^\circ$ ; I demand the bearing and distance of each of those ports, from an island that lies in the south latitude of  $18^\circ$ , and which is equally distant from both of the said ports?

*Answered by Mr. Da. Kinnebrook, jun.*

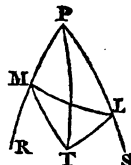
Let P be the north pole, EQ part of the equator, PE, PQ meridians, intersecting in an angle EPQ of  $50^\circ$  the difference in longitude of the two ports A and B, EA =  $30^\circ$ , and QB =  $40^\circ$ , also if the arc AB be drawn and bisected in c, by a perpendicular CD, meeting a parallel in  $18^\circ$  south latitude, at the point D, and the arcs AD, BD be drawn, it is evident from the figure that AD = BD, and that D is the place of the island. In the spherical triangle APB are given AP, BP and  $\angle APB$ , to find AB =  $41^\circ 35' 51''$ ,  $\angle PAB = 62^\circ 7' 6''$ , and  $\angle PBA = 87^\circ 49' 28''$ . Produce BA to meet the equator in F, and from D let fall the perpendicular DH; then in the right-angled triangle AEF are given EA and  $\angle EAF$ , to find  $\angle EFA = 40^\circ 2' 57''$  and FA =  $50^\circ 59' 35''$ , whence FC = FA + AC =  $71^\circ 47' 30''$ , and in the right-angled triangle GCF are given CF and  $\angle CFG$ , to find CG =  $38^\circ 36' 23''$  and  $\angle CGF = \angle DGH = 78^\circ 21' 4''$ ; also in the right-angled triangle DGH are given DH and  $\angle DGH$ , to find DG =  $18^\circ 23' 19''$ , whence CD = CG + GD =  $56^\circ 59' 42''$ , then in the right-angled triangle ACD are given AC and CD, to find AD = BD =  $59^\circ 23' 19''$ ,  $\angle CAD = \angle CBD = 77^\circ 0' 45''$ , whence  $\angle EAD = \angle FAC - \angle CAD = 40^\circ 52' 9''$ , and



$\angle QBD = \angle QBC - \angle CBD = 15^\circ 9' 47''$ , and the distance  $59^\circ 23' 19'' = 3563.3$  miles.

*The same by Mr. Rob. Wilkinson, North-Shields.*

Given  $PM = 50^\circ$ ,  $PL = 60^\circ$ ,  $PT = 108^\circ$ ,  $\angle MPL = 50^\circ$ , and  $TM = TL$ , which if given, in each of the spheric triangles  $MPT$  and  $LPT$  are given three sides, to find the angles  $MPT$  and  $LPT$ , the sum of which must be  $= 50^\circ$  the diff. of long.  $MPL$ . Now assume  $MT = TL$  any thing at pleasure, then by spherics, at a few trials, it may be approximated to the truth  $= 59^\circ 23'$ : Hence the angles  $MPT = 13^\circ 42'$ ,  $LPT = 36^\circ 18'$ ,  $PMT = 166^\circ 30'$ , and  $PLT = 139^\circ 14'$ ; the supplements of which give  $RMT = 13^\circ 30'$  easterly from  $M$  in latitude  $40$ , and  $SLT = 40^\circ 46'$  westerly from  $L$  in latitude  $30$ , supposing the port in  $40$  to lie westerly from the other; also  $59^\circ 23' = MT = LT = 3563$  nautical miles, the distance sought.

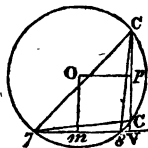


V. QUESTION 928, by Mr. John Liddell, of Habton.

Last year I made an erect direct dial, with its furniture, the angle included between the 7 and 8 o'clock hour-lines being 20 degrees. Now when I erected it, I observed the sun's altitude when south-east, which was 50 degrees. Query the latitude of the place, the day of the month, and hour of the day, the observation was made?

*Answered by Amicus.*

Since, by spherics, the tangent of the seven o'clock angle  $= \cosine$  latitude  $\times$  tangent  $75^\circ$ , and that of the eight  $= \cosine$  latitude  $\times$  tangent  $60^\circ$ , the tangents of these hour angles must be in the ratio of tangent  $75^\circ$  to tangent  $60^\circ$ , or of  $3.7320508 : 1.7320508$ , take  $7v = 3.7320508$  and  $8v = 1.7320508$ , then  $78 = 2$ ,  $7m = 8m = 1$ ; on the given line 78 describe a segment of a circle to contain an angle of  $20^\circ$  cutting  $vc$  perpendicular to  $mv$  in  $c$ ; from the centre  $o$  draw  $op$  parallel to  $mv$ , and  $om$  to  $cv$ ; then since  $7m = 1$ ,  $om$  is the tangent of  $70^\circ = pv = 2.7474774$ ,  $oc = o7 =$  the secant of  $70^\circ = 2.9238044$ , and  $cp = 1.0414080$ , consequently  $cv = pv \mp cp = 1.7060694$  or  $3.7888854 =$  the secant of the latitude  $54^\circ 7'$  or  $74^\circ 42'$ , hence, by common spherics, the declination  $= 20^\circ 45'$  answering to July 20th or May 24th, and the hour angle from noon  $= 29^\circ 5'$  or 1h. 56m. 20s.

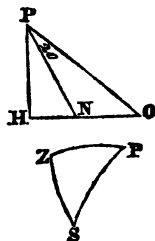


*The same by Mr. Jacob Park, of Morpeth.*

Let  $po$  and  $pn$  be the hour lines of seven and eight o'clock respectively. Now it is evident that  $pn$  and  $ho$  must be in the ratio of the tangents of  $60^\circ$  and  $75^\circ$  respectively; and the  $\angle nro$  is given  $= 20^\circ$ ;

then, by a known theorem, as  $HO - HN : HO + HN :: \sin \angle NPO : \sin (NPO + 2HPN) = 69^\circ 8' \text{ or } 110^\circ 52'$ , but the latter number is the right one in this case; then  $110^\circ 52' - 20^\circ = 90^\circ 52'$ , the half of which is  $45^\circ 26' = \angle HPN$ . Then as tangent  $60 : \text{tangent } 45^\circ 26' :: \text{radius} : \cosine \text{ latitude} = 54^\circ 7'$ .

Hence, in the spherical triangle  $PZS$ , are given  $ZP = 35^\circ 52'$  the co-latitude,  $ZS = 50^\circ$  the altitude, and  $\angle Z = 135^\circ$  the azimuth from the north, to find the  $\angle P = 29^\circ 5'$  answering to 1h. 56m. 20s. before noon, or 10h. 4m. in the morning, and  $PS = 20^\circ 45'$  the co-declination answering to May 23, or July 19.



VI. QUESTION 929, by Mr. W. Carss, jun. of Newcastle.

The surface of the Water in a reservoir is 60 feet high, from the bottom of which it is conveyed in pipes to an eminence 40 feet high, where it is discharged for the purpose of supplying a machine for drawing of coals: now supposing the reservoir always kept full, and the diameter of the pipe or conduit 4 inches, how many gallons of water will be discharged per minute, setting aside friction, and supposing the velocity to be that which is acquired by falling through the whole height of the surface above the orifice?

*Answered by Mr. John Ryley, of Leeds.*

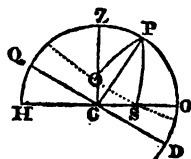
By the question, the constant height of water in the reservoir above the orifice is given, = 20 feet, or 240 inches; and as it is known that heavy bodies descend by the force of gravity  $16\frac{1}{2}$  feet, or 193 inches, in the first second of time; by the laws of falling bodies  $\sqrt{193} : \sqrt{240} :: 2 \times 193 : 8 \sqrt{(15 \times 193)} = 430.44$  inches, the velocity per second of the issuing water; which being multiplied by  $4^2 \times .7854$ , the area of the orifice, and that product by  $\frac{60}{282}$ , or  $\frac{1}{4.7}$ , the result is 1150½ ale gallons per minute, as required.

VII. QUESTION 930, by Mr. Alex. Rowe, of Reginnis.

In the year 1789, on the 24th day of July, it was observed that the sun was due east 32 minutes after his rising: required the sun's altitude, and the latitude of the place?

*Answered by Mr. Wm. Lawton, of Newcastle.*

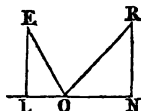
Let  $z$  be the zenith,  $P$  the pole,  $HO$  the horizon,  $s$  the sun at rising, and  $\odot$  when due east,  $QD$  the equator. Put  $t = \text{tangent of the sun's declination } 19^\circ 46'$ ,  $c = \cosine 8^\circ = \angle \odot Ps$ , and  $x = \text{tangent latitude of } P$ . Then, in the right-angled triangles  $PSO$  and  $PZ\odot$ , it is  $1 : \text{co-tangent } PS :: \text{tangent } PO : tx = \cosine \angle sPO \text{ or } \sin \angle CPS$ , and  $1 : \text{co-tangent } P\odot :: \text{tangent } PZ : t \div x = \cosine \angle ZP\odot \text{ or } \sin \angle CP\odot$ , the cosines of which are  $\sqrt{(1 - t^2 x^2)}$  and  $\sqrt{(1 - t^2)}$



—  $t^2 \div x^2$ ); then  $\sqrt{(1 - t^2 x^2)} \times \sqrt{(1 - t^2 \div x^2)} - t^2$  or  $\sqrt{(1 - t^2 x^2 - t^2 \div x^2 + t^4)} = c$ , in which the value of  $x$  is  $\sqrt{\left\{ \frac{1 - 2t^2 c - c^2}{2t^2} \right\}} \pm \sqrt{\left( \left( \frac{1 - 2t^2 c - c^2}{2t^2} \right)^2 - 1 \right)}$ , in the present case an impossible quantity,  $\frac{1 - 2t^2 c - c^2}{2t^2}$  being less than 1. The greatest possible value of  $c$  is  $1 - 2t^2 = \text{cosine } 42^\circ 7' 20''$ , therefore the time from rising till being due east ought to be at least 2h. 48½m. to make it possible to happen.

*The same answered by Amicus. (Suppl.)*

If, to make the question consistent, 3h. 2m. be put instead of 32', from the rising  $\kappa$  and easting  $\epsilon$  let fall perpendiculars  $RN$ ,  $EL$  to the equinoctial  $LN$ ; then as tangent latitude : tangent  $EL :: 1 : \text{sine } OL$ , and as co-tangent latitude : tangent  $NR = EL :: 1 : \text{sine } ON$ , therefore compoundedly tangent  $EL = \text{sine } OL \times \text{sine } ON$  which is given, and  $LN$  is given, wherefore, by problem 30th of Simpson's Algebra,  $ON = 30^\circ 40' 8''$ ,  $LO = 14^\circ 49' 52''$ ,  $OE$  the altitude  $= 24^\circ 37'$ , and  $LOE$  the latitude  $= 54^\circ 41'$ .



VIII. QUESTION 931, by Mr. John Farey, of London.

There is a mould candle 15 inches long, which will burn 9 hours; and one inch at the lesser end will be consumed in 20 minutes less than the same length at the larger end. Query the time in which an inch at the lesser end will be consumed?—*Note*, this question, though not new, has never been publicly investigated.

*Answered by Mr. John Craggs, of Hylton.*

The whole time of consuming the 15 inches length of the candle being 9 hours, or 540 minutes, if the times of burning the successive inches, be supposed in arithmetic progression, then  $540' \div 15 = 36'$  the time the 8th or middle inch will burn. Also  $20'$  being the difference of the extremes, or first and last inches, therefore  $10'$  is the difference between the middle and each of the extremes, therefore  $36' + 10' = 46'$  is the time of the bottom inch burning, and  $36' - 10' = 26'$  is the time the top inch burns, the answer near the truth, as the rate of burning in arithmetical progression is very nearly true.

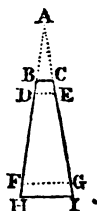
*The same answered by Amicus.*

Let the length  $= 15$  inches  $= b$ ,  $c$   $=$  the first velocity along  $b$  per minute,  $a + b$  the length for the first velocity to be infinite,  $x$   $=$  the

decrease of  $b$  in  $t$  minutes, the velocity then  $= c \times \frac{a^2}{(a+x)^3}$ , and  $\dot{t} = \frac{x \times (a+x)^2}{ca^2}$ , hence  $t = \frac{(a+x)^3 - a^3}{3ca^2}$ ; so when  $x = 1$ ,  $t = \frac{(a+1)^3 - a^3}{3ca^2}$ , when  $x = b$ ,  $t = \frac{(a+b)^3 - a^3}{3ca^2}$ , and when  $x = b - 1$ ,  $t = \frac{(a+b-1)^3 - a^3}{3ca^2}$ , which is the time for the last inch, which subtracted from the preceding value of  $t$ , gives  $\frac{(2a+b) \times (b-1)}{ca^2}$   $= 20$  per question, hence  $c = \frac{(2a+b) \times (b-1)}{20a^2}$ , but  $\frac{(a+b)^3 - a^3}{3ca^2} = 540$ , and  $c = \frac{(a+b)^3 - a^3}{1620a^2} = \frac{(2a+b) \times (b-1)}{20a^2}$ , reduced  $a^3 + (b - 54 + \frac{54}{b}) \times a = 303$ ,  $a = 42.52518$ , and  $\frac{a^3 + a + \frac{1}{3}}{ca^2} = 26.4337$ , the answer.

*The same by Mr. John Haycock. (Suppl.)*

The solid contents being as the times of consuming or burning them, which is the most natural supposition, let  $BCIH$  be the whole candle, in form of a conic frustum,  $BCED$  an inch at the smaller end, and  $FGIH$ , an inch at the larger, also  $ABC$  the part at top to complete the cone, whose height put  $= x$ ; then will the height of  $ADE$  be  $x + 1$ , of  $AFG$   $x + 14$ , and of  $AHI$   $x + 15$ . And since the contents of similar cones are as the cubes of their heights, the contents will be as follows, viz. of



$ABC$  as  $x^3$   
 $ADE$  as  $(x + 1)^3$ ,  
 $AFG$  as  $(x + 14)^3$ ,  
 $AHI$  as  $(x + 15)^3$ ; also by subtraction,  
 $BCIH$  as  $(x + 15)^3 - x^3$  or  $45x^2 + 675x + 3375$ ,  
 $FGIH$  as  $(x + 15)^3 - (x + 14)^3$  or  $3x^2 + 87x + 631$ ,  
 $BCED$  as  $(x + 1)^3 - x^3$  or  $3x^2 + 3x + 1$ ,  
 $FGIH - BCED$  as .....  $84x + 630$ ,  
 then, by the question,

$45x^2 + 675x + 3375 : 84x + 630 :: 9 : \frac{1}{3}$ ,  
 or  $15x^2 + 225x + 1125 : 28x + 210 :: 27 : 1$ ,  
 which gives  $5x^2 - 177x = 1515$ , the root of which is  $x = 42.525$ .  
 Hence  $84x + 630 : 3x^2 + 3x + 1 :: 20 : \frac{3x^2 + 3x + 1}{84x + 630} \times 20$

= 26' 433 or 26' 26'', the time required of burning one inch at the smaller end.

IX. QUESTION 932, *by the Rev. John Sampson.*

To find  $n$  affirmative integers such, that the square of the greatest may equal to the sum of the squares of all the rest.

*Answered by Mr. Christ. Cox, of Dublin.*

Let  $a, b, c, d, e$ , &c. be all known integers, denoting  $n - 2$  terms of the required series; and the sum of the squares of these put  $= s$ ; also let  $x$  and  $z$  be the two remaining terms of the series; then, by the question,  $z^2 = a^2 + b^2 + c^2 + d^2 + \dots + x^2 = s + x^2$ ; hence to have  $z$  rational, it is evident that  $s + x^2$  must be a square; assume it  $s + x^2 = (s - x)^2 = s^2 - 2sx + x^2$ ; hence  $x = \frac{s-1}{2}$ ; here  $s$  is evidently an odd number; it therefore appears that  $a^2 + b^2 + c^2 + d^2 + \dots$  to  $n - 2$  terms must be some integral squares whose sum is an odd number greater than 1, and the two remaining terms are  $\frac{s-1}{2}$  and  $(\sqrt{s^2 + (\frac{s-1}{2})^2})$  or  $\frac{s+1}{2}$ .

*Ex.* If  $n$  be 3, or there be three terms, the least of them must be greater than 2; if it be taken 3; then  $s = 3^2 = 9$ ; hence  $\frac{s-1}{2} = \frac{8}{2} = 4$ , and  $\frac{s+1}{2} = \frac{10}{2} = 5$ ; therefore the three may be 3, 4, 5.

If  $n = 4$ , or 4 terms; let the first two terms be 2, 3, the sum of whose squares is  $13 = s$ ; therefore  $\frac{s-1}{2} = 6$ , and  $\frac{s+1}{2} = 7$ , the other two terms; the four being 2, 3, 6, 7. And so on, for any number of terms.

X. QUESTION 933, *by Major Edw. Williams, of the Royal Artillery.*

Walking on the sea beach, I observed a ship firing in the offing, and wishing to know the distance, I tied a stone to a piece of pack-thread, and made a loop at a small distance to suspend it by, and found it made a given number ( $n$ ) of vibrations between the flash and report. Afterwards measuring the length of this pendulum, I found it to be  $l$  inches; from whence the distance of the ship ( $d$ ) is required?

*Answered by Mr. Alex. Rowe, of Reginnis.*

A pendulum  $39\frac{1}{8}$  inches long vibrates seconds, and the lengths of different pendulums are reciprocally proportional to the square of the number of their vibrations made in one and the same time; therefore as  $\sqrt{39\frac{1}{8}} : \sqrt{l} :: n : n\sqrt{(l \div 39\frac{1}{8})}$  the time or number of seconds in which the pendulum  $l$  performs  $n$  vibrations. Then, by Dr. Hutton's Compendious Measurer (a small book particularly useful in Schools), the velocity of sound is about 1142 feet in a second of time, or a mile in  $4\frac{2}{3}$  seconds; whence as  $1'' : 1142 :: n\sqrt{(l \div 39\frac{1}{8})} : 1142 n\sqrt{(l \div 39\frac{1}{8})} = 182\frac{1}{2} n\sqrt{l} = d$ , a general rule for the distance sought in feet.

XI. QUESTION 934, by Mr. John Ryley, of Leeds.

There is a cistern whose length is 6 feet, the breadth 4, and the depth 5 feet; it is supplied with water by a cock, which will fill it in 20 minutes; it has also another cock by which it may be emptied in 45 minutes. Now suppose the cistern empty, and both cocks set open, in what time will it be full?

*Answered by the Proposer, John Ryley, of Leeds.*

Put  $r = 24$  the area of the cistern's base,  $m = 32\frac{1}{2}$  feet the velocity acquired by a falling body in one second of time,  $n =$  the area of the aperture at the bottom,  $q =$  the quantity run in by the filling cock in a second,  $h = 5$  the height,  $t =$  the time required, and  $x =$  the variable height of the water in the cistern.

Then, by Article 1, of Dr. Hutton's Mathematical Miscellany, the time of exhaustion will be defined by  $\frac{2r}{n} \sqrt{\frac{h}{m}} = 45' = 2700''$  by the question; whence  $n$  will be had  $= 1.0021$  square inch. Also, the velocity of the issuing fluid being  $= \sqrt{mx}$ , the quantity run out in one second will be expressed by  $n\sqrt{mx}$ ; and consequently  $q - n\sqrt{mx}$  will be the rate of the vessel's filling per second; also,  $qt - nt\sqrt{mx}$  will be the fluxion of the quantity in the vessel  $= rx$ . Therefore  $\dot{t} = \frac{rx}{q - n\sqrt{mx}}$  (putting  $v^2 = x$ , and  $s = n\sqrt{m}$ )  $\frac{2rv\dot{v}}{q - sv} = \frac{2r\dot{v}}{s} + \frac{2rq}{s} \times \frac{\dot{v}}{q - sv}$ , the correct fluent of which is  $-\frac{2rv}{s} + \frac{2qr}{s^2} \times \text{hyp. log. of } \frac{q}{q - sv} = (\text{when } x = h) - \frac{2r\sqrt{h}}{s} + \frac{2qr}{s^2} \times \text{h. l. } \frac{q}{q - s\sqrt{h}} = 66^m 14^s$  or  $1^h 6^m 14^s$  the time required.

"N. B. This question was published, in the Leeds Mercury, by a pupil of mine, and two answers given to it, the former of which was intirely false, making the answer  $36'$  instead of  $66' 14''$ , the exact answer, and the latter quite unintelligible."

XII. QUESTION 935, *by Mr. John Cullyer, of Wicklewood, near Wymondham, Norfolk.*

If the equation of the curve of a boat, formed by a vertical section at its extreme breadth and depth, be  $\frac{13x}{\sqrt{(3x^2 + 5x + 7)}} = y$ , what is the area of the said curve in numbers, when the boat is the largest of the kind that can possibly sail under a bridge the arch of which is a semicircle of 10 feet radius.

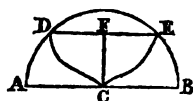
*Answered by Mr. Wm. Lawton.*

Let DCE be the boat, which it is evident will be the largest when it is inscribed in the semicircle ADEB, whose centre

is c, in which case it is the radius  $^2 = CE^2 = 100$

$$= CE^2 + FE^2 = x^2 + y^2 = x^2 + \frac{169x^2}{3x^2 + 5x + 7},$$

the root of which equation is  $x = 7.4494946$ .



Again, the fluxion of the area is  $y\dot{x} = \frac{13x\dot{x}}{\sqrt{(3x^2 + 5x + 7)}}$  the correct fluent of which is  $\frac{13}{3} \sqrt{(3x^2 + 5x + 7)} - \frac{13}{3} \sqrt{7} + \frac{65}{6\sqrt{3}}$

$\times$  h. l.  $\frac{\sqrt{7} + \sqrt{\frac{25}{3}}}{\sqrt{(3x^2 + 5x + 7)} + \sqrt{(3x^2 + 5x + \frac{25}{3})}} = \frac{13}{3} \sqrt{(3x^2 + 5x + 7)} - \frac{13}{3} \sqrt{7} + \frac{65}{6\sqrt{3}} \times$  h. l.  $\frac{\sqrt{\frac{7}{3}} + \frac{5}{6}}{\sqrt{(x^2 + \frac{5}{3}x + \frac{7}{3})} + x + \frac{5}{6}}$   
 $=$  (when  $x$  is 7.449 &c.) 39.2175, the double of which is 78.435 feet, the area of the whole section DCE of the boat.

XIII. QUESTION 936, *by Mr. G. Sanderson, of London.*

To determine whether the horizontal diameter of the moon, or its diameter augmented according to its altitude, ought to be made use of in calculating solar eclipses, by Mr. Flamsteed's method.

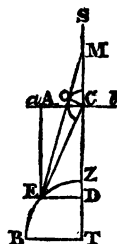
*Answered by Amicus.*

In Mr. Flamsteed's, and other graphical methods of computing solar eclipses, the shadow is supposed to fall upon the plane of the imaginary circle bounding light and darkness; and its size is made the same as if it really fell upon such a plane; whereas it ought to be the orthographical projection on such plane of the shadow as it really is upon the surface of the earth, or of the portion of the surface covered thereby, which, being nearer the moon, must be greater than the

imaginary shadow on the imaginary plane. Consequently the horizontal diameter ought to be augmented according to the altitude.

*The same answered by the Proposer, Mr. Geo. Sanderson.*

To determine which of the diameters is to be made use of in solar eclipses, let us suppose the centres of the earth, moon, and sun to be in the same right line  $ts$ , where  $t$  is the centre of the earth,  $c$  the moon's, and  $m$  the vertex of the penumbral cone, whose slant side touches the moon in the point  $o$ : Also let  $bez$  represent a part of the earth's disk,  $ed$  the semidiameter of the penumbra, or section of the penumbral cone (which in this case is a circle);  $ab$  is the plane of projection, which in Flamsteed's method touches the moon's orbit, and is perpendicular to  $ts$ , the line connecting the centres of the earth and sun: Join  $ce$ , and draw  $ea$  parallel to  $dc$  meeting  $ab$  in  $a$ .



Mr. Flamsteed, Mr. Keill, and others, make the angle  $emd$  equal to the apparent semidiameter of the sun, equal to  $ecd - oec = aeo - oec$ ; but  $ecd$  is the apparent semidiameter of the penumbra ( $ed$ ) as seen from the moon; which by reason of the great distance of the sun ( $ts$ ) in respect of  $ed$  and  $dc^*$ , is projected on the plane ( $ab$ ) into a line equal to itself, which by parallel lines is equal to  $ac$ . But the angle  $oec$  is equal to the apparent semidiameter  $oc$  of the moon as seen from  $e$ ; which angle is manifestly equal to the apparent horizontal semidiameter augmented according to its altitude; therefore the semipenumbra  $ac$  in Flamsteed's projection, must be made equal to the angles  $aeo + oec = emd + oec$ ; or to the apparent semidiameter of the sun, plus the horizontal semidiameter of the moon augmented according to its altitude.

\* Mr. Flamsteed supposes the eye to be in the centre of the sun: whence all the points on the earth's disk are orthographically projected on the plane  $ab$ .

#### XIV. QUESTION 937, by Amicus.

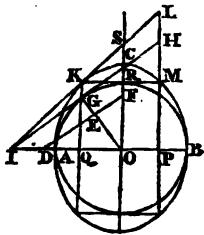
From a given ellipse to cut off a segment such, that the length of the greatest parallelogram that can be inscribed therein, may be the greatest possible.

*Answered by Amicus, the Proposer.*

From the centre  $o$ , along the transverse  $oc$  and conjugate  $oa$ , of the given ellipse, set off  $of : on ::$  side of a square : its diagonal, and with radius  $oa$  describe a circle; join  $of$ , perpendicular to which draw  $oe$ , which continue to the circle in  $e$ ; through  $e$  parallel to  $of$  draw the tangent  $xi$  meeting the transverse in  $x$ , and conjugate produced in

1; parallel to  $co$  and through  $G$  draw the ordinate of the ellipse  $kq$ ; join  $IK$ , which produce to  $L$  till  $KL = IK$ ; thro'  $L$  parallel to the transverse draw  $LF$ , and it will cut off the segment required.

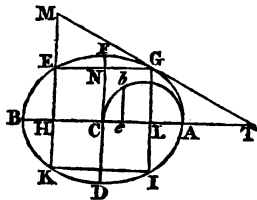
For  $IK$ , which cuts the transverse produced in  $s$ , by a known property of the ellipse, is a tangent thereto at  $k$ ; and since, by construction  $FO^2 : DO^2 :: 1 : 2 :: FE : DE$ ,  $DE = 2FE$ , therefore  $GI = 2GR$ , and  $KI = 2ks$ ; but, by construction  $KI = LK$ , therefore  $ks = sL$ , and,  $KM$  being joined,  $QKMP$  is a parallelogram, and because  $KI = KL$ , it is the greatest in the segment  $FMCKA$ , and its double the greatest in the segment of the whole ellipse, and its length  $= 2Kq$  is the greatest; because it consists of the whole chord  $2MP$  bounding the elliptic segment.



Since, by construction  $2Oq = IQ$ , and  $GO^2 = AO^2 = QO \times IO = 3QO^2$ ,  $QO = AO \sqrt{\frac{1}{3}}$ ,  $GQ = AO \sqrt{\frac{2}{3}}$ , and  $KQ = CO \sqrt{\frac{2}{3}}$ : now the length of the greatest parallelogram that can be inscribed in the whole ellipse is known to be  $= CO \sqrt{2}$ , and  $CO \sqrt{2} : 2Kq :: \sqrt{3} : 2$ .

*The same by Mr. George Sanderson. (Suppl.)*

Let  $AFBD$  be the given ellipsis,  $AB$  the transverse axis,  $FD$  the conjugate,  $c$  the centre,  $AEK$  the segment, and  $GEKT$  the parallelogram, whose sides cut  $BA$  in  $H$  and  $L$ : draw the tangent  $TGM$  cutting  $BA$  and  $HE$  produced in the points  $T$  and  $M$ .



Now it is manifest that the farther the point  $G$  is taken from  $A$ , or the nearer to  $F$ , the greater will be the subtangent  $TL$ . But the parallelogram is to be a maximum by the question; wherefore  $GE = HL = LT$ , and  $GL = HE = EM$  (Simpson on the Maxima and Minima, theorem 8.); but  $HE$  cannot be greater than the semibase or ordinate at  $H$ ; neither can it be less, for then the point  $G$  may be taken nearer to the point  $F$ , and the subtangent  $TL = HL$ , or length as well as breadth may be greater; therefore when the length of the greatest parallelogram is a maximum  $HE = GL$  must be an ordinate at  $H$ , or  $HE$  must be equal to the semibase of the segment, and  $HC = CL$ .

By a well known property of the ellipsis,  $CL : LA :: BL : LT = HL = 2CL$ , therefore  $CL : CA - CL :: BC + CL = CA + CL : 2CL$ , and  $2CL^2 = CA^2 - CL^2$ , or  $3CL^2 = CA^2 = CB^2$ . Whence  $CL$  may be easily found either by extracting the root, or by describing a semicircle on  $CA$ , and from  $c$ ,  $\frac{1}{3}$  of  $CA$ , erect  $eb$  perp. to  $CA$  meeting the circle in  $b$ ; from  $c$  apply  $CL = cb$ . Then  $ce$  or  $\frac{1}{3} CA : cb :: cb : CA$ ; therefore  $cb^2 = CL^2 = \frac{1}{3} AC^2$ , and  $3CL^2 = CA^2$ .

*The same algebraically by Mr. James Dale, at Billingham. (Suppl.)*

Put  $t$  = semidiameter  $CA$ ,  $c$  = its semiconjugate  $CF$ ,  $x = CN$  or  $LG$  half the breadth of the parallelogram  $KG$ ; then, by the nature of the ellipse (Hutton's Conic Sections, proposition 21.)  $c^2 : t^2 :: (c + x) \times (c - x) : NG^2$ , or  $c : t :: \sqrt{(c^2 - x^2)} : (t \div c) \sqrt{(c^2 - x^2)} = NG$  half the length of the parallelogram; therefore  $(stx \div c) \sqrt{(c^2 - x^2)}$  is a maximum, where  $s$  is the sine of the angle  $c$ , or  $c^2 x^2 - x^4$  a maximum, which put in fluxions, &c. gives  $x = c\sqrt{\frac{1}{2}}$ , or  $CN = CF\sqrt{\frac{1}{2}}$ . In like manner is  $CL = CA\sqrt{\frac{1}{2}}$ , or  $EG = AB\sqrt{\frac{1}{2}}$ . So that, the length  $EG$  being in a constant ratio to its parallel diameter  $AB$ , the former must be the longest when the latter is so, that is when  $AB$  is the transverse axis of the ellipsis. So that the longest of the greatest inscribed parallelograms is that, whose sides are parallel to the two axes, or perpendicular to each other, and that length is  $t\sqrt{\frac{1}{2}}$ , where  $t$  is the transverse axis of the ellipse, or the one to the other as the side of a square is to its diagonal.

PRIZE QUESTION, by *Lieut. Mudge, Royal Artillery.*

If a cylindrical piece of wood, of given dimensions and density, be perpendicularly immersed in water till its upper end be just even with the surface; and if then it be permitted to ascend by the action of the fluid; it is required to ascertain the concomitant circumstances of the motion, as to velocity and parts in and out of the water, &c.

*Solution by the Proposer.*

Let  $l$  denote the length of the cylinder,  $b$  the area of its base, and  $m$  its specific gravity, that of water being 1;  $x$  = any portion immersed in the water, when its velocity is  $v$ , also  $2g = 32\frac{1}{2}$  feet. Then  $bx$  is the weight and mass of water displaced, or force of the fluid acting against the cylinder; and  $blm$  is the weight of the cylinder, or its force against the water; therefore  $bx - blm$  is the motive force urging the cylinder upwards. Again, for the matter moved,  $bx$  is the mass of the cylinder immersed, or the mass of the water in motion; therefore  $bx + blm$  is the whole mass of matter

moved; consequently their quotient  $\frac{x - lm}{x + lm} = f$  is the accelerating

force. Then, by a well known theorem,  $v\dot{v} = 2gf$ . —  $\dot{x} = 2gx \cdot$

$\frac{lm - x}{lm + x}$ ; the correct fluent of which is  $v^2 = 4g(l - x - 2lm \times$

hyp. log. of  $\frac{lm + l}{lm + x}$ ), and  $v = 2\sqrt{g} \cdot \sqrt{(l - x - 2lm \times \text{hyp. log.}$

of  $\frac{lm + l}{lm + x}$ , the velocity when the part  $l - x$  is out of the water, or the part  $x$  is immersed.

*Observe.* 1. When the velocity is  $= 0$ , or the motion ceases, then  $l - x = 2lm \times \text{h. l. } \frac{lm + l}{lm + x}$  the greatest height that can rise out of the water. And here, if  $l$  were 10 feet, and  $m = \frac{1}{2}$ , which is nearly the density of fir, that equation becomes  $10 - x = 10 \times \text{h. l. } \frac{15}{5 + x}$ , in which equation  $x$  is found  $= 1\frac{1}{2}$  very nearly, or  $10 - x = 8\frac{1}{2}$ , the greatest height ascended.

2. When the velocity is the greatest, then is  $x = lm =$  in numbers 5, or half the cylinder is out of the water; and the said greatest velocity there is 7.786 feet per second.

3. After the cylinder has arrived at its greatest height  $l - x$ , when the upward motion ceases, it will then return and descend again in the fluid, till it arrive at its first position, with the upper end even with the surface of the water; after which it would again reascend, and so continually leap up and fall down again, between the same limits, if tenacity, friction, or some such force did not take place between the surface of the cylinder and the water.

*Additional Solution, by Mr. James Wolfenden, of Hollinwood; taken from the third Number of the Student.*

Let  $h$  denote the length of the cylinder,  $d$  the diameter of its base,  $n$  its specific gravity, that of water being 1; put  $a = .7854$ ,  $2g = 32\frac{1}{2}$  the force of gravity: also let  $x$  represent any variable height ascended by the cylinder, from the commencement of its motion, in the time  $t$ , and  $v$  its velocity at the end of that time. The motive force of the water displaced is equal to  $2gad^2(h - x)$ , and that of the cylinder  $= 2gad^2nh$ , the difference of these, since they act in opposite directions,  $2gad^2(h - nh - x)$ , is the motive force acting in the direction of the cylinder. Now to get the accelerative force, we must find the inertia of the matter moved: the inertia of the cylinder is evidently  $ad^2nh$ , its mass, and it is demonstrable, from the principles of mechanics and the nature of fluidity, that the inertia of the water in motion at any instant, is equal to the mass of water displaced at that instant, or  $ad^2(h - x)$ . Hence the force of

acceleration is  $\frac{2g(h - nh - x)}{h + nh - x}$ , and by a theorem well known

$vv = \frac{2g(hx - nhx - x^2)}{h + nh - x}$ , the correct fluents of which give

$$v^2 = 4g \left\{ x - 2nh \text{ hyp. log. } \frac{h(n+1)}{h(n+1)-x} \right\},$$

$$\text{or } v = 2\sqrt{g} \sqrt{\left\{ x - 2nh \text{ hyp. log. } \frac{h(n+1)}{h(n+1)-x} \right\}}.$$

And by converting the expression for  $v$  into a series, it will not be difficult to obtain (from the equation  $t = x \div v$ ) the time  $t$ .

*Scholium.* When  $v$  is a maximum,  $x = (1-n)h$ , and when it is nothing

or when the cylinder stops, we get  $\frac{x}{2nh} = \text{hyp. log. } \frac{h(1+n)}{h(1+n)-x}$ ;

now if we put  $l$  for the number whose hyp. log. is unity, we shall, by the nature of logarithms, have

$$l^{\frac{x}{2nh}} = \frac{h(1+n)}{h(1+n)-x}, \text{ or } l^{\frac{x}{2nh}} \times \{h(1+n) - x\} = h(1+n),$$

and from this exponential equation,  $x$ , the height to which the cylinder will ascend, may be found. It appears likewise that if  $n$ , the specific gravity of the ascending substance, be indefinitely small,

$l^{\frac{x}{2nh}}$  must be indefinitely great, consequently the factor  $h(1+n)-x$ , or  $h-x$ , will be indefinitely small, since their product is constant; hence  $x = h$  nearly the height to which the cylinder will in this circumstance rise.

*Questions proposed in 1792, and answered in 1793.*

I. QUESTION 939, by Mr. J. Holt, of Manchester.

What two numbers are those, whose sum, quotient, and difference of their squares, are all equal to each other?

*Answered by Mr. Jos. Facer.*

Put  $x =$  the greater number, and  $y$  the less; then, by the question,  $x + y = x^2 - y^2$ , and  $x + y = x \div y$ . Divide the first equation by  $x + y$ , gives  $1 = x - y$ , or  $x = y + 1$ ; substitute this for  $x$  in the 2nd equation, gives  $2y + 1 = (y + 1) \div y$ ; hence  $2y^2 + y = y + 1$ , or  $2y^2 = 1$ , and  $y^2 = \frac{1}{2}$ , or  $y = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ ; and hence  $x = 1 + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}$ .

PROOF.  $x + y = 1 + \frac{1}{\sqrt{2}}$ . And  $\frac{x}{y} = \frac{1 + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1 + \sqrt{2}$ .

Also  $x^2 - y^2 = (1 + \sqrt{\frac{1}{2}})^2 - \frac{1}{2} = 1 + 2\sqrt{\frac{1}{2}} + \frac{1}{2} - \frac{1}{2} = 1 + \sqrt{2}$ .

*The same answered by Mr. J. Hartley, Fleet Street.*

Put  $x$  and  $y$  for the required numbers: then, per question,  $x + y = x \div y$ , and  $x + y = x^2 - y^2$ . Transposing the last equation gives  $x^2 - x = y^2 + y$ ; add  $\frac{1}{4}$  to both, makes the two complete squares  $x^2 - x + \frac{1}{4} = y^2 + y + \frac{1}{4}$ , the roots of which are  $x - \frac{1}{2} = y + \frac{1}{2}$ , and here  $x = y + 1$ ; which value of  $x$  substituted in the other equation, gives  $y$  and  $x$  as above.

*The same, by Mr. James Ashton, of Harrington.*

Put  $x =$  the sum, and  $y$  the difference of the required numbers; then shall those numbers be  $\frac{1}{2}(x + y)$  and  $\frac{1}{2}(x - y)$ ; hence  $(x + y) \div (x - y)$  is their quotient, and  $xy$  the difference of their squares; whence, by the question,  $x = xy = (x + y) \div (x - y)$ ; from the first  $y = 1$ ; then by substituting  $x = (x + 1) \div (x - 1)$ , hence  $x^2 - x = x + 1$ , or  $x^2 - 2x = 1$ , and  $x^2 - 2x + 1 = 2$ ; extracting the root, &c.  $x = 1 + \sqrt{2}$ ; hence the two numbers are  $1 + \sqrt{\frac{1}{2}}$  and  $\sqrt{\frac{1}{2}}$ .

## II. QUESTION 940, by Mr. J. Hornby, Westerdale School.

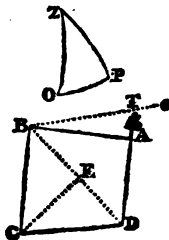
Surveying a four-sided field, in latitude  $54^\circ 30'$  north, April 20th, 1791, at half past five, afternoon, a tree at the corner A, 66 feet high, cast a shadow just the length of the side AB, and the other sides measured as follows, viz. BC = 506 links, CD = 364, and AD = 276; also the angle A a right-angle: required the area?

N. B. The above is a real case, which actually happened in surveying the aforesaid field.

*Answered by Mr. Alex. Rowe, of Reginnis.*

First, in the spherical triangle  $zro$ , are given  $rz = 35^\circ 30'$  the co-latitude,  $ro = 78^\circ 24'$  the co-declination, and the included angle at  $r$  (measured by  $5\frac{1}{2}$  hours at  $15^\circ$  each hour)  $= 82^\circ 30'$ , to find the  $3d$  side  $zo$ , which by the rule at page 164, Dr. Hutton's Mathematical Tables, comes out  $76^\circ 14'$ , which is the co-altitude, and theref. the altitude itself of the sun is  $13^\circ 46'$ .

Then, as tangent altitude  $AB \odot = 13^\circ 46'$ : radius  $:: AT = 66$  feet or 100 links: 408 links  $=$  the shadow's length  $= AB$ . Hence the diagonal  $AD = \sqrt{(AB^2 + AD^2)} = 493$ ; and  $DC$  being  $= 364$ ,



BC = 506, the perpendicular CE is found = 343 links. Consequently the area of the trapezium ABCD is = 1 acre 1 rood  $25\frac{1}{2}$  perches, as required.

*The same answered by Mr. Tho. Woolston, Master of the Boarding School, Adderbury, Oxfordshire.*

For the solution of this question there are given, first, in an oblique spherical triangle, two sides and the angle between them, viz. the co-lat.  $35^{\circ} 30'$ , the co-declination  $78^{\circ} 19' 25''$ , and the angle at the pole, or time from noon,  $82^{\circ} 30'$ , to find the co-altitude, or sun's zenith distance, the complement of which gives  $13^{\circ} 49' 39''$  for the altitude of his centre; and as his semidiameter at the time was  $15' 57''$ , this makes the altitude of his upper limb  $14^{\circ} 5' 36''$ , but by the proper corrections for refraction and parallax (the first being for the given height  $3' 44''$ , and the latter  $8''$ ), the apparent altitude will be found  $14^{\circ} 9' 12''$ . Then, by plane trigon. as tang.  $14^{\circ} 9' 12''$  : radius :: 100 links : 397 links = the side AB. Hence are given all the sides of the trapezium, and one angle a right angle, which agrees with the first quest. Dr. Hutton's Mensuration, pa. 152, 2d edit. and by the method used there the area is found = 1 acre, 1 rood,  $21\frac{1}{2}$  perches, as required.

### III. QUESTION 941, by Mr. R. Wilkinson, North Shields.

Being in a garden on the 1st of April, 1791, in  $55^{\circ}$  north lat. in which was a grass plat truly horizontal, and on which at some distance from me stood an empty cylindrical vessel. Having a cane in my hand, I set it up perpendicularly, exactly between the sun and the vessel, and found its shadow reached exactly to the bottom of the vessel on the outside next to me; but going 14.632 feet nearer, I observed that the shadow of the cane's top, after exactly touching the top or upper edge of the vessel, struck the opposite inside  $10\frac{1}{4}$  inches from the top. The time was 4hrs. 57min. afternoon: Quere the length of the cane, my distance from the vessel at first, and its content in ale gallons?

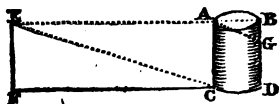
*Answered.*

This question, like the last, may be resolved either by allowing, in the sun's altitude, for the semidiameter, refraction, and parallax, or not. And, first, without that allowance,

*By Mr. Thomas Ridout, of Canterbury.*

First by spherics, as in the last quest. having given the latitude  $55^{\circ}$  the time April 1, 1791, at  $4^h 57^m$  the true altitude of the sun's centre will be =  $12^{\circ} 51' 13''$ . Now let ABCD be the vessel, AG the line of

shadow within it,  $EF$  the cane,  $CF$  its shadow : then, the angles  $GAB$  and  $ECF$  are equal ; and the  $\triangle s$   $ABG$ ,  $CFE$ , are therefore equiangular. Then in the right-angled triangle  $ABG$ , are given all the angles, and the leg  $BG = 10\frac{1}{2}$  inches, to find  $AB$  the diameter of the base of the cylinder  $= 44.919$  inches ; from which its area is found  $= 1584.75$ . Now, to find the height ; by similar triangles, say as  $44.919 : 10.25 :: 14.632 : 3.3386 = 40.0632$  inches  $=$  the least possible length of the cane, and height of the cylinder ; then  $1584.75 \times 40.0632 \div 282 = 63490 \div 282 = 225.142$  ale gallons.

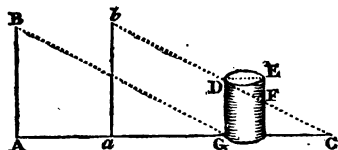


But the exact length of the cane, and first distance from the vessel, cannot be determined ; for that length may be any thing not less than  $40.0632$  inches, and then the first distance will be in proportion to the increased length.

2nd. With that allowance, by the Proposer, Mr. Robert Wilkinson, North Shields.

This question is not properly limited, owing to the omission of a few words, in proposing it, which should have been inserted thus, " but going  $14.632$  feet nearer", where I was just  $1\frac{1}{2}$  times the length of the cane from the vessel, " I observed that the, &c."

Now here are given  $Aa = GC = 14.632$  feet,  $EF = 10.25$  in. And  $AG = 1\frac{1}{2} AB$ .



Lat.	$55^{\circ} 00'$	} Hence true alt. centre $12^{\circ} 52' 41''$	
Decl.	$4^{\circ} 38'$		Semidiam., refr. paral. $19^{\circ} 57'$
Hour $\angle 74^{\circ} 15'$			App. alt. upper limb $13^{\circ} 12' 38''$

$= \angle BGA = \angle EDF = DCG$ , the tangent of which call  $t$ . Hence, in the  $\triangle DEF$ , as  $t$  : radius  $:: EF : DE = 43.6916$  diam.

in the  $\triangle DCG$ , as radius :  $t :: GC : DG = 41.1918$  the alt.

so that  $.7854 DE^2 \times DG \div 282 = 219.0312$  gallons, the content.

Lastly, as  $t : 1 :: AB : AG = Aa + 1\frac{1}{2} AB$  :

hence  $AB = 5.2965$  feet, the length of the cane,

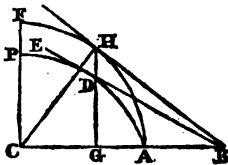
and  $AG = 22.5767$  feet, the distance from the vessel at first.

#### IV. QUESTION 942, by Fidelio.

Otaheite is an island in the South Seas, lat.  $17^{\circ} 45'$  ; now supposing the earth to be an oblate spheroid, the equatorial diameter being  $7974$  miles, and polar diameter  $7940$ , and that a person is at the equator, at the point nearest to Otaheite, required how high above the earth's surface he must be raised to see the said island ?

*Answered by Mr. Wm. Pearson, North Shields.*

Let PDA be a quarter of the elliptical meridian of Otaheite,  $CA = 3987$  the greater semiaxis, or radius of the earth's equator,  $PC = 3970$  the earth's semiaxis,  $P$  the south pole, and  $A$  the place of the person at the equator. Then (by Robertson's Navigat. B. 8, Sec. 8, Prob. 2) let  $BE$  be drawn to touch the ellipsis in  $D$ , making with  $CA$  continued an angle  $CBD$  equal to the compl. of the given lat.; and with  $CA$  describe the circular quadrant  $AF$ , and draw  $GDH$  parallel to  $CP$ , and join  $BH$  and  $CH$ ; then  $BH$  is a tangent to the circle, and  $BD$  a tangent to the ellipsis, which will (by writers on conics) both terminate at  $B$ .



*Calculation.* As  $GD : GH$  or  $CP : CA :: \text{tangent } \angle CBD : \text{tangent } CBH = 72^\circ 19' 15''$ , the comp. of the lat. on the sphere. Then, in the right-angled triangle  $CHB$ ,  $\sin CBH : \text{radius} :: CH : CB = 4184.63$ , and  $CB - CA = AB = 197.63$  miles, the height required.

*The same answered by Amicus.*

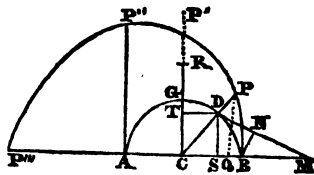
Given the angle  $BDG = 17^\circ 45'$ ,  $CA = CF = 3987$ , and  $CP = 3970$ . But  $GD : GH :: CP : CA :: \text{tangent } DBG : \text{cotangent } HCB$ , and the secant of  $HCB = 4184.632 = CB$ , whence  $CB - CA = AB = 197.632$  the answer.

V. QUESTION 943, by Mr. A. Buchanan, Sedgefield.

If in the quadrant of a given circle,  $CBG$ , whose centre is  $C$ , there be drawn any radius  $CD$ , and produced to  $P$ , so that  $DP$  be always = the corresponding versed sine  $BS$ ; it is required to find the nature, area, &c. of the curve  $BPP$ , described by the point  $P$ ?

*Answered by Amicus.*

Let  $ADB$  be the circle,  $BPP'P''P'''$  the curve. Draw the tangent  $DM$ , perpendicular to which draw  $BN$ , which by the property of the circle is  $= SB = DP$ , therefore  $CP^2 = CB^2 + 2CB \times SB + SB^2 = CB^2 + 4CB \times BN - DS^2$ ; fluxion of arc  $BD$  is  $(CB \div DS) S'B$ , and fluxion of the area  $CPB$  is  $(CP^2 \div 2CB) D'B = \frac{1}{2}CB \times D'B + 2BN \times D'B - \frac{1}{2}DS \times S'R$ ; the fluent = sector  $CDB$  of the circle + 4 segment  $BDB - \frac{1}{2}$  segment  $DSBD = \frac{3}{2}BDB + \frac{1}{2}$  trapezium  $TDEC$  = the quadrature of  $BPC$ ; and when  $SB = CB$ , this becomes  $\frac{3}{4}$  quadrant  $CGDB - 2CB^2 = CP^2/PB$ , and  $BPP'P''P''' = \frac{3}{4}$  the semicircle.



Drawing the right ordinate  $PQ = y$ ,  $BQ = x$ , and  $BC = a$ ; as  $CD = CB : DS = \sqrt{(2CB \times SB - SB^2)} :: CP = CD + BN : PQ = y$ ; but  $CD$

$= CB : SC = CB - SB :: CP = CB + SB : CB - QB$ , therefore  $CB : SB :: SB : QB = x$ , and  $SB = \sqrt{ax}$ , also by the first analogy  $ay = (a + \sqrt{ax}) \times \sqrt{(2a\sqrt{ax} - ax)}$  or  $y^2 = 3ax - x^2 + 2a\sqrt{ax}$  is the equation of the curve, which is therefore ovaliform, and a line of the 4th order. When  $x = a$ ,  $y = 2a = CP$ ; when  $x = 2a$ ,  $y = a\sqrt{2} + 2\sqrt{2} = AP''$ ; when  $x = 4a = BP''$ ,  $y = 0$ : when  $y$  is a maximum,  $ds \cdot CP$  is a max. and  $SB = \frac{1}{2}(1 + \sqrt{3})CB$ , or  $\sqrt{x} = \frac{1}{2}(1 + \sqrt{3})\sqrt{a}$ .

If  $SB$ , instead of being set off on  $CD$  produced, be set off on  $SD$  produced from  $D$  to  $R$ , the quadrature of the curve  $SRB$ , which in this case is a conic ellipsis, will evidently  $=$  segment  $SDB + \frac{1}{2}SB^2$  and area  $RDBR = \frac{1}{2}SB^2$  or strictly quadrable.

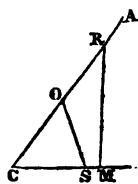
If  $SB$  be set off from  $D$  towards  $s$ , as in Kirby's Doctrine of Ultimators, the curve, which he calls *piriformis*, will be an ellipsis equal and similar to the other; as may be seen at large in Hutton's Miscellany, and the quadrature  $= SDB - \frac{1}{2}SB^2$ .

#### VI. QUESTION 944, by Mr. John Cullyer, Wicklewood.

There are two men  $A$  and  $B$ ;  $A$  can reach, when his arms are properly extended in raising a ladder, seven feet from the place on the ground where he stands, and  $B$  can in like manner reach six feet; but  $B$ 's strength is to that of  $A$  as 6 to 5; quere which of these men can raise a ladder with the most ease to himself, and what is the greatest weight each will have to sustain, supposing the ladder 30 feet long, its weight 60lb. and the centre of gravity 12 feet from its lowest end?

Answered by Mr. Jos. Garnett, jun. Richmond.

Let  $CA$  represent the ladder,  $\pi$  its centre of gravity, and  $os$  a prop or supporter; from  $\pi$  let fall  $\pi M$  perpendicular to  $cs$  the plane of the horizon; and put  $os = a$ ,  $cn = r$ , the weight of the ladder  $= w$ , and  $cm = x$ ; then, by art. 450 Simpson's Fluxions, the position of the prop  $os$ , to sustain the ladder with the greatest ease, is when the angles at  $o$  and  $s$  are equal; and the weight sustained by the said prop  $(2wx \div a) \sqrt{((r - x) \div (r + x))}$ ; whence, when a maximum, is had  $x = \frac{1}{2}(\sqrt{(5 - 1)}r)$ ; which being written for



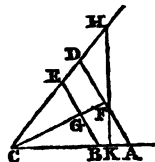
$x$  in the said expression for the weight on the prop it becomes  $\frac{rw}{a}$

$\sqrt{\frac{28 - 12\sqrt{5}}{1 + \sqrt{5}}} = \frac{rw}{a} \sqrt{(10\sqrt{5} - 22)} =$  (in the present case,  $r$  being  $= 12$ , and  $w = 60$ )  $720 \sqrt{(10\sqrt{5} - 22)} \div a$ ; where  $a$  being severally expounded by 7 and 6, we get 61.7725lb. and 72.0679lb. for the greatest weights sustained by  $A$  and  $B$  respectively. Now as 6 : 5 :: 72.0679 : 60.0566lb. the weight that  $A$  can sustain with the

same ease as B can 72·0679 ; but A has 61·7725lb to sustain ; therefore he cannot rise the ladder with so great ease as B.

*The same answered by Mr. Wm. Burdon, Acaster Malbis.*

It is shewn, at article 450 of Simpson's Fluxions, that the ladder  $CH$  will make an angle with the horizon of  $51^{\circ}50'$  when the men  $AD$  and  $BE$  have the greatest weight to sustain ; also  $CB = CE$ , and  $CA = CD$ . Then, as radius :  $FD :: \cotangent FCD : CF = 7\cdot2026$ , and radius :  $GE :: \cotangent GCE : CG = 6\cdot1736$  : now if  $H$  be the centre of gravity,  $CH = 7\cdot4154$  ; hence by mechan.  $(CH \div CF) 60 = 61\cdot7727lb$ , and  $(CH \div CG) \times 60 = 72\cdot0688lb$ . the greatest weight sustained by A and B respectively ; consequently B's strength to A's ought to be as 6 to 5·1042 to raise the ladder with as much ease.

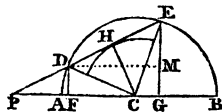


VII. QUESTION 945, by Mr. David Kinnebrook, jun.

Given the diameter  $AB$  of a circle, and the distance  $AP$  in the diam. produced ; to draw a right line  $PDE$ , to cut the circle in the points  $D$  and  $E$ , so that the difference between  $EG$  and  $DF$ , the sines of the arcs  $BE$  and  $AD$ , may be a maximum.

*Answered by Mr. John Howard.*

Suppose the thing done, and  $CH$  drawn perpendicular to  $PE$ . Then, by the similar triangles  $EPG$ ,  $HPC$  and  $DPF$ , we have  $PC : CH :: PD : DF = CH \times PD \div PC$ , and  $PC : CH :: PE : EG = CH \times PE \div PC$ , therefore  $EG - DF = CH \times (PE - PD) \div PC = CH \times DE \div PC$  ; or, because  $PC$  is given,  $CH \times DE$ , or the triangle  $DCE$  a maximum ; which, by Simpson on the Maxima and Minima, prop. 4, will be so, when  $CH = CD \sqrt{\frac{1}{2}}$  ; whence this



*Construction.* With radius  $CH = CD \sqrt{\frac{1}{2}}$ , and centre  $C$ , describe an arch ; to which, from the point  $P$ , draw the tangent  $PDE$  ; and the thing is done.

*Algebraical Solution by Mr. Jacob Park.*

Draw  $DM$  parallel to  $PC$ . Put  $PC = a$ ,  $CH = x$ ,  $CD$  or  $CA = r$ , then is  $DE = 2\sqrt{(r^2 - x^2)}$  ; but, by the similar tri.  $PC : CH :: DE : EM$  ; that is  $a : x :: 2\sqrt{(r^2 - x^2)} : 2x\sqrt{(r^2 - x^2)} \div a = EM$  a maximum, therefore  $r^2x^2 - x^4 = a$  a maximum, the fluxion of which made  $= 0$ , gives  $r^2x - 2x^3$ , or  $r^2 = 2x^2$ , and hence  $x = r\sqrt{\frac{1}{2}}$ .

*Cor.* Since  $\sqrt{\frac{1}{2}}$  is the sine of  $45^{\circ}$ , therefore the angles  $CDH$  and  $CEH$  are each equal to  $45^{\circ}$  ; and also the angle  $DCE$  is a right one.

VIII. QUESTION 946, by *Lieut. Wm. Dixon, of the Royal Artillery.*

To determine how far a man, who pushes with a force of 100lb. can introduce a sponge into a piece of ordnance whose diameter is five inches, and length 10 feet, when the barometer stands at 30 inches: the vent, or touch-hole, being stopped, and the sponge having no windage, that is fitting the bore quite close.

*Answered by Mr. Chr. Cox, Dublin.*

A column of quicksilver 30 inches high and 5 in diameter is  $5^2 \times 30 \times .7854 = 589.05$  inches, which, at 8.102 oz. each inch, weighs 4772.48oz. or 298.28lb. which is the pressure of the atmosphere alone, and equal to the elasticity of the air in its natural state; to this adding the 100lb. gives 398.28lb. the whole external pressure. Then, as the spaces which a quantity of air possesses, when under the influence of different pressures, are in the reciprocal ratio of those pressures, it will be as  $398.28 : 298.28 :: 10 \text{ feet or } 120 \text{ inches} : 89.87 \text{ inches}$ , the space occupied by the air; and therefore  $120 - 89.87 = 30.13$  inches, is the distance sought.

*The same answered by Lieut. T. H. Fenwick, of the Royal Artillery.*

Let  $x$  = the length required;

Then is  $10 - x$  the space in which the air is confined.

And if  $a = 298.16$ lb. the pressure of the atmosphere on a circle whose diameter is 5 inches (the caliber of the gun) the barometer standing 30 inches high;

Then is  $a + 100$ lb. the force which retains the air in the above-mentioned space; and these forces being reciprocally as the spaces, therefore  $a : a + 100 :: 10 - x : 10$ ;

$$\text{and hence } x = \frac{1000}{a + 100} = 2 \text{ feet } 6 \text{ inches.}$$

*Observation.* If  $f$  denote any force in general, then will  $x = \frac{10f}{a+f}$

be the distance the sponge can be pushed down by the force  $f$ ; which being always less than 10, it is evident that the force may be increased continually, without ever being able to get the sponge entirely home; if the vent be stopped, and no air escape by the side of the sponge.

IX. QUESTION 946, by *the Rev. Mr. J. Ewbank, Thornton-Steward.*

A man wants to travel from A unto D,

And, before he gets there, cross the river BC.

On the side marked with D, can walk three miles per hour;

But on that marked with A, 'tis quite out of his power;

Only two he can go, and he begs Lady Di.

Will the question insert, that her artists may try,

If they can direct, where the same must be crost,

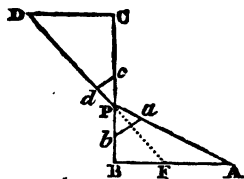
In his journey to D, that no time may be lost.

The perp. AB being seven miles, the perp. DC = 5 miles, and BC = 20 miles.

*Answered by Amicus.*

As this has been done so many times before, the following new and easy method is given for the sake of variety.

Let P be the point where the river is to be crossed, and because CB is given, CP the variation of CP = BP that of BP: let fall cd and ba; then because  $\frac{1}{2}DP + \frac{1}{2}AP$  is a minimum,  $\frac{1}{2}dP = \frac{1}{2}aP$ ; but  $dP : aP :: \text{sine PDC} = \text{sine PDA} = \text{PAB} :: \frac{1}{2} : \frac{1}{3}$ , or  $\frac{2}{3} \text{sine PDC} = \text{sine PAB}$ ; and hence, from the common mathematical tables, 'by a few trials, CD being 5, AB = 7, and BC = 20, PB is found = 5.670405873.



Nearly in the same manner was the geometrical solution by Ferdinando, and by Mr. Rodham, who farther observes, that, having found  $m(2) : n(3) :: \cos. EPA : \cos. CPD$  or  $\text{sine } A : \text{sine } D$  or  $\text{sine } F :: PF : PA$ , where DF is produced to F, then the algebraic solution may be thus: Put  $x = BP$ ,  $a = AB$ ,  $b = CD$ , and  $c = BC$ ; then, by similar triangles,  $PC : PB :: CD : BF = bx \div (c - x)$ ; hence  $b^2x^2 \div (c - x)^2 + x^2 (PF^2) : a^2 + x^2 (PA^2) :: m^2(4) : n^2(9)$ ; whence, putting  $d = n^2 - m^2 = 5$ , is obtained  $dx^4 - 2cdx^3 + (n^2b^2 + c^2d - m^2a^2) \times x^2 + 2cm^2 \times a^2x - m^2a^2c = 0$ ; where  $x = BP$  will be found = 5.6704 miles.

*The same by Mr. Kinnebrook, jun. Norwich.*

Put  $a = 7 = AB$ ,  $b = 5 = DC$ ,  $2s = PC + PB$ , and  $2x = PC - PB$ , then  $PC = s + x$ , and  $PB = s - x$ ; now by Eucl. 47 of 1,  $PD = \sqrt{(b^2 + (s + x)^2)}$ , and  $PA = \sqrt{(a^2 + (s - x)^2)}$ ; whence  $\frac{1}{2} \sqrt{(b^2 + (s + x)^2)} + \frac{1}{2} \sqrt{(a^2 + (s - x)^2)}$  or  $2\sqrt{(b^2 + (s + x)^2)} + 3\sqrt{(a^2 + (s - x)^2)}$  is to be a minimum; which, in fluxions, gives  $x^4 - 194\frac{1}{2}x^3 - 1684x = -10580$ , hence  $x = 4.3296$ , and consequently  $PC = 10 + x = 14.3296$ , and  $PB = 10 - x = 5.6704$ .

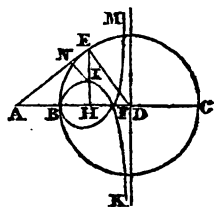
X. QUESTION 948, by Mr. S. Woolcott, South-Molton.

If tangents drawn to a circle, meeting the diameter produced, be divided in the given ratio of  $m$  to  $n$ , and at the points of division if perpendiculars be erected, and intersect the ordinate of the circle, produced when necessary, drawn to the points of contact of the said

tangents; to determine the species, &c. of the curve which is the locus of these intersections.

*Answered by Mr. S. Woolcott.*

Let AE be a tangent drawn to the circle at E, meeting the diameter AB produced in A: divide the tangent in N in the given ratio of  $m$  to  $n$ , and erect the perpendicular NI intersecting the ordinate EH in I; then I will be a point in the curve. Put the radius of the circle BD =  $a$ , DH =  $x$ , and HI =  $y$ . Then, by the property of the circle, HD : HE :: DE : AE =  $(a \div x) \sqrt{(a^2 - x^2)}$ : but AN : NE ::  $m : n$ , or AE : NE ::  $m + n : n$ , therefore NE =  $n \div sAE = (na \div sx) \sqrt{(a^2 - x^2)}$ , putting  $m + n = s$ ; also, by the similar triangles NEI and MDE, HD : DE :: NE : EI, or  $x : a :: NE : (a \div x) NE = (na^2 \div sx^2) \sqrt{(a^2 - x^2)} = EI$ ; therefore HI or  $y = \sqrt{(a^2 - x^2)} - (na^2 \div sx^2) \sqrt{(a^2 - x^2)} = ((sx^3 - na^2) \div sx^2) \sqrt{(a^2 - x^2)}$  the equation of the curve.



Hence, when  $x = a\sqrt{(n \div s)}$  or  $a\sqrt{(n \div (m + n))}$ , then  $y = 0$ ; and when  $x = a$ , then  $y = 0$  also; but when  $x = 0$ ,  $y$  is infinite. Let the ratio of  $m$  to  $n$  be what it will, the curve will have a nodus between B and D, and F its punctum duplex, and passing through which it branches out into two infinite legs FK and FM, to which a perpendicular drawn through D, the centre of the circle, will be an asymptote. If  $m = 3n$ , then  $x = \frac{1}{2}a = BF$  the diameter of the nodus, and the curve is Newton's 41st species; but if  $m = 0$ , the nodus becomes a cuspis, and the curve is the *cissois veterum*, or the 42d species.

*The same by Mr. Joseph Garnett, jun.*

After giving a solution and observations, exactly as above, this gentleman adds, The area of the curve may be easily found. For the fluxion of the area BHI is  $HI \times B'H = y\dot{x} = x \sqrt{(a^2 - x^2)} - (na^2 \div sx^2) x \sqrt{(a^2 - x^2)}$ , the fluent of which is easily found from the circle.

#### XL QUESTION 949, by Jack Western.

The Rev. Mr. Hellins, in his Mathematical Essays, page 107, from the equation  $z = x \sqrt{(aa + xx)}$  where  $x$  and  $z$  begin together, derives

$$z = \frac{1}{2}xx + \frac{1}{2}aa \times \text{h. l. } x, + \frac{a^4}{2.4.2.x^2} - \frac{3.a^6}{2.4.6.4.x^4} + \frac{3.5a^8}{2.4.6.8.6.x^6} \&c. + d, \text{ where } d \text{ is a constant quantity which he}$$

has shewn how to determine: it is proposed therefore to compute the value of  $z$  by this series, when  $d = 0$ , and  $x = 100$ .

*Answered by Mr. J. Holt, Master of the Mathematical School, Manchester.*

Mr. Hellins, in his Mathematical Essays, page 108, determines  $d = a^3 \times (.5965736 - \frac{1}{2} \text{hyp. log. } a)$ ; consequently when  $d = 0$ , then  $\text{hyp. log. } a = 1.1931472$ , and  $a = 3.297446$ ; which value of  $a$ , and 100 for  $x$ , being used in the series in the question,  $z$  comes out = 5025.03748 as required. Or if the same values be substituted in the correct fluent of  $\dot{z} = \dot{x} \sqrt{(aa + xx)}$ , viz.  $z = \frac{1}{2}x \sqrt{(a^2 + x^2)} + \frac{1}{2}a^2 \times \text{hyp. log. } \frac{x + \sqrt{(a^2 + x^2)}}{a}$ , it brings out the same value of  $z$ .

*The same answered by Mr. Da. Kinnebrook, jun. (Suppl.)*

The fluxion  $\dot{z} = \dot{x} \sqrt{(a^2 + x^2)}$  is  $= \frac{a^2 \dot{x}}{\sqrt{(a^2 + x^2)}} + \frac{x^2 \dot{x}}{\sqrt{(a^2 + x^2)}}$  and the correct fluent is  $z = \frac{1}{2}x \sqrt{(a^2 + x^2)} + \frac{1}{2}a^2 \times \text{hyp. log. } \frac{x + \sqrt{(a^2 + x^2)}}{a}$ ; and by the question

$$x - d = \frac{1}{2}x^3 + \frac{1}{2}a^2 \times \text{h. l. } x + \frac{a^4}{2.4.2x^3} - \frac{3a^6}{2.4.6.4x^5} \&c.$$

$$\text{Then by subtraction } d = \frac{1}{2}x \sqrt{(a^2 + x^2)} - \frac{1}{2}x^3$$

$$+ \frac{1}{2}a^2 \times \text{h. l. } \frac{x + \sqrt{(a^2 + x^2)}}{ax} - \frac{a^4}{2.4.2x^3} \&c.$$

and by putting the surds into a descending series, it is

$$d = \frac{1}{2}x^3 + \frac{a^2}{4} - \frac{a^4}{16x^3} \&c. - \frac{1}{2}x^3 + \frac{1}{2}a^2 \times \text{hyp. log.}$$

$$\frac{2x + \frac{a^2}{2x} - \frac{a^4}{8x^3} \&c.}{ax} - \frac{a^4}{2.4.2x^3} \&c. = \frac{a^2}{4} - \frac{a^4}{16x^3} \&c. + \frac{1}{2}a^2 \times$$

$$\text{h. l. } \frac{2}{a} + \frac{a}{2x^3} - \frac{a^3}{8x^5} \&c. - \frac{a^4}{2.4.2x^3} \&c. \text{ Now by taking } x \text{ inde-}$$

finitely large in comparison with  $a$ , all the terms but two vanish,

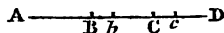
$$\text{leaving } d = \frac{a^2}{4} + \frac{1}{2}a^2 \times \text{h. l. } \frac{2}{a}; \text{ and when } d = 0, \text{ then } \frac{1}{2}a^2 \times \text{h. l.}$$

$$\frac{a}{2} = \frac{a^2}{4}, \text{ or } \frac{1}{2} \text{ h. l. } \frac{a}{2} = \frac{1}{4}; \text{ hence } a = 3.2974; \text{ and when } x = 100,$$

then  $z = 5025.037$ .

## XII. QUESTION 950, by Mr. John Maseres.

It is required to divide the given line AD into three such parts AB, BC, CD, that the sum of the three rectangles under these parts, viz. under AB . BC, and AB . CD, and BC . CD, shall be a maximum, or greater than if the same line be divided into any other three parts, *ab, bc, cd*?



*Answered by Mr. Chr. Cox, Dublin.*

It is evident, from Eucl. 2, 5, that wherever the point *c* is, the rectangle AB × BC will be the greatest, when AC is bisected in *B*, and for the same reason the rectangle BC × CD will be the greatest when the line BD is bisected: therefore when the given line AD is trisected, the sum of the rectangles of the parts will be a maximum. Mr. Cox also brings out the same conclusion algebraically.

*The same by Mr. W. Virgo.*

It is demonstrated by Mr. Simpson, in his Geometry, Theor. 1. on the Maxima and Minima, that, of all the rectangles that can possibly be contained under the two parts of a given right line, any how divided, that will be the greatest whose sides are equal to each other. Hence it follows, that the sum of all the rectangles, whatever be their number, under all the parts of a given right line, can only be a maximum when all those parts are equal to each other.

*Algebraically by Mr. J. Nicholson.*

Put  $a = AD$ ,  $x = AB$ ,  $y = BC$ , and  $z = CD$ ; then is  $xy + xz + yz = a$  maximum, and  $x + y + z = a$ ; hence  $z = a - x - y$ , which substituted in the maximum gives  $ay - y^2 + ax - x^2 - xy = a$  maximum. This in fluxions, making first  $y$  constant, and then  $x$  constant, gives  $ax - 2xx - yx = 0 = a - 2x - y$ , and  $ay - 2yy - xy = 0 = a - 2y - x$ ; putting these last two equal, gives  $x = y$ . And in the same manner  $x = z = y$ , so the parts are all equal.

## XIII. QUESTION, 951, by Mr. George Sanderson.

To determine on which day of the year 1792; the time between noon and sunset will be the greatest possible at Petersburg, latitude  $59^\circ 56'$  north.

*Answered by Mr. John Dalton, of Kendal.*

By the word noon, in this question, I suppose is meant 12 o'clock, the clock being duly regulated. It is well known that at the summer

solstice the clock is before the sun, and consequently the equation of time is then to be added to the time per clock, or mean time, to reduce it to apparent time; hence, the equation of time being upon the increase, the afternoons are longer than the forenoons, and are increasing on that account. And as the days decrease very slowly after the solstice, by the slow decrease of the declination, and the equation of time increasing fast, the length of the afternoons will continue to increase after the solstice, till the decrease of the lengths of the days balances the equation of time. Hence the calculation will be easy to any one furnished with a Nautical Ephemeris. Having the latitude and sun's declination, the length of the day is had by finding the sun's ascensional difference; thus, as radius : tang. lat.  $59^{\circ} 56'$  :: tang. declin. : sin. of ascensional difference. Hence we have

June	Sun's Decl.	Ascen. Dif.	Semi-day.	Eq. time.	Afternoon.
			<i>h. m. s.</i>		<i>h. m. s.</i>
20 - 23	$27^{\circ} 48''$	$- 48^{\circ} 34' 18''$	$- 9\ 14\ 17\cdot2$	$- 1' 12''\cdot7$	$- 9\ 15\ 29\cdot9$
21 - 23	$27\ 43$	$- 48\ 34\ 16$	$- 9\ 14\ 16$	$- 1\ 25\ \cdot8$	$- 9\ 15\ 41\cdot8$
22 - 23	$27\ 13$	$- 48\ 32\ 30$	$- 9\ 14\ 10$	$- 1\ 38\ \cdot5$	$- 9\ 15\ 48\cdot8$
23 - 23	$26\ 19$	$- 48\ 29\ 42$	$- 9\ 13\ 58\cdot8$	$- 1\ 51\ \cdot7$	$- 9\ 15\ 50\cdot5$
24 - 23	$25\ 0$	$- 48\ 25\ 37\frac{1}{2}$	$- 9\ 13\ 42\cdot5$	$- 2\ 4\ \cdot5$	$- 9\ 15\ 47\cdot$

From which it appears that the afternoon of the 23d of June will be longest of any in the year at that place.

#### XIV. QUESTION 952, by *Lieut. W. Mudge.*

A cylindrical vessel, full of water, having its axis perpendicular to the horizon, is whirled about its axis with such a velocity, that its circumference passes through 40 feet per second of time; it is required to determine how much water will run over; the diameter of the cylinder being two feet, and its depth four feet?

*Answered by Amicus.*

I do not see that any thing can be determined with precision, concerning this physical question, till the water has acquired the same rotatory velocity as the vessel, which, if we suppose to be covered close, till the water has acquired that circulatory velocity; then, the cover being taken off, it will fall in the middle, rise at the edge of the vessel, and fly off in the direction of tangents to its top, forming itself within into a hollow curved surface, till, its bounding curve being every where perpendicular to the direction of the single force compounded of gravity and the centrifugal one; when no one particle of the surface having any tendency either to rise or fall, it must rest in equilibrio; the effluent horizontal velocity continuing uniform till then, but the quantity thrown out per second diminishing till it vanishes, no more water can be forced over. And, by Bernoulli's *Hydrodynamica*, sect. 11th, if  $g = 32\cdot2$  the force of gravity, the water in that circumstance will form itself into the hollow surface of a paraboloid, whose equation is

$gx = 800y^3$ ; and at the top of the vessel,  $y$  being  $= 1$ ,  $x = 800 \div g = 24.84472$ , and at the bottom of the vessel, when  $x = 20.84472$ ,  $y = .91597$  the radius of the circle left dry at the bottom of the vessel: hence it will easily be found that .9195 of the whole quantity of water will flow over.

*The same answered by Mr. George Stevenson, Howdon.*

It is proved at Scholium 1st to the solution of question 797 in the Ladies' Diary for 1783, or question 14 Gentleman's Diary for 1789, that the curve  $AKB$  will be a parabola, or the concave surface of the water a paraboloid. The equation there given is  $n^3fx = 2p^3y^2$ , where  $n$  is the time of one revolution in seconds,  $f = 32\frac{1}{2}$ ,  $p = 3.1416$ ,  $x$  the absciss of the parabola, and  $y$  its ordinate. In the present case  $n$  being  $= 2py \div 40 = 157$ , and  $y = AM = 1$ , therefore  $x = KM = 2p^3y^2 \div n^3f = 24.87$ , which being greater than the depth of the vessel, a part of the bottom will be dry, and the empty part of the vessel will be the frustum of a paraboloid, whose height  $ML$  is 4 feet, and its content 11.556 feet  $= 70.82$  ale gallons, the quantity which will overflow.



*The same by Mr. John Ryley, of Leeds.*

When the vessel is put in motion, it is evident that every particle of the fluid will be agitated, and endeavour to recede from the axis of the vessel; and while part is forced over, the concavity will continually increase till an equilibrium obtains, and then the water will remain permanent. Now let  $AKB$  be a section of the cavity through the axis  $KM$ ; and put  $KL = x$ ,  $LN = y$ ,  $KN = z$ ,  $32\frac{1}{2} = m$ ,  $3.1416 = p$ , and the time of one revolution in seconds or  $2p \div 40 = n$ : then, by the resolution of forces,  $mxz^{-1}$  is the force of gravity in direction of the curve at  $N$ ; and  $4p^3yn^{-2}$  the centrifugal force at  $N$  in the direction  $LN$ , and the same in direction of the curve is  $4p^3yn^{-2}z$ ; and these two opposite forces must be equal in the case of an equilibrium. Hence then,  $mx = 4p^3yn^{-2}$  or  $mn^2x = 2p^3yy$ , the fluent of which is  $mn^2x = 4p^3y^2$ , the equation to a parabola. Now when  $x$  becomes  $= KM$ , then  $y = 1$ , and  $x = 2p^3 \div mn^2 = 800 \div m = 24.8704$ . Also by the nature of the curve  $KL : KM :: AM^2 : LN^2 = .83917$ , and the content of the frustum  $ABON$  is  $= 11\frac{1}{2}$  very nearly, which is the quantity of water that runs over, and is only one foot less than the content of the vessel, so that there will be only one foot of water left in it.

PRIZE QUESTION, by Amicus.

To how many terms, past the first, must the series of squares whose roots are 1, 2, 3, 4, 5, 6, &c. be carried, so that the sum of them all

may be a square number?

*Answered by Mr. Thomas Leybourn.*

The sum of  $n$  terms of the proposed series  $1^2, 2^2, 3^2, 4^2$ , &c. it is well known is equal to

$$\frac{1}{6}n \cdot n + 1 \cdot 2n + 1,$$

which is to be a square number greater than 1. Let us suppose that one of the factors,  $\frac{1}{6}n$ , is a square number  $= z^2$ , the other two factors will then be

$$6z^2 + 1, \quad 12z^2 + 1;$$

and their product must be a square number, a condition which will be satisfied if we make each a square number. Because  $6 = 2 \times 3$  and  $12 = 3 \times 4$  it is manifest that both factors are included in the expression

$$v(v+1)z^2 + 1 = z^2v^2 + z^2v + 1.$$

Now this will manifestly be a square, whatever be the value of  $v$ , if we make  $z = 2$ ; for it is then  $(2v+1)^2$ , therefore  $z = 2$  makes at once the three factors  $\frac{1}{6}n$ ,  $n+1$ ,  $2n+1$  squares, and  $n = 24$ , and the sum of the series is  $4900 = 70^2$ .

*The same answered by Mr. Charles Brady.*

It is well known that  $1^2 + 2^2 + 3^2$  &c. ....  $+ n^2 = \frac{1}{6}n \cdot n + 1 \cdot 2n + 1$ , which must be a square number; therefore put  $6r^2 = n$ , which substituted above gives  $r^2 \cdot (6r^2 + 1) \cdot (12r^2 + 1) =$  a square, and consequently  $(6r^2 + 1) \cdot (12r^2 + 1) = 72r^4 + 18r^2 + 1 = 81r^4 + 18r^2 + 1 - 9r^4 = (9r^2 + 1)^2 - (3r^2)^2$  must be a square number; where it is plain that  $9r^2 + 1$  and  $3r^2$  must be the hypothenuse and one leg of a right-angled triangle; therefore take the other leg  $= 9r^2 - 1$ ; then will  $(9r^2 + 1)^2 - (9r^2 - 1)^2 = (3r^2)^2$ , which gives  $r = 2$ ; and hence  $n = 6r^2 = 24$  the number of terms of the series.

*Questions proposed in 1793, and answered in 1794.*

1. QUESTION 954, by Mr. Tho. Nield, Master of a Boarding-School, Hawarden.

There are three numbers in continued proportion, the product of which is 4096, and the sum of the two extremes is 68. The double of the mean is my age in years, half the least extreme is the month; and one-fourth of the greatest the day of the month that gave me birth. Required my age, with the month and day in which I was born?

*Answered by Mr. George Baron, South Shields.*

As the three numbers are in continued proportion, the product of the two extremes is equal to the square of the mean, and consequently

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the continual product of all the three is equal to the cube of the mean; therefore  $\sqrt[3]{4096} = 16$  is the mean, or the second of the three numbers. Put now  $x =$  one of the extremes, then is  $68 - x =$  the other, and their product  $68x - x^2 = 16^3 = 256$  the square of the mean; then, by completing the square, the two roots of this equation are 64 and 4, which are the two extremes; so that the three numbers are 4, 16, 64. Hence 32 is the age in years, 2 the month, and 16 the day of the month; that is, the gentleman was 32 years of age on the 16th day of February.

*The same answered by Mr. Wm. Virgo.*

Let  $x, y, z$ , denote the three numbers. Then  $xyz = 4096 = a$ ,  $x + z = 68 = b$ , by the question, and  $xz = y^2$  by the nature of the proportionals. By substituting  $y^2$  for  $xz$  in the first equation, it becomes  $y^3 = a = 4096$ , and so  $y = 16$ , consequently  $xz = y^2 = 16^2 = 256$ . From the square of the second equation subtract  $4xz$  or 1024, there remains  $x^2 - 2xz + z^2 = 3600$ , the square root of which is  $x - z = 60$ , then by adding and subtracting this with the second equation; and dividing by 2, we have  $x = 64$ , and  $z = 4$ . And the age as above.

*The same answered by Mr. C. Brady.*

By the question, the mean term is given  $= \sqrt[3]{4096} = 16 = b$ . Put  $a = 34$  half the sum of the extremes, and  $x =$  half their difference; then will those extremes be denoted by  $a + x$  and  $a - x$ , hence their product  $(a + x) \times (a - x) = a^2 - x^2 = b^2$ , and therefore  $x = \sqrt{a^2 - b^2} = 30$ ; consequently the three numbers are 4, 16, and 64.

## II. QUESTION 955, by Mr. J. Hartley, Fleet-street.

A person is indebted 250*l.* who by industry has saved 8*l.* 5*s.* and promises his creditors he will add five shillings each quarter to the above 8*l.* 5*s.* placed at compound interest, which he has an opportunity of doing at the rate of 4*l.* per cent. per annum, or 1 per cent. per quarter, the interest becoming due quarterly. Required in what time will the debt be discharging, by a general theorem that will answer the same for as many millions?

*Answered by Mr. G. H.*

The industrious man hinted at by the ingenious proposer of this question, I suspect to be no other than poor honest John Bull; for I find that, putting millions for pounds; the circumstances of this case are the same as those of the national debt. The question may be thus solved; put  $a = 250$  the debt,  $p = 8.25$  the present sum saved,  $s = .25$  the future quarterly saving,  $r = 1.01$  the quarterly ratio at 4 per cent. and  $x =$  the time in quarters; the equation will be  $pr^x +$

$(ur^2 - u) \div (r - 1) = a$ , which reduced, the time will be found to be nearly 53 years, in less than which time I will venture to predict the national debt will not be paid.

*The same answered by Mr. Edward Wear, Assistant at the English Academy, Canterbury.*

Add the present worth of the annuity of one pound, payable quarterly, which is 25 pounds, to both the debt and principal, which will then become 275*l.* and 33*¼l.*; then find how long that principal  $p = 33\frac{1}{4}l.$  will be in discharging the debt  $d = 275l.$  at  $r$  the given rate of 1 per cent. per quarter compound interest; the general theorem for which is  $(\log. \text{ of } p - \log. \text{ of } a) \div \log. \text{ of } r = t = 212.32$  quarters, or 53.08 years the time required.

III. QUESTION 956, *by Mr. Robert Wilkinson, North Shields.*

Having to survey a field in form of a trapezium (which I was not allowed to enter) at one end of its longest side I found that the angle subtended by the opposite corner of the field and that side was equal  $42^\circ 28'$ , and I was informed the said opposite corner was a right-angle; going now further off in the direction of the said side to 520 links from my former station where I found the opposite and longest side would meet and form an angle of  $30^\circ 16'$ , from whence my distance to the nearest end of the said opposite side was 730 links. Required the area of the field?

*Answered by Mr. Robert Wilkinson.*

Let ABCD represent the field; A and E the first and second stations; then there are given

$$\angle BAC = 42^\circ 28'$$

$$\angle BEC = 30 \quad 16$$

$$\angle BCE = 90 \quad 0.$$

$$\text{line EA} = 520 \text{ links}$$

$$\text{line ED} = 730. \text{ Hence are found}$$

$$\angle ECA = \text{BAC} - \text{BEC} = 12^\circ 12'$$

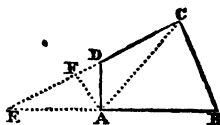
$$\angle EBC = 90^\circ - \text{BEC} = 59 \quad 44. \text{ Then, by trigonometry; as } s. \angle ECA : \text{EA} :: s. \angle CAE \text{ or } s. \angle BAC : \text{EC} = 1661.34, \text{ and } s. \angle EBC : \text{EC} :: s. \angle ECB : \text{EB} = 1923.54.$$

Then, by rule 2, page 96, Hutton's Mensuration, 2d edit.

$$\text{EB} \times \text{EC} \times \frac{1}{2} s. \angle E = \text{area of the triangle EBC, and}$$

$$\text{EA} \times \text{ED} \times \frac{1}{2} s. \angle E = \text{area of the triangle EAD, therefore}$$

$$(\text{EB} \times \text{EC} - \text{EA} \times \text{ED}) \times s. \angle E = 709693 = 7 \text{ ac. Or } 15\frac{1}{2} \text{ poles, is the area of ABCD required.}$$

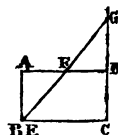


## IV. QUESTION 957, by Mr. B. Benson.

In the rectangular parallelogram ABCD, there is given AB = 26, and AD = 39; to find the position of the right line EFG = 65, such that the right-angled triangle FGD, whose area is required, may be equal the space ABEF.

*Answered by Mr. John Jackson.*

Bisect AD in F, and draw BFG, and it is done. For, as AF = FD, and the two triangles ABF, DGF equiangular, these are equal in all respects. Therefore  $BG^2 = BC^2 + CG^2 = BC^2 + 4CD^2 = 65^2$ , as by the question; and the area of the triangle DGF = ABF =  $\frac{1}{4}$  of ABCD =  $13 \times 19\frac{1}{2} = 253\frac{1}{2}$ .



*The same by Mr. George Stevenson. (Suppl.)*

As the area of the space ABEF is to be equal to the area of the triangle FGD, and the part DFEC being common, consequently the area of the triangle ECG will be = the parallelogram AC; hence the hypotenuse and area are given, to construct the triangle; which reduces the case to problem 33, page 345, Simpson's Algebra, where the construction and calculation are given. The calculation for the angles is, "As the square of the hypotenuse, is to the area, so is radius, to the sine of double the lesser of the two acute angles;" whence the  $\angle ECG = 36^\circ 52'$ ; also the base EC = 39 = BC, and CG = 52 = 2CD; consequently the hypotenuse must be drawn from E, and CD = DG.

## V. QUESTION 958, by Mr. Mich. Mooney, Dublin.

In a certain north latitude last spring when the sun's declination was double his altitude at six; the difference of the sines of his meridian altitude and midnight depression was equal to half the sine of the latitude. Query the time and place?

*Answered by Amicus.*

By the question,  $c6 = 2q6$ , and  $\sin. (zp + c6) = \sin. (zp - c6) = \frac{1}{2} \sin. PH = 2 \cos. zp \times \sin. c6 = 2 \sin. PH \times \sin. c6$ , or  $\frac{1}{2} = \sin. c6$  the declination  $14^\circ 28' 38''$  on April 28; the latitude  $30^\circ 15' 55''$ .



*The same answered by Mr. John Cavill, Bighton.*

It is well known that the difference of the sines of the sun's meridian altitude and midnight depression is equal to twice the sine of his altitude at six o'clock; and since that difference is equal to half the sine of the latitude, by the question; the ratio of the sines of the lati-

tude, and his altitude at six, is that of four to one; therefore as  $4 : 1 :: 1 : \frac{1}{4} = \text{sine of } 14^\circ 28'$ , the sun's declination nearly; hence his altitude at six is  $7^\circ 14'$ . Again,  $\sin. 14^\circ 28' : \sin. 7^\circ 14' :: \text{rad. } 1 : \sin. 30^\circ 16'$  the latitude required; and the time is the 28th day of April.

VI. QUESTION 959, by Mr. James Ashton, Harrington.

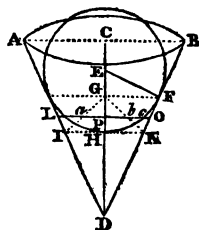
It is required to resolve questions 58 and 59, at page 315 of Hutton's Compendious Measurer, without the assistance of Algebra.

*Answered by the Proposer, Mr. James Ashton, of Harrington, near Liverpool.*

1st. For Quest. 58, which is as follows.

If a heavy sphere, whose diameter is 4 inches, be let fall into a conical glass,  $\frac{1}{2}$  full of water, whose diameter is 5, and altitude 6 inches; required what part of the axis of the sphere is immersed in the water?

Suppose the surface of the water to rise to LO. Here are given  $AB = 5$ ,  $CD = 6$ ,  $EF = EH = 2$ , E being the centre of the sphere, and F the point of contact of the sphere and cone; to find FH the height of the segment immersed. Now  $BD = \sqrt{(CB^2 + CD^2)} = 6.5$ ; then  $CB : BD :: EE : ED = 5.2$ , and  $CB : CD :: EF : DF = 4.8$ ; hence  $DE : DF :: DF : DG = 4.43077$ , and  $HD = ED - EH = 3.2$ , also  $GH = GD - HD = 1.23077$ ; likewise  $DC : DH :: AB : IK = 2\frac{2}{3} = \frac{8}{3}$ . Now  $cof$  (the ungula above the water, where the sphere is immersed) is = the cone  $gab$ , as may be proved; whence  $IK^3 \times .7854 \times \frac{1}{3}GD - \frac{1}{3}$  of the whole cone  $ABD$  is =  $cof$  (round the sphere) = the small cone  $gab = .393683$ ; then as cone  $IGK : \text{cone } gab :: GH^3 : GF^3$ , or  $\sqrt[3]{IGK} : \sqrt[3]{gab} :: GH : GF = .684$ ; hence  $FH = GH - GF = .5467$ , the answer.



2nd. For Quest. 59, which is thus:

The cone being still the same, and  $\frac{1}{2}$  full of water; required the diameter of a sphere that may be just all covered with the water?

Let  $LVM$  be the whole cone, and  $FVG = \frac{1}{2}$  of it =  $7.854$  cubic inches. Here  $VM = 6$ ,  $LM = 5$ ; then  $\sqrt[2]{(6^2 - 5^2)} = VH = 3.508821$ ; and  $VN : VM :: LM : FG = 2.924$ , then  $GN = 1.462$ . But by similar triangles  $VH : HG :: KT : KO$ , that is  $3.508821 : 1.462 :: KT : KO :: 1 : .416664$ ; therefore if  $KT = 1$ , then  $KO = .416664$ ; then  $KO : KT :: KT : KV = 2.4$ , and  $VO = VK + KO = 2.816664$ . But  $or = \sqrt{(KT^2 + KO^2)} = 1.083332$  hence  $VE = VO + OE = 3.9$ , and  $VK :$



exactness by considering it as a plane triangle, in which the angles and the side  $TM$  are given; let fall a perpendicular  $MR$  upon  $TO$ ; then it will be as rad. :  $TM :: \sin MT : MR = .16815$ , also  $\sin. TOM : TM :: \sin. TMO : TO = 27.0934$ ; hence  $\frac{1}{2}TO \times MR = 2.27787$  miles, or 1458 acres, the area sought.

*The same answered by Mr. John Liddell, of Hovingham.*

Let  $P$  represent the north pole,  $H$  the place of Hovingham, and the great circle  $PH$  its meridian, or its co-latitude at right angles to which let the great circle  $THM$  be drawn, upon which set off  $HM = 9$ , and  $HT = 18$  miles; then will  $T$  be the place of Thirsk, and  $M$  that of Malton.—Draw the meridians  $PT$  and  $PM$  and  $TO$  perpendicular to  $PT$ , forming the spherical triangle  $TMO$ , the area of which is required. Now as 25,000 miles :  $360^\circ :: 9$  miles :  $7' 48'' 10''' = HM$ , the double of which is  $15' 36'' 20''' = TH$ : therefore in the right-angled triangles  $PHT$  and  $PHM$ , the legs in each are given, to find the rest, which will be best found by the problems in pages 159 and 160 of Dr. Hutton's Tables, by which we find as follows:

$PT = 35^\circ 50' 3'' 0'''$ , its comp. =  $54^\circ 9' 57'' 0'''$  the lat. of Thirsk,  
 $PM = 35^\circ 50' 0'' 40$ , its comp. =  $54^\circ 9' 59'' 20$  the lat. of Malton,  
 $\angle PTH = 89^\circ 38' 24'' 00$ , its comp. =  $0^\circ 21' 36'' 00 = \angle OTM$ ,  
 $\angle PMH = 89^\circ 49' 11'' 40$ , its suppl. =  $90^\circ 10' 48'' 20 = \angle OMT$ ,  
 hence in  $\triangle TMO$  is found..... =  $89^\circ 27' 35'' 41.78 = \angle TOM$ ,

sum of the three angles is.....  $180^\circ 0' 0'' 1.78$ .

Hence, by Dr. Hutton's Mensuration, page 202,  $180'' : 25000^2 \times .0795774715 :: 1''' .78 : 2.277$  miles, the area; which multiplied by 640, gives 1457 acres.

#### VIII. QUESTION 961, by Mr. Tho. Woolston, Adderbury.

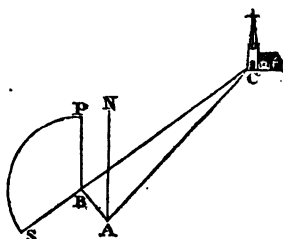
Being out one day in the spring, 1792, surveying some land, some of my pupils were desirous of knowing the distance of a steeple which we saw a considerable way off, in the north-east quarter: to gratify them, I fixed my theodolite, and having ordered a mark to be set up for a second station, in a convenient place, which bore exactly  $NW\frac{1}{4}W$ . from us, I observed that the steeple and the mark subtended an angle of  $103^\circ 52'$ , and the distance between the two stations measured 3650 links; and upon fixing the instrument at the second station, when I had adjusted the sights to the bearing of the steeple, I observed the shadow of the sight exactly to coincide with the direction of the index; I therefore immediately took the sun's altitude, and found it  $39^\circ 49\frac{1}{2}'$ . The declination at that time was  $18^\circ 9'$  north, and the latitude of the place  $52^\circ 5'$  north. Required from hence, the time when the observation was made, and the bearing and distance of the steeple from our second station?

*Answered by the Proposer, Mr. Thomas Woolston.*

The solution of this question presents us with a case in spherics, where the three sides are given, viz. the co-latitude  $37^{\circ} 55'$  the sun's co-altitude  $50^{\circ} 10\frac{1}{2}'$ , and his co-declination  $71^{\circ} 51'$ , to find the angle at the zenith, or sun's azimuth, and the angle at the pole, or time from noon; these resolved give  $114^{\circ} 14' 15\frac{1}{2}''$  his azimuth from the north, and  $47^{\circ} 28' 27\frac{1}{2}''$  for the angle at the pole; this reduced into time gives three hours ten minutes nearly for the time from noon; and as the shadow falls between the north and east, it must be in the afternoon; and the declination answers to the 11th day of May; also the point which the sun was on being  $65^{\circ} 45' 44\frac{1}{2}''$  from the south; gives NE by  $E\frac{1}{4}E$  for the bearing of the steeple: Now the angle subtended by the steeple and first station will be found  $66^{\circ} 25\frac{1}{2}'$  nearly, therefore the sum of the angles at the two stations is  $170^{\circ} 17\frac{1}{2}'$ , the supplement of this, or  $9^{\circ} 42\frac{1}{2}'$  is the angle at the steeple; hence by Plane Trigonometry, as sine  $9^{\circ} 42\frac{1}{2}'$  : 3650 :: sine  $103^{\circ} 52'$  : 21014 links = 210.14 chains, or 2 miles and 5 furlongs, the distance sought.

*The same answered by the Rev. Mr. L. Evans, Little Bedwin.*

Let A and B be the first and second stations, C the steeple, P the pole, and S the sun. Now there are given the latitude =  $52^{\circ} 5'$  north, declination  $18^{\circ} 9'$  north, and the altitude of the sun's centre =  $39^{\circ} 49\frac{1}{2}'$ , the complements of which give PB =  $37^{\circ} 55'$  the co-lat. PS =  $71^{\circ} 51'$  the co-declin. and BS =  $50^{\circ} 10\frac{1}{2}'$  the co-altitude; to find the angle of time, or at the pole P =  $47^{\circ} 28' 30''$ , answering to 3h. 9m. 54s. afternoon, the time of observation on the 11th of May; and the angle PBS =  $114^{\circ} 14' 15''$  the sun's azimuth from the north, whose supplement gives the angle PBC =  $65^{\circ} 45' 45''$  = NE by  $E\frac{1}{4}E$  here, the bearing of the steeple at the second station. Let NA be another meridian or parallel to PB, then since there are given the  $\angle NAB$  the bearing NW  $\frac{1}{4}W$  =  $47^{\circ} 48' 45''$ , the  $\angle BAC$  =  $103^{\circ} 52'$ , therefore PBA or supplement of NAB =  $132^{\circ} 11' 15''$ , from which taking PBC, there remains  $\angle ABC$  =  $66^{\circ} 25' 30''$ ; lastly, the sum of ABC and BAC taken from  $180^{\circ}$ , leave the  $\angle C$  =  $9^{\circ} 42' 30''$ ; then, in the plane triangle ABC, being known all the angles, and the side AB = 3650 links =  $36\frac{1}{2}$  chains, it will be as sin.  $\angle C$  : sin.  $\angle A$  :: AB : BC = 210  $\frac{1}{4}$  chains =  $2\frac{1}{8}$  miles.



IX. QUESTION 962, by Mr. John Ryley, Leeds.

As I was walking out one evening last summer, I observed a person at a distance dropping a stone into a coal-pit; the noise that it

made by striking against the bottom, reached my ear exactly in  $\frac{1}{4}$  part of the time that the stone was in falling, and the person at the brink of the pit heard the sound one second sooner than I did. Required the depth of the pit?

*Answered by Mr. James Adams, School Master, Plymouth Dock.*

Put  $a = 1142$ ,  $b = 16\frac{1}{12}$ , and  $x =$  the depth of the pit in feet. Then as  $\sqrt{b} : \sqrt{x} :: 1'' : \sqrt{(x \div b)}$  the time of the stone's descent, and  $a : x :: 1'' : (x \div a)$  the time the sound is ascending to the brink of the pit; therefore, by the question,  $x \div a + 1 = \frac{1}{4}\sqrt{(x \div b)}$ ; this reduced gives  $x^3 + (2 - \frac{1}{4}(a \div b)) \cdot ax = -a^2$ , in numbers  $x^3 - 6725.76857x = -1304164$ ; here the value of  $x$  is 199.844 or 6525.925; but the latter number, being near one mile and a quarter, is too great; therefore the pit is 199.844 or near 200 feet deep.

*The same answered by Mr. Thomas Cram, jun. Killingworth.*

Put  $b = 16\frac{1}{12}$  feet, the space a body falls through in the first second;  $c = 1142$  feet, the space sound moves uniformly through in each second; and  $x$  the time of a body's falling. Then as  $1'' : x^3 :: b : bx^3 =$  the depth descended; and  $\frac{1}{4}x - 1 =$  the time in which the sound passed from the bottom to the top, therefore  $\frac{1}{4}cx - c =$  the space passed over by sound in this time, which space must be equal to the former, that is  $bx^3 = \frac{1}{4}cx - c$ , which equation solved gives  $x = 3.52498$ ; and hence  $bx^3 = 199.84$  feet, the depth of the pit.

x. QUESTION 963, by the Rev. Mr. John Sampson.

Given  $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m \dot{x}}{ay^n}$ , to find the relation of  $x$  and  $y$ .

*Answered by the Rev. J. Sampson.*

Multiply the given equation by  $anx^n y^n$ , and it becomes  $anx^{n-1} \dot{x} y^n + anx^n y^{n-1} \dot{y} = nx^{m+n} \dot{x}$ ; the fluents of which are

$$ax^n y^n = \frac{n}{m+n+1} x^{m+n+1}. \quad \text{Q. E. I.}$$

*Note.* Mr. Thomas Simpson, after laying down a rule for the resolution of fluxional equations, gives this equation as an example, art. 262 of his Fluxions; and after giving a solution in the particular case where  $n = 1$ , adds, "this appears to be the only case of the given equation, where this method is of use." But it appears from the above that the method is general, whatever be the value of  $n$ . I readily confess that this was shewn me several years ago by an ingenious school-fellow of mine (now a Fellow of St. John's, Cambridge), who had it

from that able mathematician Mr. Dawson, of Sedbergh, at the time he was his pupil, and I was induced to make it public by inserting it here, thinking it might be serviceable to the readers of Simpson's Fluxions, if generally known.

*The same answered by Mr. John Dalton, Kendal.*

By reduction  $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{xy + y\dot{x}}{xy} = \frac{x^m \dot{x}}{ay^n}$ ; put  $xy = z^2$ ; then  $2z\dot{z} = xy + y\dot{x}$ , and  $\frac{2\dot{z}}{z} = \frac{xy + y\dot{x}}{xy} = \frac{x^m \dot{x}}{ay^n} = \frac{x^m \dot{x}}{az^{2n} \div x^n} = \frac{x^{m+n} \dot{x}}{az^{2n}}$ ; hence  $2az^{2n-1}\dot{z} = x^{m+n}\dot{x}$ ; and taking the fluents we get  $\frac{az^{2n}}{n} = \frac{x^{m+n+1}}{m+n+1}$ ; or by restitution  $\frac{a}{n} x^n y^n = \frac{x^{m+n+1}}{m+n+1}$ ; or  $\frac{a}{n} y^n = \frac{x^{m+n}}{m+n+1}$ , the relation required.

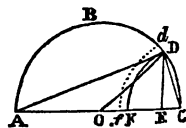
This question is similar to prob. 47, page 485, Emerson's Miscellanies; and to question 502 of this work. It is also resolved, generally, in Agnesi's Analytical Institutions, 1748.

# XI. QUESTION 964, by Ferdinando.

ABC is a given semicircle, o its centre, and with any line CD as a radius there is described the arch DF; it is required to find the length of the said radius so, that the quadrature of the space DFC, cut off the semicircle by the said arch DF, may be equal to any given quantity.

*Answered by Mr. A. Buchanan, Sedgfield.*

In order to find the quadrature of the space CDF. Describe, on the same centre c, the arch *df* indefinitely near *DF*; join *AD*; and demit the perpendicular *DE*; then by similar triangles (putting *AC* = *2r*, and *CD* = *CF* = *x*), *AC* : *CD* :: *CD* : *CE* =  $x^2 \div 2r$ ; consequently *EF* = *CF* - *CE* =  $x - x^2 \div 2r$ . Now *fdDF* is the fluxion of the area DFC = arch *DF* × *xf* = arch *DF* ×  $x$ ; but by prop. 14, book 1, Emerson's Trigon. or prob. 7, sect. 1, part 2, Hutton's Mensuration, the arch *DF* is =



$x\sqrt{\left(2\left(1 - \frac{x}{2r}\right)\right)} \times \left(1 + \frac{1 - \frac{x}{2r}}{3.4} + \frac{3\left(1 - \frac{x}{2r}\right)^2}{4.5.8} \&c.\right)$ , whence, putting  $1 - x \div 2r = z^2$ , the fluxion of the segment DFC is =

$- 8r^2 z^2 \cdot (1 - z^2) \sqrt{2} \times (1 + \frac{z^2}{3 \cdot 4} + \frac{3z^4}{4 \cdot 5 \cdot 8} + \frac{3 \cdot 5z^6}{4 \cdot 7 \cdot 8 \cdot 12} \&c.),$

and the correct fluents, or the segment DFC is =

$$8r^2 \sqrt{2} \times \left\{ \frac{1}{3} - \left( \frac{1}{5} - \frac{1}{5 \cdot 3 \cdot 4} \right) - \left( \frac{1}{7 \cdot 3 \cdot 4} - \frac{3}{7 \cdot 4 \cdot 5 \cdot 8} \right) \&c. \right\} -$$

$$8r^2 \sqrt{2} \times \left\{ \frac{1}{3} z^3 - \left( \frac{1}{5} - \frac{1}{5 \cdot 3 \cdot 4} \right) \cdot z^5 - \left( \frac{1}{7 \cdot 3 \cdot 4} - \frac{3}{7 \cdot 4 \cdot 5 \cdot 8} \right) \cdot z^7 \&c. \right\}.$$

But the sum of the former part of this series is  $= 8r^2 \sqrt{2} \times \cdot 13884009$ ; hence then, equating this with any given possible area ( $a$ ), we shall

$$\text{have } 8r^2 \sqrt{2} \times \cdot 13884009 - 8r^2 \sqrt{2} \times \left\{ \frac{1}{3} z^3 - \left( \frac{1}{5} - \frac{1}{5 \cdot 3 \cdot 4} \right) \cdot z^5 \&c. \right\}$$

$$= a, \text{ or } \frac{1}{3} z^3 - \left( \frac{1}{5} - \frac{1}{5 \cdot 3 \cdot 4} \right) z^5 - \left( \frac{1}{7 \cdot 3 \cdot 4} - \frac{3}{7 \cdot 4 \cdot 5 \cdot 8} \right) z^7 \&c. =$$

$$\cdot 13884009 - \frac{a}{8r^2 \sqrt{2}}.$$

From whence, by reverting the series, or otherwise,  $z$  in all cases may be found, and consequently the required radius  $x$  (being  $= (1 - z^2) \cdot 2r$ ) will also be known.

**EXAMPLE.** Suppose  $r = 39 \cdot 25073055$ , &c. and  $a = 1210$ , which is the case proposed at question 13, Carnan's Diary 1788; then  $z$  will be found  $= \cdot 648563$ , and consequently  $x = 45 \cdot 481$  nearly, which will be found to correspond with the answer by any other method.

**COROL. 1.** The sum of the series mentioned to be equal  $8r^2 \sqrt{2} \times \cdot 13884009$ , appears to be equal  $2r^2 \times \cdot 7853981633$ , &c. and consequently  $\cdot 13884009 \times 4\sqrt{2} = \cdot 7853981633$ , &c. For, when  $x = 2r$ , the area cut off is evidently the whole semicircle  $= 2r^2 \times \cdot 785398$ , &c. but  $z$  is then  $= 0$ , and consequently the value of  $a$  barely  $= 8r^2 \times \cdot 13884$ , &c.: therefore, &c.

**COROL. 2.** If it were required to find the radius  $cd$ , when the arch DF is a maximum. Then the length of that arch being universally  $= 2rz \cdot (1 - z^2) \sqrt{2} \times (1 + \frac{z^2}{3 \cdot 4} + \frac{3z^4}{4 \cdot 5 \cdot 8} + \frac{3 \cdot 5z^6}{4 \cdot 7 \cdot 8 \cdot 12} \&c.),$  by taking the fluxion, and equating it to 0, the values of  $z$  and  $x$  may be determined.

## XII. QUESTION 965, by Mr. John Rodham, Richmond.

A vessel being filled with water, it is observed, on emptying it by a hole in the bottom, whose area is one inch, that the perpendicular descent of the surface of the water, is uniformly 4 inches per minute; the perpendicular height of the vessel is 5 feet, required its content in ale gallons.

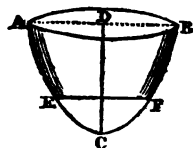
*Answered by Amicus.*

If  $x$  = any variable height of the water,  $d = 193$  inches, and  $4 \div 60 = c$ , then  $\sqrt{d} : 2d :: \sqrt{x} : 2\sqrt{dx}$ ; and  $\sqrt{2dx}$  = the effluent velocity,  $\sqrt{(2dx) \div py^2} = c$  the velocity of the descending surface ( $p = 3.14159$ ) which also gives the biquadratic parabola for the form of the vessel, and  $py^2x = \sqrt{2d} \times c^{-1} \times x^{\frac{5}{2}}$ ; fluent  $= \frac{2}{3}\sqrt{2d} \times c^{-1} x^{\frac{3}{2}} = 323.796$  ale gallons.

*The same answered by Mr. John Dalton, of Kendal.*

Since the descent of the surface of the water is uniform, the vessel cannot be cylindrical, but of such a form, that the area of the surface of the water, is always directly as the velocity of the effluent water; or, which amounts to the same thing, as the square root of the perpendicular height of the surface of the water above the hole; whence, putting  $p = .7854$ ,  $x$  = any variable height of the surface, and  $y$  = the corresponding diameter of the vessel, we shall have  $y^2$  as  $\sqrt{x}$ , or  $y'$  as  $x$ ; from which it appears that the vessel is formed by the revolution of a parabolic curve of the 4th order round its axis. Moreover, as the surface descends  $\frac{1}{15}$ th of an inch the first second, and the depth = 5 feet, the quantity of water discharged in the first second will be  $\frac{1}{15} \sqrt{(10 \times 16\frac{1}{2})} = .08807$  feet, which gives the diameter of the top of the vessel, or  $AB = 4.49$  feet =  $d$ . Hence we shall have  $\sqrt{5} : d^2 :: \sqrt{x} : y^2 = EF^2 = d^2 \sqrt{\frac{1}{5}x}$ , and the fluxion of the content  $= pd^2x \sqrt{\frac{1}{5}x}$ , the fluent of which  $= \frac{2}{3}pd^2 \sqrt{\frac{1}{5}} \cdot x^{\frac{3}{2}}$  = the solidity; and when  $x = 5$ , this becomes  $\frac{2}{3}pd^2 = 52.83$  feet =  $323.7$  ale gallons, the content.

If we take the supposition that the velocity of the effluent water is equal to that acquired by falling through the *whole* height of the surface, instead of half of it, the top diameter will come out 5.34 feet, and the content  $\sqrt{2} \times 323.7 = 457.8$  ale gallons.



### XIII. QUESTION 966, by Vertigo.

Let  $AB$  be a given horizontal plane. It is required to find the length and position of another plane  $AC$ , meeting  $BC$  perpendicular to  $AB$  in  $c$ ; such that a heavy body, descending freely along it from  $c$ , after quitting it at  $A$ , may strike another horizontal plane  $MD$  at the greatest distance possible from  $D$ ; and also to determine that distance.

*Answered by Amicus.*

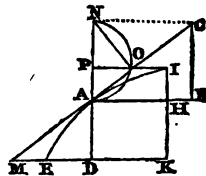
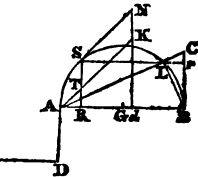
If  $BL$  be let fall perpendicular to  $AC$ , and  $LR$  drawn parallel to  $AB$ , then, if the body instead of descending along  $CA$ , moved in a parabola

so as to have the same direction  $CA$  and velocity at  $A$ , it is well known that  $cr$  would be the height of that parabola above  $AB$ , and  $Lr = \frac{1}{4}$  of its random, or  $2Lr$  the semiordinate to the abscissa  $cr$ . And, by conics, if the ball strike  $DM$  at  $M$ ,  $\sqrt{cr} : 2Lr :: \sqrt{(cr + AD)} : 2Lr + MD$ , or  $\sqrt{cr} : \sqrt{(cr + AD)} - \sqrt{cr} :: 2Lr : MD :: Lr : \frac{1}{2}MD$ ; produce  $RL$  till it cuts a semicircle described on  $AB$  in  $s$ , and let fall  $SR$  perpendicular to  $AB$ , then  $AR = Lr$ , and  $SR = Br$ , therefore  $Lr = \sqrt{(cr \cdot Br)}$ , and  $\sqrt{cr} : \sqrt{(cr + AD)} - \sqrt{cr} :: \sqrt{(cr \cdot Br)} : \frac{1}{2}MD$ , or  $1 : \sqrt{(cr + AD)} - \sqrt{cr} :: \sqrt{Br} = \sqrt{RS} : \frac{1}{2}MD = \sqrt{(AR^2 + AD \cdot RS)} - AR$ , or  $\frac{1}{4}MD^2 = AD \cdot RS - MD \cdot AR$ . On  $AB$  take  $Ad = AD$ , and the perpendicular  $dK = MD$ , and  $NK = MD^2 \div 4Ad = RS - TR = ST$ . If therefore  $MD$  be a given quantity, having described the semicircle, taken  $Ad = AD$ ,  $dK = MD$ , and  $NK$  as above, through  $N$  parallel to  $AK$  draw a right line, which in all possible cases will either cut the circle in two points, or touch it in one as  $s$ , in which latter case  $ST$  and consequently  $MD$  is a maximum because  $AD$  is given. But to find under what limits this latter case obtains, since  $\frac{1}{4}Ad \cdot dK^2 = Ad^2 \cdot RS - Ad^2 \cdot TR = Ad^2 \cdot RS - Ad \cdot dK \cdot AR = Ad^2 \cdot RS - Ad \cdot dK \cdot AG + Ad \cdot dK \cdot RG$  ( $G$  being the circle's centre, and  $GS$  perpendicular to  $SN$  and  $AK$ )  $= Ad^2 \cdot RS + dK^2 \cdot RS - Ad \cdot dK \cdot AG = AK^2 \cdot RS - Ad \cdot dK \cdot AG = AK^2 \cdot RS + dK^2 \cdot RS - Ad \cdot dK \cdot AG = AK \cdot AG \cdot Ad - dK \cdot AG \cdot Ad = \frac{1}{2}AB \cdot Ad \cdot (AK - dK)$ ;  $dK^2 = 2AB \cdot (AK - dK)$ , and  $dK^4 = 4AB^2 \cdot AD^2 - 4dK^3 \cdot AB$ , which therefore is in general *solid*.

But if  $2AD = AB$ ,  $dK = \frac{AB\sqrt{2}}{(1+\sqrt{2})(\sqrt{2}-1)}$ , and  $\frac{dK}{Ad} = \text{tang. of } dAK = 50^\circ 15'$ , and  $CAB = ASR = \frac{1}{2}AKd = 19^\circ 52\frac{1}{2}'$ , &c.

*The same by Mr. J. Nicholson, Teacher of Mathematics, Newcastle. (Supplement).*

Let  $AC$  be the required plane. Produce  $DA$  so that  $AN = BC$ , and on  $AN$  describe the semi-circle  $AON$  cutting  $AC$  in  $O$ ; draw  $POI$  perpendicular to  $AN$ , in which take  $OI = OP$ , so shall  $I$  be the vertex of the parabolic curve  $IAE$ , of which the part  $AE$  is described by the heavy body after running down the tangent  $CA$ , and quitting it at  $A$ . Put  $AB = a$ ,  $AD = c$ , and  $BC = x$ ; then  $AC = \sqrt{(a^2 + x^2)}$  suppose  $z$  for the present; now, by similar triangles,  $AC : BC :: AN = BC : AO = x^2 \div z$ ;  $AO : AP = HI = x^3 \div z^2$ ; and  $AC : AB :: AO : PO = ax^2 \div z^2 = OI$ ,  $AH = 2ax^2 \div z^2$ ; then, by the nature of the parabola,  $\sqrt{IH} : \sqrt{IK} :: HA : KE$ , or  $\sqrt{IH} = \sqrt{IK} :: HA : DE = \frac{2ax^2}{x^2} \times \frac{\sqrt{(x^3 + cz^2)} - \sqrt{x^3}}{\sqrt{x^3}} = \frac{2ax^2}{x^2} \times \frac{\sqrt{(x^3 + czx^2)} - x^3}{x^2} =$



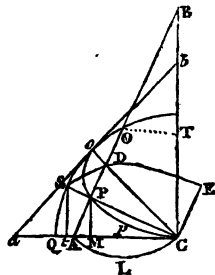
$\frac{2a}{x^2} \times \frac{cxz^3}{\sqrt{(x^4 + cxz^2) + x^2}} = \frac{2acx}{\sqrt{(x^4 + cxz^2) + x^2}} =$  (restoring the value of  $z^2$ )  $\frac{2acx}{\sqrt{(x^4 + cx^3 + ca^2x) + x^2}}$  a maximum; its fluxion being put = 0, by reduction gives  $x^4 - (8a^2 \div c)x^3 - 2a^2x^2 + a^4 = 0$ . Hence, when  $a = 3$ , and  $c = 9$ , we have  $BC = x = 1.687$ ,  $AC = 3.4419$ ,  $DE = 5.502$ , and  $\angle BAC = 29^\circ 21' 15''$ .

#### XIV. QUESTION 967, by Amicus.

If from the fixed right angle of a triangle, a perpendicular be let fall to the hypotenuse (which is given and constant) in every possible position of it, required the equation of the curve which is the locus of the end of the perpendicular, together with its quadrature and rectification, by means of those of the conic sections?

*Answered by the Proposer, Amicus.*

Let  $ab$  be the position of the given hypotenuse when  $ao = ob$  and the perpendicular  $co = ao = ob$  and  $AB$  any other position, perpendicular to which draw  $CP$ , then  $o$  and  $P$  are in the curve, with  $oc$  radius describe a circle cutting  $CP$  produced in  $s$  and  $ac$  in  $q$ , draw  $PM$  and  $st$  parallel to  $BC$ , take  $AO = BO$ , draw  $OT$  parallel to  $AC$ , and with the radius  $AO = BO$  and centre  $o$  describe a circle  $CLA$ , perpendicular to  $sc$  draw  $CE = \frac{1}{2}sc$ , and to the semiaxes  $sc$  and  $CE$  describe an ellipsis cutting  $PO$  in  $D$ , then must  $PD = \frac{1}{2}PO$ . Also  $\frac{1}{4}$  the area of the circular segment  $PALC$  = the curved area  $rpc$ . And the elliptic arc  $DE =$  the curve  $rpc$ .



For since the  $\angle ACS = \angle ABC$ , and  $sc = BO = AO$ ,  $st \perp OT$ ,  $TC = BT$ , and the fluxion of the area  $rpc = (PC^2 \div 2BO) s'q = (PC^2 \div 2BT) o't = PM \times o't = \frac{1}{2}PM \times A'C = \frac{1}{2}PM \times (AB \div 2AC) A'P = \frac{1}{4}PC \times A'P$ , and  $rpc = \frac{1}{4}PAC$ . Also the fluxion of the rectification =  $\sqrt{(P'C^2 + (P'C^2 \div BO^2) s'q)} = \sqrt{(P'C^2 + (PC^2 \div BT^2) o't)} = \sqrt{(P'C^2 + \frac{1}{4}A'P^2)} = \sqrt{(P'C^2 + \frac{1}{4}o'P^2)} = \sqrt{(P'C^2 + P'D^2)} = D'E$ , and the rectification =  $DE$ . Moreover, since  $AC : AB = 2OB :: OT : OB :: PM : PC = \sqrt{(PM^2 + MC^2)} = \sqrt{(MC \times AC)} = AB \times PM \div AC$ ,  $AC = MC + PM^2 \div MC =$

$PC^2 \div MC$ ,  $PC^2 : AB \times MC :: PM : PC$ , and  $AB^{\frac{2}{3}} = \frac{PM^{\frac{2}{3}}}{MC^{\frac{2}{3}}} + \frac{MC^{\frac{2}{3}}}{PM^{\frac{2}{3}}}$  the curve's equation.

*The same by Mr. Joseph Garnett. (Suppt.)*

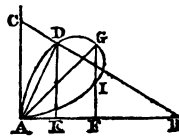
Let  $cb$  be a position of the hypotenuse; draw  $ad$  perpendicular

to CB, and from D demit DE perpendicular to AB, the curve being ADA IA. Put  $a = CB$ ,  $x = AE$ , and  $y = DE$ ; then  $\sqrt{(x^2 + y^2)} = AD$ , and by similar triangles  $AD : AE :: BC : AC$ , and

$$AD : DE :: BC : AB, \text{ which give } AC = \frac{ax}{\sqrt{(x^2 + y^2)}},$$

$$\text{and } AB = \frac{ay}{\sqrt{(x^2 + y^2)}}; \text{ but } AC \times AB = BC \times$$

$$AD, \text{ that is } \frac{a^2xy}{x^2 + y^2} = a\sqrt{(x^2 + y^2)}, \text{ hence } axy =$$



$(x^2 + y^2)^{\frac{3}{2}}$  the equation of the curve. And as it is equally affected with  $x$  and  $y$ , the curve will return into itself, and will have its convex and concave parts exactly similar.

To find the area: put  $vy = x$ ; then the equation will become  $avy^2 = (v^2y^2 + y^2)^{\frac{3}{2}}$ ; hence  $y = \frac{av}{(1 + v^2)^{\frac{3}{2}}}$ , and  $x = \frac{av^2}{(1 + v^2)^{\frac{3}{2}}}$ ; consequently  $x = \frac{2av - av^3}{(1 + v^2)^{\frac{1}{2}}}$ , and the fluxion of the area  $yx =$

$\frac{2a^2v^2 - a^2v^4}{(1 + v^2)^{\frac{1}{2}}}$  the fluent of which is  $\frac{a^2}{48} \times : \frac{24v^2}{(1 + v^2)^3} + \frac{6v^2}{(1 + v^2)^2} + \frac{3v^2}{1 + v^2} - 3v + 3 \times .017453 \times \text{deg. in the arch whose tangent is equal } v, \text{ and radius equal } 1, \text{ which needs no correction.}$

. Bisect the angle CAB with AG equal half the hypotenuse, and demit GF perpendicular to AB; then will AG divide the curve into two equal parts, and AF = FG. But when the abscissa and ordinate are equal,  $v = 1$ , therefore  $\frac{1}{16} \times .7854a^2 = \text{area of ADGFA}$ ; consequently  $\frac{1}{16} \times .7854a^2 - \frac{1}{16}a^2 (\triangle AFG) = \frac{1}{16} \times .7854a^2 = \text{area of ADGA, or } .7854a^2 \times \frac{1}{8} = \text{the area of ADGIA, the whole curve,} = \text{twice the area of a circle whose diameter is } \frac{1}{4}a.$

PRIZE QUESTION, by Mr. D. Kinnebrook, jun.

Suppose an uniform slender rod AB, parallel to the horizon, (considered without regard to its weight) to have two equal bodies fixed to it, one at each end, and to revolve round the point c in that rod as a centre in any given time; the length of the rod being given, as also cd the distance of the middle of the rod from the centre of motion c: it is required to determine the velocity with which the rod passes through the point c, as also the time of description. Supposing AB = 100 feet, cd = 1 inch at first, and the time of each revolution 20 seconds.

*Answered by Mr. John Nuttall, Schoolmaster, Bury.*

Let the line  $AB$  represent the rod,  $n$  its middle point,  $c$  the point or centre about which it revolves, and through which it slides. Put  $AD = DB = a = 50$  feet, the variable distance  $CD = x$ , where  $t$  is the time it has been in motion, and when  $v$  is the velocity of the motion through the point  $c$ ; also  $c = \frac{1}{12}$  of a foot the distance  $DC$  at the beginning, and  $b =$  the weight or quantity of matter in each of the equal bodies, and  $d = 10$  seconds, also  $p = 3.1416$ . Then, by the nature of centrifugal forces,  $p^2 b (a + x) \div d^2 =$  the motive centrifugal force in the direction  $CA$ , and  $p^2 b (a - x) \div d^2 =$  that in the direction  $CB$ ; therefore the difference of these two forces being divided by the quantity of matter in both the bodies, gives  $p^2 x \div d^2 = n^2 x$  for the accelerative force of the rod in the direction  $CA$ , putting  $p^2 \div d^2 = n^2$ ; then, by the laws of motion  $n^2 x \dot{x} = v \dot{v}$ , the fluents of which give  $n^2 x^2 = v^2$ ; but when  $x = c$ ,  $v = 0$ , therefore the fluents corrected are  $v^2 = n^2 (x^2 - c^2)$ . Again, by the said laws,  $\dot{t} = \frac{\dot{x}}{v} = \frac{\dot{x}}{n \sqrt{(x^2 - c^2)}}$  the fluxion of the time; the fluent of which is  $\frac{1}{n} \times \text{hyp. log. of } (x + \sqrt{(x^2 - c^2)})$ , or corrected as above  $t = \frac{1}{n} \times \text{hyp. log. of } \frac{x + \sqrt{(x^2 - c^2)}}{c}$  is the true time of describing the distance  $x - c$ . And this, when  $x = 50$ , gives  $t = 22.56$  seconds, the time of the shorter end passing through  $c$ .



*Questions proposed in 1794 and answered in 1795.*

I. QUESTION 969, by Mr. James Wilding, *High Ercaik*.

Jack Gauge the exciseman's a good-natured fellow,  
Except when he's crosst, or is more than half mellow,  
Which happen'd last night, when I offer'd a wager,  
(For he prides himself much on being a gauger,)   
That I'd shew him a rule by which he may know,  
The content of a tub from the measures below.  
At first he look'd wise, chang'd his face, and then swore,  
He never had seen such a question before.  
What! the top, but no bottom diameter given!  
I'll double the wager, or lay ten to seven.  
Thus in confidence trusting the question's not true,  
He hazards the bet, and leaves it to you.

Top diameter 48 inches =  $AB$ ; diagonal 50 inches =  $AD$ ; stave's length 30 within =  $AC$ .



First, when the principal only is to be diminished by the influx at compound interest.—Put  $x$  = time sought,  $a = 50$ ,  $r = 1.05$ , and  $n = 5$ ; then by the common theorem  $\frac{r^n - 1}{r - 1} \times n = a$ , or by

$$\log r. x = \log. \frac{ar - a + n}{n} \div \log. r = 8.31 \text{ years.}$$

Secondly, by considering the principal at compound interest to be constantly diminished by the influx of the yearly profit without interest.—Let  $r = 1.05$ ,  $n = 5$ ,  $a = 50$ , and  $x$  = the time required. Then the yearly decrement of the principal will be thus expressed :

$ra - n$  the first year's principal,

$r^2a - rn - n$  the second year's,

$r^3a - r^2n - rn - n$  the third year's,

$r^4a - r^3n - r^2n - rn - n$  the fourth year's,

$ar^x - nr^{x-1} - nr^{x-2} \dots - nr^{x-x}$  the  $x$  year's, which by the

nature of the question must be  $= 0$ ; hence  $r^n = \frac{n}{a + n - ar}$ ,

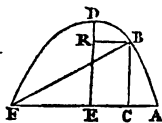
$$\text{and } x = \log. \frac{n}{a + n - ar} \div \log. r = 14.206 \text{ years.}$$

### III. QUESTION 971, by Mr. Wm. Armstrong, Carlisle.

There is a field in the form of a parabola, whose base is 10 chains, and its abscissa 15 chains; required the length of a line drawn from one end of the base, so that it may divide the field into two parts having the ratio of 3 to 2, the greater part lying next the base?

*Answered by the Rev. Mr. Ewbank, of Thornton-Steward.*

Put  $DE = a$ ,  $AE = b$ ,  $AC = x$ ; then  $EC = 2b - x$ ; and by Conics,  $db^2 : a :: (2b - x) \times x : a \times (2bx - xx) \div db^2 = bc$ ; this multiplied by  $x$ , and the fluent taken, gives  $ax^3 \div b - ax^3 \div 3b^2$  = the area of the space  $ABC$ ; also the area of the triangle  $BCF = \frac{1}{2}FC \times BC = ax(2b - x) \div 2b^2 = 2ax - 2ax^2 \div b + ax^3 \div 2b^2$ ; this added to the former, or space  $ABC$ , gives  $2ax - ax^2 \div b + ax^3 \div 6b^2 = \frac{1}{10}x^3 - 3x^2 + 30x = 60$  by the question, being the area of the space  $ABF$ . Hence  $x = 2.63$  chains; and then  $\sqrt{(FC^2 + BC^2)} = BF = 13.76$  chains, the length of the line required.



*The same answered by the Proposer, Mr. Wm. Armstrong.*

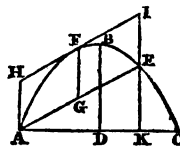
Let  $rs$  represent the required line. Draw  $sa$  parallel to  $AF$ , and let fall the perpendicular  $ac$ . Put  $a = AE = 5$ ,  $b = DE = 16$ ,  $x = ac$ , and  $y = bc$ . Then, by Conics, as  $b : a :: a : p = \frac{4}{5}$  the para-

meter; and as  $p : a + x :: a - x : y = (a^2 - x^2) \div p = (a^2 - x^2) \times b \div a^3$ . Now the area of the semiparabola ADE is  $= \frac{2}{3}ab$ ; and the area of the part RDB  $= \frac{2}{3}DR \times RB = (b - y) \times \frac{2}{3}x$ ; also the area of the rectangle ERBC is  $= xy$ ; therefore  $\frac{1}{3}xy + \frac{2}{3}bx =$  area of the space EDBC, which taken from ADE or  $\frac{2}{3}ab$ , leaves  $\frac{2}{3}ab - \frac{1}{3}xy - \frac{2}{3}bx =$  area of the part ABC, to this adding the triangle FBC  $= \frac{1}{2}FC \times BC = (a + x) \times \frac{1}{2}y = \frac{1}{2}ay + \frac{1}{2}xy$ , gives  $\frac{2}{3}ab + \frac{1}{2}ay + \frac{1}{2}xy - \frac{2}{3}bx =$  the space FBA, which is also, by the question,  $= \frac{2}{3}$  of FDA  $= \frac{2}{3} \times \frac{1}{2}ab = \frac{1}{3}ab$ , that is  $\frac{2}{3}ab + \frac{1}{2}ay + \frac{1}{2}xy - \frac{2}{3}bx = \frac{1}{3}ab$ , or  $\frac{1}{2}xy - \frac{2}{3}bx = \frac{1}{3}ab - \frac{2}{3}bx$ ; in this equation substitute the value of  $y$ , viz.  $(a^2 - x^2) \times b \div a^3$ , and it becomes  $(a^2 - x^2) \times b \div 2ab + (a^2 - x^2) \times bx \div 6a^2 - \frac{2}{3}ab = \frac{1}{3}ab$ , or by reduction  $x^3 + 3ax^2 + 3a^2x = 2\frac{1}{3}a^3$ , and by adding  $a^3$  to complete the cube, it is  $x^3 + 3ax^2 + 3a^2x + a^3 = 3\frac{1}{3}a^3$ , the cubic root of which gives  $x + a = a^{\frac{2}{3}} \sqrt[3]{3\frac{1}{3}}$ , and  $x = a^{\frac{2}{3}} \sqrt[3]{3\frac{1}{3}} - a = 2.36806$ . Hence  $FC = a + x = a^{\frac{2}{3}} \sqrt[3]{3\frac{1}{3}} = 7.36806$ , and  $BC = y = (a^2 - x^2) b \div a^2 = 11.6354$ ; then  $FB = \sqrt{(FC^2 + BC^2)} = 13.76$  chains, as required.

*The same by Mr. Cullen O'Connor. (Suppt.)*

**Lemma.** The area of any right segment ABC of a parabola, is as the cube of its base AC; and the area of an oblique segment AFE, is as the cube of AK, the segment of the right base cut off by EK parallel to the axis, or perpendicular to the base. For, any segment, as AFE, is two-thirds of its circumscribed parallelogram AHIE, which is equal to AH or FG  $\times$  AK; but the absciss FG is as the square of the base AE, which is as the square of AK; therefore the area is as  $AK^3 \times AK$  or  $AK^4$ . In like manner the area ABC is as  $AC^3$ .

**Solution.** Let ABC be the parabola; its base AC = 10, its axis BD = 15, and AE the required line dividing the area (100) in the ratio of 3 to 2, or to make AFE = 40, and AEC = 60. Draw EK perpendicular to AC; then, by the lemma, as ABC : AFE or as 5 : 2 ::  $AC^3$  or  $10^3$  :  $AK^3$  =  $\frac{2}{5}$  of  $10^3$ ; therefore  $AK = \sqrt[3]{400} = 7.368063$ ; hence  $KC = 10 - AK = 2.631937$ . Now the parameter being a 3d proportional to the absciss and ordinate, as BD : AD :: AD :  $\frac{25}{15} = \frac{5}{3}$  = the parameter; then by Hutton's Conic Section's, prop. 2, as the parameter  $\frac{5}{3}$  : AK :: KC : KE = 11.63537; hence, by right-angled triangles, AE =  $\sqrt{(AK^2 + KE^2)} = 13.7608$ , as required.



IV. QUESTION 972, by Mr. Olinthus Gregory, Yaxley.

There is an inflexible rod, void of gravity, 26 inches long, at one end of which is suspended 1cwt. 1qr.  $23\frac{1}{2}$ lb. of sugar, in a barrel that weighs 21lb.; at 1 foot distance from this end hangs a weight of  $8\frac{1}{2}$ lb. and 4 inches farther a weight of  $6\frac{1}{2}$ lb.; also at 7 inches from the other end a weight of 5lb., and at this other end a weight of  $4\frac{1}{2}$ lb. Query

the point of the rod, which being made a fulcrum, these weights, &c. will remain in equilibrio?

*Answered by Mr. J. Knight, Schoolmaster, at Esher, in Surry.*

Let  $x = AF$ , the small distance from the end A, where the cask is suspended to the fulcrum, weighing  $184\frac{1}{2}$  lb. which is to be balanced by all the other weights on the other side of it; and because the sums of the products are equal on each side in case of an equilibrium; therefore these products on one side must be equal to  $184\frac{1}{2}x$  the product on the other side. Whence  $209x = 418$ , and  $x = 2$  inches from A, as required.

$$\begin{array}{r} (12 - x) \times 8\frac{1}{2} = 102 - 8\frac{1}{2}x \\ (16 - x) \times 6\frac{1}{2} = 104 - 6\frac{1}{2}x \\ (19 - x) \times 5 = 95 - 5x \\ (26 - x) \times 4\frac{1}{2} = 117 - 4\frac{1}{2}x \\ \hline 418 - 24\frac{1}{2}x \end{array}$$

*The same by Mr. George Brown, of South Shields.*

1st wt. ....	$184\frac{1}{2} \times 26$ (dist. from the other end)	$= 4797$
2d .....	$8\frac{1}{2} \times 14$ (or $26 - 12$ )	$= 119$
3d .....	$6\frac{1}{2} \times 10$ (or $14 - 4$ )	$= 65$
4th .....	$5 \times 7$	$= 35$
5th .....	$4\frac{1}{2} \times 0$	$= 0$
	<u>209</u>	<u>5016</u>

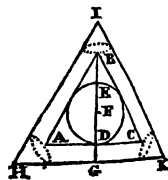
Then the sum of the products 5016, divided by the sum of the weights 209, gives 24 inches, the distance of the centre of gravity from the other end.

v. QUESTION 973, by Mr. R. Mountjoy, Chittlehampton.

If an equilateral triangle, whose area is given = 10000 square feet; be surrounded with a walk every where of the same breadth, and equal to the area of its inscribed circle; it is proposed to determine the breadth of the walk.

*Answered by Mr. Tho. Simpson Evans, Odiham.*

The side of an equilateral triangle being 1, it is evident that its perpendicular BD is  $\sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$ , its area =  $AD \times BD = \frac{1}{4}\sqrt{3} = .4330$ , the diameter DE of its inscribed circle =  $\frac{2}{3}BD = \frac{1}{3}\sqrt{3}$ , and the area of the inscribed circle =  $\frac{1}{3} \times .7854 = .2618$  = the area of its similar surrounding walk; the sum of these two areas is .6948, which is the area of a like equilateral triangle surrounding the former. Then, linear dimensions being as the roots of the areas, it will be  $\sqrt{.4330} : \sqrt{.6948} :: r$



$= \frac{1}{3} \sqrt{3} = .2887 : .3657 = FG$ , the radius of a circle inscribed in the outer triangle; and the difference .0770, is the breadth of the similar walk; and so as  $\sqrt{.4330} : \sqrt{10000}$  or  $100 :: .0770 : 11.701$ , the breadth of the walk sought, when it is considered as the space between the sides of two complete equilateral triangles.

But if the proposer, by the words, "every where of the same breadth," means that the corners of the outer triangle should be rounded off by circular arcs, as in the figure marked with dotted arcs, then the walk must be a little wider, to make up the same space as the area of the inscribed circle. Now, in this case, it is evident that the walk is made up of three parallelograms and three small circular sectors, of the common breadth of the walk, and the length of the parallelograms equal to the side of the given triangle. It is farther evident that each of the circular sectors is one third of a circle, and therefore the three together just equal to a circle whose radius is the breadth of the walk. Hence, if  $x$  denote the breadth of the walk, and  $p = .7854$ , then  $4px^2$  is = that circle, or the three corner parts; also if  $a = \sqrt{(10000 \div .433)} = 151.9671 = AB$  or  $AC$  the side of the given triangle, then  $3ax =$  the three parallelograms; consequently  $4px^2 + 3ax =$  the whole area of the walk in this case. But as  $1 : \frac{1}{3}\sqrt{3} :: a : \frac{1}{3}a\sqrt{3} =$  the diameter of the inscribed circle, and therefore the area of that circle is  $\frac{1}{3}pa^2$ , which is also equal to the area of the walk, by the question; hence then  $4px^2 + 3ax = \frac{1}{3}pa^2$ , a quadratic equation which gives  $x = 12.2307$ , the breadth of the walk in this latter case.

VI. QUESTION 974, by Mr. James Cunliffe, *Westhoughton*.

To find three whole numbers such, that the sum of every two of them may be a square number, and also that their squares may be in arithmetical progression.

*Answered by Amicus.*

Let  $a^2 + 2a^2b^2 - b^4$ ,  $a^4 + b^4$ , and  $b^4 + 2a^2b^2 - a^4$  be the three numbers; then the squares are necessarily in arithmetical progression, their common difference being  $4a^2b^2 \times (a^4 - b^4)$ : hence the three sums will be  $a^4 \times (2a^2 + 2b^2)$ ,  $b^4 \times (2a^2 + 2b^2)$ , and  $4a^2b^2$ .

The last is evidently a square, and the two first will be squares when  $2a^2 + 2b^2$  is a square, which, as is shewn by many writers on this subject, will be when  $a = 2mn + m^2 - n^2$  and  $b = 2mn - m^2 + n^2$ ; these substituted for  $a$  and  $b$  in the above values of the three numbers, will give general theorems for infinite answers,  $m$  and  $n$  being any whole numbers at pleasure, provided  $2a^2b^2$  be greater than  $a^4 - b^4$ , or  $5n$  exceed  $4m$ .

Ex. Gr. let  $n=6$ , and  $m=5$ ; then  $a=71$ ,  $b=49$ , and 43853762, 31176482, and 4560002 are the three numbers, their three sums being respectively equal to the squares whose roots are 8662, 6958, and 5978.

VII. QUESTION 975, by Mr. Wm. Robinson, *Alnwick*.

Required to find the least quantity of sheet lead, of  $\frac{1}{2}$  of an inch thick, to make a cistern to contain 85 gallons of ale measure, where the length, breadth, and thickness, are in arithmetical proportion?

*Answered by Mr. John Ryley, of Leeds.*

If the length, breadth, and depth of the cistern be in arithmetical progression; put the content in inches  $= 23970 = a$ , the length  $= x + y$ , the breadth  $= x$ , and the depth  $= x - y$ ; then the content  $x^3 - xy^2 = a$ , and the internal surface  $5x^2 - xy - 2y^2$  to be a minimum. From the former  $y^2 = (x^3 - a) \div x$ , and  $y = \sqrt{((x^3 - a) \div x)}$ ; which being substituted for  $y^2$  and  $y$  in the expression for the minimum, it becomes  $3x^2 - \sqrt{(x^3 - ax)} + 2a \div x$ ; this put into fluxions and reduced, we obtain  $128x^3 - 232ax^2 + 111a^2x - 16a^3 = 0$ .

Now take  $v = x^4$ , and put the equation into numbers, so shall it be transformed to  $v^3 - 43445\frac{1}{4}v^2 + 498252030\frac{1}{4}v = 1721528096625$ . Hence  $v = 27806\cdot00997$ ,  $x = 30\cdot2956$ , and  $y = 11\cdot2525$ ; therefore the dimensions of the cistern are  $41\cdot5481$ ,  $30\cdot2956$ , and  $19\cdot0431$ ; and the content of the lead  $= 798\cdot09538$  cubic inches.

*The same otherwise answered by Amicus.*

Since the lead is so thin, its quantity will be a minimum, nearly, when the internal surface is a minimum. Let  $x + y$ ,  $x$ , and  $x - y$ , be the three dimensions of the cistern; then its content  $23970 = x^3 - xy^2$ , hence  $3a^2x - y^2x - 2xyy = 0$ , or  $\frac{y}{x} = \frac{3x^2 - y^2}{2xy}$ ; also its surface  $= 5x^2 - xy - 2y^2$  a minimum, hence  $\frac{y}{x} = \frac{10x - y}{4y + x} = \frac{3x^2 - y^2}{2xy}$ , and  $\frac{3x^3}{y^3} - \frac{8x^2}{y^2} + \frac{x}{y} = 4$ , or  $\frac{x}{y} = 2\cdot7235$ , which gives the cistern in specie; whence the dimensions are readily found.

VIII. QUESTION 976, by Mr. Ralph Burton, of *Salton*.

There are certain low grounds in my neighbourhood, which at present are drained by a sewer running two miles in length, and has 12 feet perpendicular descent; but, if consent can be obtained, it is proposed to make the drainage only one mile in length, with three feet perpendicular descent. Quere which of the two sewers will evacuate soonest, supposing their sections equal and similar?

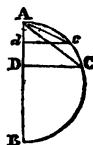
*Answered by Mr. John Wright, Westhoughton.*

The particles of the fluid (setting aside all consideration of friction, &c.) are impelled down the drains, in the same manner as a heavy body falling down an inclined plane. And it is well known that the times of falling down different inclined planes, are to each other, as their lengths directly, and the square roots of their heights inversely. Therefore the time of the water's running down the longer drain, will be to that of the shorter, as  $2 \div \sqrt{12} : 1 \div \sqrt{3} :: 1 : 1$ . Hence it appears that the water will run the length of the longer sewer, in the same time it does that of the shorter; but it is evident that the longer sewer will have double the effect of the shorter, in as much as the water runs through it with twice the velocity.

*The same answered by Mr. Wm. Pearson, North Shields.*

Let  $AC = 10560$  feet (or two miles) the length of the longer sewer, and  $AD = 12$ , its perpendicular fall; then by similar triangles  $AD : AC :: AC : AB = 9292800$ . Now, by the laws of falling bodies, a body will descend through the chord  $AC$  in the same time that another descends through the diameter  $AB$ . Hence  $16\frac{1}{2} : 1^2 :: 9292800 : 577790\cdot67$ , the square root of which is  $760''$ , or  $12^m 40^s$  the time of falling.

Again, let  $ac = 5280$  feet (or 1 mile) the length of the shortest sewer, and  $ad$  its descent  $= 3$ ; and by similar triangles as before,  $ad : ac :: ac : ab = 9292800$ , the same as before. From hence it is evident that each sewer will evacuate an equal quantity of water in the same time; but the shortest sewer will have the advantage, on account of repairing and cleaning, &c.



IX. QUESTION 977, by the Rev. Mr. L. Evans.

In turning out the lessons for the morning service, on a certain day last year, I observed, that the digits composing the chapter for the first lesson were inverted in that for the second; also the difference of the squares of the two chapters (the first being the greater number)  $= 1485$ , and the difference of their cubes  $= 66177$ . Hence the two lessons may be known, and the day of the month.

*Answered by the Rev. Mr. Ewbank, of Thornton-Steward.*

Let  $x$  and  $y$  represent the digits; then  $10x + y$  expresses the first chapter, and  $10y + x$  the second. But  $(10x + y)^2 - (10y + x)^2 = 99x^2 - 99y^2 = 1485$ , or  $x^2 - y^2 = 15$ . If  $y = 1$ , and the equation be reduced,  $x = 4$ , and the required chapters will be 41 and 14. As these numbers answer the other condition of the question, they are consequently the true ones. Hence the first lesson was Isaiah 41

chapter, and the second Acts the 14th chapter; the time being the 14th day of December.

*The same by the Rev. T. S. Rector of Ormside.*

Put  $x$  for the digit in the ten's place, and  $y$  for that in the unit's. Then will  $10x + y$  represent the first lesson, and  $10y + x$  the second. But, by the question,  $(10x + y)^2 - (10y + x)^2 = 1485$ , that is  $99x^2 - 99y^2 = 1485$ , or  $x^2 - y^2 = 15$ . From this equation, it is very evident, without farther investigation, that as  $x^2 - y^2$  must be 15, and  $x$  and  $y$  whole numbers, that the numbers can be no other than 4 and 1.

Theref.  $10x + y = 41$  the first lesson { appointed, in the morning service,  
and  $10y + x = 14$  the 2nd lesson } for the 14th day of December.

*The same answered by Amicus.*

Since  $1485 = x^2 - y^2$ , and  $66177 = x^3 - y^3 = (x - y) \times (x^2 + xy + y^2)$ , their greatest common measure must be  $x - y$ , and the greatest common measure of the numbers being 27,  $x - y = 27$ ,  $x + y = 55$ ,  $x = 41$ ,  $y = 14$ , and the day December the 14th.

X. QUESTION 978, by Mr. John Dalton, of Kendal.

There is a rain gauge (or vessel with a circular aperture set to receive the falling rain); by some accident the gauge has been turned aside a little, so as the plane of the aperture makes a given angle ( $5^\circ$ ) with that of the horizon, and the direction of the common section of the two planes is also given; now admitting that  $q =$  the quantity of water caught any day by the gauge in such a position, and that the direction of the wind (S. W.), and the angle made by the falling rain with the horizon ( $30^\circ$ ) are both given: It is required to determine, by a general theorem, the quantity of rain that would have been caught by the same gauge if truly horizontal?

*Answered by Mr. John Dalton, of Kendal.*

The quantity of rain that falls into any given vessel, must evidently be as the area of the orthographic projection of the vessel's aperture upon a plane at right angles to the falling rain; when the aperture is a circle, its projection is necessarily an ellipse, whether the vessel be on a level or not, whose transverse axis is always equal to the diameter of the circle, and whose conjugate is to its transverse as the cosine of inclination of the planes of the projection and of the aperture, to radius. The question then is, to determine the ratio of the cosines of two angles, the one made by the intersection of the plane of projection and plane of the aperture of the vessel, the other by that of the projection and horizon; for the quantity of water caught and the acquired quantity, will be respectively as their cosines.

In order to which it may be convenient to consider the three planes as great circles of a sphere, having a common centre, the intersection of the circumferences of which circles constitute a spheric triangle, the angles of which are those of the intersections of the planes. Let  $ABC$  be the triangle; the angle at  $A$  = the given inclination of the plane of projection to the horizon, the side  $AB$  = the given arc of the horizon, between the point of the wind and the common intersections of the planes of the gauge and horizon; the angle  $C$  = the required angle of intersection of the planes of the projection and gauge. Putting  $s$  and  $c$  for the sine and cosine of angle  $A$ ,  $d$  and  $b$  for the sine and cosine of angle  $B$ ,  $c$  for the cosine of the side  $AB$ ; then, by spherics,  $dsc - bc$  = the cosine of the angle at  $C$ , radius being 1. Whence  $dsc - bc : q :: b : bq \div (dsc - bc)$  = the required quantity of water; which is a general theorem for all cases. To apply this to the case in question, we have  $\angle A = 5^\circ$ , and  $s = .0871557$ ,  $c = .9961947$ ;  $\angle B = 60^\circ$ , and  $d = .8660254$ , and  $b = .5$ ; side  $AB = 45^\circ$ , and  $c = .7071068$ ; whence  $dsc - bc = .4447256$  = cosine of  $\angle C$ ; but the supplement must be used here, and consequently the sine must be changed; and  $bq \div (dsc - bc) = 1.12429 \times q$  as required.



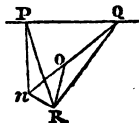
*Corol. 1.* If it be required to find in what direction the wind must blow, when the quantity caught is neither too little nor too great, we must put  $dsc - bc = b$ , in which case one of the cosines,  $b$  or  $c$ , must be negative; and the resulting equation is  $c = b(1 - c) \div ds$ .

*Corol. 2.* The redundancy or deficiency will be greatest when the direction of the wind is at right angles to the common section of the planes of the gauge and horizon; for, from the nature of the case,  $d$  is less than  $c$ , and  $s$  less than  $b$ ; wherefore  $ds$  and consequently  $dsc$ , is less than  $bc$ ; the difference then will be a maximum or minimum, when  $dsc = 0$ , that is, when  $c = 0$ , or  $AB = 90^\circ$ .

*Corol. 3.* The higher the wind, or the more obliquely to the horizon the wind falls, all other circumstances being the same, the greater is the error: and if the direction of the wind be as in the last corollary, and the rain strike the horizon at an angle equal to or less than the inclination of the plane of the gauge to the horizon, and on the side where the gauge is too high, there will be no rain at all caught, whatever quantity fall.

*The same answered by Amicus. (Suppl.)*

Suppose  $norq$  to be the horizontal plane,  $rpq$  the inclined one,  $ro$  their intersection,  $oq$  the given direction of the wind,  $or$  that of the falling rain,  $rq$  a line or section of the inclined gauge,  $nr$  perpendicular to  $qo$  produced; and  $nr$  to  $ro$ . Join  $rp$ , which is necessarily perpendicular to  $rq$ , then the angle  $nrp$  ( $5^\circ$ ) and  $pnr$  right, being given, the triangle  $pnr$  is given in specie, and the ratio of  $nr$  to  $nr$  is given; and because  $nqr$  the direction



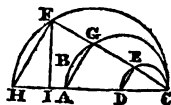
of the wind is given, and the angle  $nqr$  right, the ratio of  $nq$  to  $rq$  is given, and consequently the ratio of  $nq$  to  $rq$  is given, and the triangle  $nqr$  is given in specie; but  $nq$  is given in length, and consequently the triangle  $nqr$  is given; but the angle  $noq$  ( $30^\circ$ ) made by the falling rain is given, consequently  $on$  is given, and  $nq \mp on = oq$  is given, which is the line on the horizontal plane receiving the same quantity of water as  $rq$  on the inclined one; and as this holds for every line parallel to  $qr$  on the surface of the gauge, and their number the same whether it be inclined or horizontal, we have as  $oq : q :: rq : \text{the quantity required, let the form of the gauge be what it will.}$

XI. QUESTION 979, by Mr. A. Buchanan, of Sedgefield.

On the radius of a given semicircle  $ABC$  there is described the semicircle  $DEC$ , and from the point  $c$  there is drawn any line  $ceg$  to cut the semicircles in  $e$  and  $g$ . Now if  $ceg$  be produced to  $r$ , so that  $rc$  may be always equal to  $eg$ ; required the equation and area of the curve which is the locus of  $r$ .

*Answered by Mr. John Cavill.*

It is well known that if any line  $ceg$  be drawn from the point of contact  $c$ , cutting the given circles in  $e$  and  $g$ ; the parts  $ce$ ,  $eg$ , intercepted by the peripheries, will always be in the constant ratio of  $cd$  to  $da$ . For the same reason, if  $cg$  be produced to  $r$  as per question, the locus of the point  $r$  will also be a circle whose diameter is triple of  $cd$ , and consequently the area is nine times the area of  $ced$ . See Theor. 1. Cor. 4 and 6, in the solution to question 12, art. 35, of Hutton's Mathematical Miscellany, where Captain Williams has elegantly demonstrated several of the like nature.



*The same by Mr. Geo. Baron, South-Shields.*

Because  $cd = da = ah$ , and  $ce = eg = gf$ , by the question, therefore the lines  $de$ ,  $ag$ ,  $hf$  (being drawn) are parallel. And as the angles  $e$  and  $g$ , being in semicircles, are right angles, therefore  $r$  is also a right angle, and consequently is in a semicircle, that is, the locus  $cfh$  is a semicircle.

Or thus: Draw  $fi$  perpendicular to  $ch$ ; and put  $ch = 3cd = d$ ,  $ci = x$ , and  $if = y$ ; then  $hi = d - x$ , and by similar triangles,  $ci : if :: if : ih$ , that is  $x : y :: y : d - x$ , or  $dx - x^2 = y^2$ , the equation of the circle, whose area is well known.

Mr. W. Armstrong refers to Simpson's Apollonius Loc. Plan. where it is shewn the circles and the point  $c$  may have various other positions.

XII. QUESTION 980, by Mr. John Liddell, *Hovingham*.

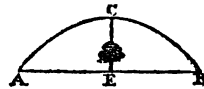
Adjoining the canals at Hovingham, lat.  $54^{\circ} 10'$ , a spruce fir grows perpendicular to the plane of the horizon, and on the 12th of May 1793, the transverse axis of the curve described by the shadow of its top, on the surface of the water, was 245 feet. Quere the height of the tree, and the area of the hyperbolic space over which its shadow passed, during the sun's apparent diurnal revolution, from the east and western points of the prime vertical.

*Answered by Amicus.*

The sun's declination on the given day was  $18^{\circ} 19'$  north, the meridian altitude in lat.  $54^{\circ} 10'$  was  $54^{\circ} 9'$  its complement  $35^{\circ} 51'$ , whose tangent is  $\cdot 7225502$ ; and when the declination was  $18^{\circ} 19'$  south the complement of the meridian altitude was  $72^{\circ} 29'$ , whose tangent is  $3\cdot 1683808$ ; and as  $3\cdot 1683808 - \cdot 7225502 = 2\cdot 4458306$  : transverse axis = 245 :: 1 :  $100\cdot 170466$  = the tree's height; also as 1 to tree's height :: tangent  $35^{\circ} 51'$  : its distance from the vertex of the hyperbola = mid-day shadow =  $72\cdot 6738756$  = the abscissa. Moreover, the amplitude =  $32^{\circ} 28'$ , and cotangent amplitude  $\times$  transverse = the conjugate axis =  $385\cdot 0671$ ; whence the area to the abscissa  $72\cdot 6738756$  becomes known.

*The same answered by Mr. T. Bulmer, Sunderland.*

Put  $c$  = cosine of the lat.  $d$  = sine and  $p$  = cosine of the declination, and  $x$  the height of the tree. Then, by Simpson's Dissertations, pa. 156, we have  $2pdx \div (c^2 - d^2) = 245$  the given transverse, whence  $x = 245 (c^2 - d^2) \div 2pd = 100\cdot 18$  feet, the height of the tree. Now the altitude of the sun's upper limb when on the meridian (allowing for refraction and parallax) is  $54^{\circ} 24' 24''$ , and the altitude when due east (allowing as before) is  $23^{\circ} 2' 21''$ . Let  $A$  and  $B$  be the east and west points of the horizon,  $E$  where the tree stands; then the ordinate  $AE$  or  $EB$ , the length of the shadow, when east or west, will be  $235\cdot 5472$ , and the absciss  $CE = 71\cdot 69989$ , or length of the shadow at twelve o'clock; and the conjugate diameter to the curve is  $382\cdot 966545$ ; and hence the area  $21462\cdot 3239$ .



N. B. The foregoing calculation is made by allowing for semidiameter, refraction, and parallax.

## XIII. QUESTION 981, by the Rev. Mr. John Hellins.

It is proposed to find the correct value of  $z$  from the equation  $\dot{z} =$

$\frac{x}{1+x^3}$ , where  $x$  and  $z$  begin together, in series which shall converge when  $x$  is greater than 1; and to perform the whole by means of series.

*Answered by the Rev. Mr. John Hellins.*

The value of  $z$  may be brought out in this series

$$\frac{x}{1+x^3} + \frac{3x^4}{4 \cdot (1+x^3)^2} + \frac{3 \cdot 6x^7}{4 \cdot 7 \cdot (1+x^3)^3} + \frac{3 \cdot 6 \cdot 9x^{10}}{4 \cdot 7 \cdot 10 \cdot (1+x^3)^4},$$

&c. which it is evident needs no correction, and always converges; and therefore may be considered as an answer to the question. But yet when,  $x$  is much greater than 1, its convergency will be very slow; for instance, if  $x$  be only = 10, the quantity whose powers are to be used in this series will be  $1000 \div 1001$ , which will converge too slowly to be useful; we must therefore look for some other form of series in this case. Now it will appear, by operations similar to those in my Essays, page 105 and 106, that the correct value of  $z$  is =

$$1 \cdot 2092 - \frac{1}{2x^2} + \frac{1}{5x^3} - \frac{1}{8x^4} + \frac{1}{11x^5} \text{ \&c. a few terms of which series,}$$

where  $x = 10$ , will give the result more exact than 10000 terms of that above exhibited.

But still, when  $x^3$  is =  $\frac{1}{2}(1 + \sqrt{5})$ , or  $13 \div 8$  nearly, neither of these series will converge very swiftly; but since, when  $x$  is greater than 1, the correct value of  $z$  is =  $1 \cdot 2092 - x \times \frac{1}{4} \times (x^3 + 1)^{-1} + \frac{3}{2 \cdot 5} \times (x^3 + 1)^{-2} + \frac{3 \cdot 6}{2 \cdot 5 \cdot 8} \times (x^3 + 1)^{-3} + \frac{3 \cdot 6 \cdot 9}{2 \cdot 5 \cdot 8 \cdot 11} \times (x^3 + 1)^{-4}$  &c. we shall, even in this case, obtain a proper degree of convergency; for  $x^3$  being =  $13 \div 8$  nearly,  $(x^3 + 1)^{-1}$  will be =  $(8 \div 21)$ , nearly, which differs but little from  $(8 \div 13)^{-2}$ ; so that, with this value of  $x^3$ , the last series converges about twice as fast as either of the former ones.

*Remark 1.* We have three forms of series for computing the value of  $z$ , the first of which will always converge when  $x$  is affirmative; the second, on account of the simplicity of its co-efficients, is the most convenient for computing the value of  $z$  when  $x$  is much greater than 1; and the third is the best form for computing the value of  $z$  when  $x$  is but a little greater than 1.

*Remark 2.* With respect to the constant quantity in the second and third forms, by which the fluent is corrected, it is obtained by taking the difference of  $z$  given by the first and third series when  $x=1$ . It is also equal the sum of the two series  $1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{16}$ , &c. and  $\frac{1}{2} - \frac{1}{2} + \frac{1}{8} - \frac{1}{16}$ , &c. which may be easily computed by the method explained in Dr. Hutton's Tracts, tract 2, pa. 11.

*The same answered by Amicus.*

The given expression, by substituting  $1 + u = \frac{1 + 2x + x^2}{1 - x + x^2}$ , and  $y = \sqrt{\frac{1}{4} \times \frac{x}{\sqrt{(1 - x + x^2)}}}$ , becomes  $z = \frac{1}{2} \times \frac{u}{1 + u} + \frac{1}{\sqrt{3}} \times \frac{y}{\sqrt{(1 - y^2)}}$ , where, both the members being such well known forms, the application of series seems unnecessary; but if it must needs be done by them, the fluent of the latter member, since  $y$  is less than unity, will converge, and may be made to do so very swiftly, by applying art. 350 of Simpson's Fluxions. In the former member, if  $x$  be not greater than  $2 + \sqrt{3}$ ,  $u$  will be greater than unity, in which case, let  $1 + u = c + w$ , so that  $w$  may not be greater than unity, and the correct fluent of  $\frac{w}{c + w}$  in series will be  $\frac{w}{c} - \frac{w^2}{2c^2} + \frac{w^3}{3c^3} - \frac{w^4}{4c^4}$ , &c.  $-\frac{d}{c} + \frac{d^2}{2c^2} - \frac{d^3}{3c^3} + \frac{d^4}{4c^4}$ , &c. where  $d$  is the value of  $w$  when  $x = 0$  and  $u = 0$ ; thus, if  $w = 1$  and  $c = 2$ , the correct fluent of  $\frac{1}{2} \times \frac{w}{c + w}$  is  $\frac{1}{2} \times (\frac{1}{2} + \frac{1}{3 \cdot 8} + \frac{1}{5 \cdot 32} + \frac{1}{7 \cdot 128} + \&c.)$ , and  $z = 1.0900009$  nearly.

*The same answered by Mr. Colin Campbell, Kendal. (Suppl.)*

First, when  $x$  does not exceed 1,  $z = \dot{z} - x^2 \ddot{z} + x^4 \dddot{z}$  &c. and  $z = x - \frac{1}{4}x^4 + \frac{1}{10}x^7 - \frac{1}{10}x^{10}$  &c. which is correct; therefore when  $x = 1$ ,  $z = 1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10}$  &c.  $= \frac{3}{1.4} + \frac{3}{7.10} + \frac{3}{13.16}$  &c.

Again, when  $x$  is not less than 1, then

$$z = \frac{\dot{z}}{x^3} - \frac{\ddot{z}}{x^2} + \frac{\ddot{z}}{x^5} \&c. \text{ and } z = d - \frac{1}{2x^2} + \frac{1}{5x^3} - \frac{1}{8x^4} \&c. \text{ and}$$

when  $x = 1$ ,

$$z = d - \frac{1}{2} + \frac{1}{5} - \frac{1}{8} + \frac{1}{11} \&c. = d - \frac{3}{2.5} - \frac{3}{8.11} - \frac{3}{14.17}$$

&c. Now the two values of  $z$  in this case being equal, we have  $d - \frac{3}{2.5}$

$$- \frac{3}{8.11} - \frac{3}{14.17} \&c. = \frac{3}{1.4} + \frac{3}{7.10} + \frac{3}{13.16} \&c. \text{ hence}$$

$$d = 3 \times : \frac{1}{1.4} + \frac{1}{7.11} + \frac{1}{13.17} \&c. + 3 \times : \frac{1}{2.5} + \frac{1}{8.11}$$

$$+ \frac{1}{14.17} \&c. = 1.21104, \text{ therefore } z = 1.21104 - \frac{1}{2x^2} + \frac{1}{5x^4} - \frac{1}{8x^6} \&c.$$

*Cor.* When  $x$  is infinite,  $z = d$ .

*Schol.* The sum of the two compound series

$\frac{1}{2.5} + \frac{1}{8.11} + \frac{1}{14.17} \&c.$  and  $\frac{1}{1.4} + \frac{1}{7.11} + \frac{1}{13.17} \&c.$  may be found by Increments. But, they will be much easier and quicker found, by finding the sums of their several constituting simple alternate series  $1 - \frac{1}{4} + \frac{1}{4} \&c.$  by Dr. Hutton's method in the second of his Tracts.

#### XIV. QUESTION 982, by Amicus.

To find, without series, the value of  $x$  in the equation  $\ddot{x} - xz^2 = \frac{2xz}{t}$ ; where  $z$  is the circular arc whose tangent is  $t$ , and radius unity.

*Answered by Amicus, the Proposer.*

*Answer,*  $x = 2 \cosine z$ . For then  $\dot{x} = 2 \cosine \dot{z}$ ,  $\ddot{x} = 2 \cosine \ddot{z}$ ,  $\ddot{z} = \cosine \dot{z} \div \sin z - \cosine \dot{z} = \dot{z} \times \sin z$ ,  $\dot{x} = -2\dot{z} \times \sin z$ ,  $z, -\cosine \ddot{z} = \sin \dot{z} \times \dot{z} = \dot{z} \times \cosine z$ ,  $\ddot{x} = -\dot{z}^2 \times 2 \cosine z$ ,  $-xz^2 = -\dot{z}^2 \times 2 \cosine z$ , and  $\frac{2xz}{t} = -2\dot{z}^2 \times \frac{2 \sin z}{t} = -4\dot{z}^2 \times \cosine z$ . Therefore, &c.

*The same answered by Capt. Mudge, of the Artillery.*

The given equation being  $\ddot{x} - xz^2 = \frac{2xz}{t}$ ; if there be taken  $y = \cos. z$ , then  $t = \frac{-\dot{y}}{y^2}$ , hence  $\ddot{x} - xz^2 = \frac{-2\dot{y}\dot{z}^2}{y}$ ; but  $\dot{z} = \frac{-\dot{y}}{\sqrt{(1-y^2)}}$  and  $-\dot{y} = \dot{z} \sqrt{(1-y^2)}$ ; making  $\dot{z}$  constant, this gives  $\ddot{y} = \frac{y\dot{y}\dot{z}}{\sqrt{(1-y^2)}}$   $= -y\dot{z}^2$ ; and substituting for  $\dot{z}^2$  its value  $\frac{-\dot{y}}{y}$ , we get  $\ddot{x} + \frac{x\dot{y}}{y} = \frac{2x\dot{y}}{y}$ , which gives  $x = 2y$ .

## PRIZE QUESTION, by James Glenie, Esq.

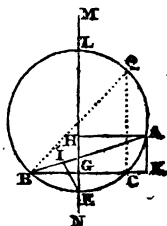
In the palace of one of the Persian kings it is said there was a triangular area, such, that the cubes on two of the sides were together equal to thrice the cube on the third side, which was 200 feet in length; and that the area itself contains exactly 10000 superficial feet. Supposing this to have been really the case, it is required to construct the triangle by common or plane geometry?

N. B.—Whoever solves this problem will find himself sufficiently rewarded for his trouble, by the discovery of a number of new propositions which the solution will naturally suggest to him.

*Answered by James Glenie, Esq.*

Let the given base  $ac$  be bisected in  $g$  by the perpendicular  $nm$ ; upon which take  $gx = \frac{1}{2}ac$ ,  $gx = gc$ ,  $HL = 2GN = bc$ ; upon the diameter  $LE$  describe a circle cutting  $HA$  parallel to  $ac$  in  $A$ , the vertex of the required triangle  $ABC$ .

For it is evident that  $EL = \frac{1}{2}ac$ ,  $gx = \frac{1}{2}bc$ ,  $HE = \frac{3}{2}bc$ , and  $HA$  or  $gx = bc \sqrt{\frac{1}{2}}$  (by 13, Euc. 6). Also  $BK = bc \times (\frac{1}{2} + \sqrt{\frac{1}{2}})$ , and  $CK = bc \times (\sqrt{\frac{1}{2}} - \frac{1}{2})$ . Therefore  $BA^2 = bc^2 \times (\frac{1}{2} + \sqrt{\frac{1}{2}})$  (47 Euc. 1.) and  $BC = \frac{1}{2}bc \times (2 + \sqrt{\frac{1}{2}})$ . In like manner,  $AC = \frac{1}{2}bc \times (2 - \sqrt{\frac{1}{2}})$ . Whence the whole is manifest, for  $(2 + \sqrt{\frac{1}{2}})^2 + (2 - \sqrt{\frac{1}{2}})^2 \div 8 = 3$ .



Cor. 1.  $gx = BA - AC$ .

Cor. 2.  $BA + AC = 2bc$ .

Cor. 3. If  $cq$  be perpendicular to  $bc$ , the cubes on  $bq$  and  $cq$  are together equal to seven times the cube on  $bc$ . Also  $bq + cq = 3bc$ , and  $bq - cq = \frac{1}{2}bc$ .

*The same answered by Mr. W. Armstrong, Newcastle-upon-Tyne.*

Draw  $bc =$  the given base, which bisect in  $g$ ; make  $gx = bc \sqrt{\frac{1}{2}}$ , and at  $x$  erect the perpendicular  $KA = Bg$ ; join  $AB$ ,  $AC$ , and  $ABC$  will be the required triangle.

For (by 47 Euc. 1)  $AB^2 = BK^2 + AK^2 = BG^2 + 2BG \times GK + GK^2 + BG^2 = bc^2 \times (\frac{1}{2} + \sqrt{\frac{1}{2}})$ , or  $AB = bc \sqrt{(\frac{1}{2} + \sqrt{\frac{1}{2}})} = bc \times (1 + \sqrt{\frac{1}{2}})$ . Also,  $AC^2 = CK^2 + KA^2 = BG^2 - 2BG \times GK + GK^2 + BG^2 = bc^2 \times (\frac{1}{2} - \sqrt{\frac{1}{2}})$ , or  $AC = bc \times (1 - \sqrt{\frac{1}{2}})$ . Now it is obvious that if  $AB$  and  $AC$  be each cubed and added together, all the terms except the first and third in each will destroy each other, viz.  $AB^3 + AC^3 = bc^3 \times (2 + 1) = 3bc^3$ .

*The same, by Mr. Tho. Thompson, of Norley.*

Let  $bc$  be the given base  $= 200$ ; take a line  $x$  a mean proportion-

al between  $BC$  and  $\frac{1}{2}AC$ ; on  $BC$  describe a triangle such, that the side  $AB = BC + M$ , and  $AC = BC - M$ , and the thing is done.

For let fall the perpendicular. It is well known that  $AB^2 + AC^2 = 2BC^2 + 6M^2 \times BC$ ; and by construction,  $6M^2 = BC^2$ , therefore  $AB^2 + AC^2 = 3BC^2$ , one of the conditions. Again,  $BC : AB + AC (2BC) :: AB - AC (2M) : 4M$ , hence  $BK = \frac{1}{2}BC + 2M$ ; and (by 47 Euc. 1.)  $AK^2 = AB^2 - BK^2 = (BC + M)^2 - (\frac{1}{2}BC + 2M)^2 = \frac{1}{4}BC^2 - 3M^2 = \frac{1}{4}BC^2$  or  $AK = \frac{1}{2}BC = 100$ , the given perpendicular.

*The same answered by Nondum, Esq.*

Questions of this kind may be made as follows: Let the base of a triangle be  $= b$ ; the sum of the other two sides  $= mb$ , and the sum of their cubes  $= nb^3$ : Then the difference of the sides will be  $= b\sqrt{((4n - m^3) \div 3m)}$ ; where  $(4n - m^3) \div 3m$  may be any positive number with the following limitations: Since, from the nature of the question,  $m$  must always be greater than unity, it follows that  $n$  must be greater than  $\frac{1}{4}$ , and because  $b^3 \times (\frac{1}{4}m^3 + \frac{1}{4}m^3)$  is the greatest limit of the sum of the cubes of the sides, it is evident that  $n$  must be less than  $\frac{1}{4}m^3 + \frac{1}{4}m^3$ .

The two least whole numbers for  $m$  and  $n$  (unity excepted) will answer the question: Let  $m = 2$ , and  $n = 3$ ; then the difference of the sides will be  $2b \div \sqrt{6}$ , and the sides themselves  $b + b \div \sqrt{6}$  and  $b - b \div \sqrt{6}$ , and the perpendicular  $= \frac{1}{2}b$  as given by the question. For by Trigonometry,  $b : 2b :: b \div \sqrt{6} : 4b \div \sqrt{6} =$  twice the distance of the perpendicular from the middle of the base: hence the greater segment  $= \frac{1}{2}b + 2b \div \sqrt{6}$ ; therefore  $(b + b \div \sqrt{6})^2 - (\frac{1}{2}b + 2b \div \sqrt{6})^2 = \frac{1}{4}b^2$  the square of the perpendicular. In this manner, when the values of  $m$  and  $n$  are chosen within the above limits, the perpendicular, and thence the area may be determined.

If  $m = 3$ , then the least whole number for  $n$  will be 7; and the three sides of the triangle will be  $b$  and  $1\frac{1}{2}b$ , and  $1\frac{1}{2}b$ , and the triangle is right angled.



*Questions proposed in 1795, and answered in 1796.*

I. QUESTION 984, by Miss Nancy Mason, of Clapham.

Dear ladies fair, I pray declare,

In Dia's page next year,

When first it was I 'gan to pass

My time upon this sphere.

My age so clear; the first o'th' year,

In years, in months, and days,

With ease you'll find, by what's subjoin'd\*,

Exact the same displays.

$*xy + z = 238$	} Where $x =$ the years, $y =$ the months, and $z =$ the days of my age, the first of January, 1795.
$zx + y = 158$	
$x + y + z = 39$	

*Answered by Mr. J. Knight, Schoolmaster, Esher, Surry.*

Given, First,  $xy + z = 238$ ; second,  $xz + y = 158$ ; third,  $x + y + z = 39$ .

By subtracting the third equation from each of the other two, and dividing the remainders by  $x-1$ , gives  $y = (99+x) \div (x-1)$ , and  $z = (119+x) \div (x-1)$ ; then the sum of these two taken from the third leaves  $x = 39 - (318+2x) \div (x-1)$ . Hence  $x = 19 \pm \sqrt{4} = 21$  or  $17$ ;  $y = 11$  or  $13\frac{1}{2}$ , and  $z = 7$  or  $8\frac{1}{2}$ . But as  $13\frac{1}{2}$  is too much for the months,  $x$  must be  $21$ ,  $y = 11$ , and  $z = 7$ . Therefore this ingenious lady's age the 1st of January, 1795, was 21 years, 11 months, 7 days.

*The same answered by the Rev. T. S. Rector of Ormside.*

The sum of the two first equations is  $xy + xz + y + z = (y+z) \times (x+1) = 396$ ; hence  $y+z = 396 \div (x+1) =$  (by the third)  $39 - x$ ; therefore  $39x - x^2 + 39 - x = 396$ , or  $x^2 - 38x = -357$ ; hence  $x = 21$ ; then  $y = 11$ , and  $z = 7$ ; so that the lady's age is 21 years, 11 months, 7 days.

• *The same answered by Mr. W. Virgo, of Thornbury.*

By the third equation,  $z = 39 - x - y$ ;  
This value of  $z$  substituted in the first, gives  $xy + 39 - x - y = 238$ .  
The same substituted in the second, gives  $39x - x^2 - xy + y = 158$ .  
Then the sum of these two gives  $x^2 + 38x = -357$ .  
Hence  $x = 21$ ,  $y = 11$ , and  $z = 7$ .

II. QUESTION 985, by Mr. J. Jackson, of Hutton-Rudby.

If a person borrow any sum of money at five per cent. per annum, compound interest; and lend it again at the same time at ten per cent. per annum, simple interest, in what time will the balance of interest be at the least?

*Answered by Mr. Charles Pretty, Thornham, Suffolk.*

The simple interest will at first have the advantage, but every year less; after some years their annual interests become equal; after which the compound interest, still increasing, gradually gains upon the simple interest, till at one time the whole amount of the compound interests become equal to the whole amount of the simple interests, which is evidently the time sought in the question. Therefore let the sum borrowed be unity or 1, and put  $x$  for the time sought. Then by the rules of interest, the whole compound interest is  $1.05^x - 1$ , and the whole of the simple interest is  $\frac{1}{10}x$  therefore  $1.05^x - 1 = \frac{1}{10}x$ , or  $1.05^x - \frac{1}{10}x = 1$ . Hence by a table of logarithms  $x$  is found be-

tween 26 and 27 years ; but by approximation. I find  $x$  nearly = 26.5838 years, or 26 years, 7 months, 13 hours.

*The same answered by Mr. Colin Campbell, of Kendal.*

Let  $s$  = the sum borrowed,  $r = 1.05$ ,  $n = \frac{1}{10} = .1$ , and  $x$  = the required time in years. Then will the compound interest of  $s$  for  $x$  years at 5 per cent. =  $s \times (r^x - 1)$ ; and the simple interest for the same at 10 per cent. =  $srx$ ; therefore  $s(r^x - 1) = srx$  by the question, or  $r^x - 1 = nx$  or  $r^x = nx + 1$ , and  $x \times \log. r = \log. (nx + 1)$ . Put  $nx + 1 = 1 \div (1 - y)$ , (the hyp. log. of which =  $y + \frac{1}{2}y' + \frac{1}{3}y^2 + \frac{1}{4}y^3 + \&c.$ ), and we get  $2 - \frac{2 \text{ hyp. log. } r}{n} = 1.024195 = y + \frac{y^2}{3} + \frac{y^3}{6} + \frac{y^4}{10} + \frac{y^5}{15} + \&c.$ ; hence, by reversion of series,  $y = .726768$ , and consequently  $x = y \div n(1 - y) = 26.6$  nearly.

*The same answered by Mr. William Davies, of Gluvias.*

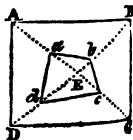
Let  $p$  = the principal,  $n = 1.05$ ,  $r = 0.1$ , and  $t$  = the time. Then  $pn^t$  = the amount in  $t$  years at compound interest, and  $p + prt$  = that at simple interest. Therefore, by the question,  $pn^t = p + prt$ , or  $n^t = 1 + rt$ ; or by logarithms,  $t \times \log. n = \log. (rt + 1)$ . By a few trials it appears that  $t$  is between 26 and 27 ; hence, by working a few suppositions by the rule of false position, we find  $t = 26.58$  years.

III. QUESTION 986, by Mr. Robert Wilkinson, North-Shields.

Within a rectangular garden, whose length is four chains, and breadth three chains ; there is a piece of water in form of a trapezium, whose opposite angles are in a direct line with those of the garden, and the respective distances of the angles of the one, from those of the other, are 20, 25, 40, and 45 yards, in a successive order, required the area of the water ?

*Answered by Mr. John Surtees, of Sunderland.*

Let ABCD represent the garden, and abcd the pond. Now by the question, AB = 4 chains = 88 yards, and AD = 3 chains = 66 yards, therefore the diagonal AC = BD =  $\sqrt{(AB^2 + AD^2)} = 110$  yards; also  $20 + 40 = 60 = AD + ab$ , and  $25 + 45 = 70 = AC + cc$ ; these being subtracted from the diagonals BD and AC, leave the diagonals of the pond ac and bd = 40 and 50. Then say, as AC : radius 1 :: DC : sine



$\angle DAC = \frac{88}{110} = \frac{44}{55}$ ; and as  $DE : \sin \angle DAE :: AD : \cdot 96 = \sin \angle DEA$ . Hence, by Hutton's Mensuration, rule 2, page 101,  $\frac{1}{2} \times 40 \times 50 \times \cdot 96 = 1000 \times \cdot 96 = 960$  yards, is the area of the pond.

*The same answered by Mr. A. Rouillé, of Deptford.*

Let ABCD be the given rectangle, AB = 4 chains, BC = 3 chains, aa = 25 yards, bb = 40, cc = 45, dd = 20 yards. In the triangle ABC (Euclid 1, 47,)  $AC = \sqrt{(AB^2 + BC^2)} = \sqrt{(16 + 9)} = \sqrt{25} = 5$  chains = 110 yards; also  $40 + 20 = 60$ , and  $110 - 60 = 50 = bd$  = one of the diagonals of the trapezium; also  $45 + 25 = 70$ , and  $110 - 70 = 40 = ac$  the other diagonal. Then to find the angle ACD, we have AC (5) : radius :: AD (3) :  $\sin \angle ACD = 36^\circ 52\frac{1}{4}'$ ; its double  $73^\circ 45\frac{1}{2}' = \angle AED$ . Hence, by Dr. Hutton's Mensuration, as above,  $50 \times 40 \times \frac{1}{2} \times \sin \angle AED = 1000 \times \cdot 96 = 960$  yards, the area of the water sought.

*The same answered by Amicus.*

Since the diagonals of the parallelogram and trapezium make the same angles of intersection, they must be to each other in the ratio of the rectangles under their diagonals; and the diagonals of the pond being, (found as before) 40 and 50 yards; therefore  $5 \times 5 : 3 \times 4$  its area, or  $25 : 12 :: 40 \times 50 = 2000 : 960$  square yards, the area of the water.

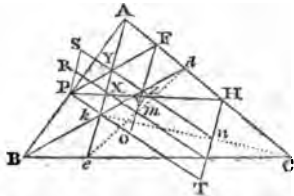
#### IV. QUESTION 987, by Mr. J. T. Connor, *Lewes Academy.*

In a triangle ABC, drawing any two lines  $ae$ ,  $bd$ , from the extremities of one side, to terminate in the other two sides; and thereby form a trapezium  $cdke$ ; if the diagonals  $de$  and  $ck$  of that trapezium be bisected in the points  $m$  and  $n$ , and the first side of the triangle in the point  $p$ , these three points  $p$ ,  $m$ ,  $n$ , will be in a straight line. Required the demonstration?

*Answered by Major Henry Haldane, of the Royal Engineers.*

At the points  $n$  and  $m$ , and through the point  $p$ , construct the two parallelograms  $nspt$ ,  $mrpo$ , having their sides parallel to  $ac$  and  $ae$ . I say these parallelograms are similar, and, having the common angle  $p$ , are about the same diameter  $nmp$ , which must therefore be a straight line.

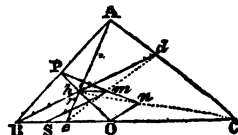
For produce  $om$  and  $tn$  until they intersect  $ac$  in  $r$  and  $h$ , and join  $rf$  and  $rh$ . Now (by 2 Euclid 6), because  $kc$  is bisected in  $n$ ,  $ac$  is bisected, by  $tn$  produced, in  $h$ , and  $ak$ , by  $ns$ ,



in  $\gamma$ . Because  $de$  is bisected in  $m$ ,  $Ad$  is bisected by  $om$  produced in  $r$ , and  $Ac$  by  $mn$  in  $x$ . And because  $As$  is bisected in  $r$ ,  $PF$  and  $PH$  are parallel to  $ad$  and  $bc$  respectively, and also bisect  $Ak$  and  $Ae$ ; therefore  $PF$  must pass through the point  $\gamma$ , and  $PH$  through the point  $x$ . Then by the parallelism of the sides, the triangle  $PXY$  is equiangular and consequently similar to the triangle  $PVF$ . For the same reason the triangle  $XYZ$  is similar to the triangle  $VFH$ . Whence it follows, that  $PK : PV :: XY : VF$ , and  $XY : VF :: XZ : VH$ ; therefore (by 11 Euc. 5)  $PX : PV :: XZ : VH$ , and (by 16 Euc. 5)  $PX : XZ :: PV : VH$ , that is (by 2 Euc. 6, and 11 Euc. 5)  $PR : RS :: PO : OT$ , and (by 18 Euc. 5)  $PR : RS :: PO : PT$ ; whence (by Def. 1, Euc. 6) the parallelograms  $PRPT$  and  $MRPO$  (being equiangular by construction) are similar; and, having the common angle  $P$ , are (by 26 Euc. 6) about the same diameter  $mnp$ , which therefore must be a straight line. Q. E. D.

*The same answered by Mr. John Ryley, of Leeds.*

Let  $ABC$  be the given triangle,  $Ac$  and  $Ad$  lines drawn from the angular points  $A$  and  $B$ , meeting the opposite sides in  $e$  and  $d$ ;  $cekd$  the trapezium, whose diagonals  $ck$ ,  $ed$  are bisected in the points  $m$  and  $n$ ; draw  $no$  and  $ms$  parallel to  $bd$ , also to  $P$  the middle of  $AB$  draw  $or$ ; join the points  $pm$ ,  $mn$ , and  $rm$ . Now, because  $cn = nk$ ,  $co = ob$ , and  $BP = PA$ ;  $OP$  is parallel to  $CA$ , and  $bd$  is bisected in the point  $r$ , and  $ed$  in  $m$ , consequently  $rm$  is parallel to  $nc$ ; then by similar triangles  $cn : ck :: co : cb :: no : kb :: BP : BA :: BR : bd :: dm : de$ ; hence it appears that  $pmn$  is one continued straight line.



V. QUESTION 988, by Mr. T. Woolston, *Adderbury Academy*.

An honest old malster had a bushel much above the standard, but as he loved good ale, he was unwilling to hurt the cause by short measure; however, there having long been a clamour about buying and selling by the legal Winchester bushel, he, with some reluctance, ordered a workman to shave down his old bushel. Having reduced the depth to precisely eight inches, he went about proving it, by the weight of water it should hold; but having forgot what the weight ought to be, he left it standing brimful, till the exciseman came, who when he went to gauge it, the sun shining upon it, he observed that the shadow of the upper edge, nearest the sun, cut the water exactly in the diagonal of the bushel; he therefore instantly pulled out his watch, and observed it was just  $19\frac{1}{4}$  minutes past five o'clock in the morning, being the 2d of September, 1793. Being a curious and clever fellow, and knowing the latitude of the place to be  $52^{\circ} 6'$  north, he

told the malster he required no other dimensions. Now supposing his watch to shew true time, the bushel exactly round, and every where of an equal depth, required the true content; allowing for the sun's semidiameter, refraction and parallax?

*Answered.*

The ingenious proposer of this question informs us, that he made a mistake in copying out this question, which according to the data as printed will give the diameter of the bushel not quite the half of what it should be; but that if, instead of the words, "*the shadow of the upper edge nearest the sun cut the water exactly in the diagonal of the bushel*", be substituted the words, *the shadow, &c. fell exactly upon the centre of the bottom of the bushel*, the diameter will then come out near the true quantity.

*The Principles of the Solution to this question are thus given by Amicus.*

The time and latitude being given, the sun's altitude is given; and the complement of altitude is the angle of incidence of the sun's rays upon the water;  $\frac{3}{4}$  of its sine is the sine of refraction out of air into water, which is also given; and this angle of refraction, being that which the diagonal makes with the sides of the vessel, is given, and the depth given, the vessel itself is given also.

*The same answered by the Rev. Mr. L. Evans, of Little Bedwin.*

In this question we have the latitude of the place =  $52^{\circ} 6'$ , the apparent time of observation = Sept. 1st. 17h. 19m. 59sec. and the sun's declination at this time =  $7^{\circ} 47' 9''$  north; to find the

Sun's zenith distance =	$90^{\circ} 4' 9''.8$	
Its supplement is . . . .	89 55 50.2	
Refraction . . . . . +	33 0	} According to Dr. Maskelyne.
Sun's semidiameter . +	15 55	
Sun's parallax . . . —	9	

---

90 44 36.2

Subtract . . . . . 90 0 0 leaves

Ap. alt.  $\odot$ 's upper limb 0 44 36.2

Its complement is . . . . 89 15 23.8, the natural sine of it  $\cdot 9999158$ , which is the sine of incidence, and the sine of refraction, in this case, will be  $\frac{3}{4}$  of it =  $\cdot 7499368$ , answering to  $48^{\circ} 35' 5''.6$ , the angle of refraction. Therefore as radius : 8 the bushel's depth :: tang  $48^{\circ} 35' 5''.6$  : 9.06 its diameter, which is not quite half the true diameter, and of course the vessel not quite the quarter of a bushel. It therefore appears the ingenious proposer committed an error somewhere in the formation of this pretty question.

*The same answered by Mr. Wm. Robinson, London.*

First, in an oblique-angled spherical triangle, there are given two sides and an included angle, viz. the co-declination =  $82^{\circ} 19'$ , the co-latitude =  $37^{\circ} 54'$ , and the time from noon = 6hrs.  $40\frac{3}{4}$ min. =  $100^{\circ} 6'$ ; to find the co-latitude of the sun's centre =  $90^{\circ} 6'$ ; consequently his centre was  $6'$  below the horizon when the observation was made. But the ray that struck the edge of the bushel must have passed from the sun's upper limb, and therefore after correction for semidiameter, &c. we get the true altitude of the sun's upper limb =  $42^{\circ} 50''$ ; its complement  $89^{\circ} 17' 10''$  is the angle of incidence, its nat. sine =  $\cdot 9999224$ ; but when a ray of light passes out of air into water, the sine of the angle of incidence is to that of refraction as 4 to 3; hence  $4 : 3 :: \cdot 9999224 : \cdot 7499418$  = sine of  $48^{\circ} 35' 7''$  = the angle of refraction, or the angle made by the height and diagonal of the vessel; from whence, and the height = 8, the diameter comes out =  $9\cdot 069$  inches, and its solid content  $516\cdot 8$ .

VI. QUESTION 989, by Mr. Olinthus Gilbert Gregory, *Yazley*.

The sum of all the chords drawn in a semicircle, from one end of the diameter, to those points which divide the circumference into a certain number of equal parts (together with the diameter) is  $429\cdot 7877$ . Now, if the diameter be 100, what is the number of chords, and the length of each?

*Answered by Mr. Newton Bosworth, Peterborough.*

The following theorem from page 8, of the Introduction to Dr. Hutton's Mathematical Tables, from which Mr. Gregory seems to have formed this question, will enable us to give a solution to it. "As the least chord, in a semicircle, is to the diameter, so is the sum of the said least chord and diameter and greatest chord, to double the sum of all the chords, including the diameter as one of them." Put  $x$  = the least chord, then, by the nature of the semicircle and right-angled triangles  $\sqrt{(100^2 - x^2)}$  is the greatest chord; and by the theorem above, as  $x : 100 :: x + 100 + \sqrt{(100^2 - x^2)} : 429\cdot 7877 \times 2$ , whence this equation  $859\cdot 5754x = 100x + 1000 + 100 \sqrt{(100^2 - x^2)}$ ; the resolution of which quadratic equation gives the value of  $x = 25\cdot 8819$ , for the least chord; hence  $\sqrt{(100^2 - x^2)} = 96\cdot 5925$  is the greatest chord. Then, by trigonometry, the least chord is found to belong to  $15^{\circ}$ , one of the equal parts of the arch; consequently  $90 \div 15 = 6$  is the number of parts the arch is divided into, and the number of chords is 5, beside the diameter; and as two of them are already discovered, the other three which may be found in divers ways, come out 50, and  $70\cdot 7106$ , and  $86\cdot 6025$ .

*The same answered by Mr. Terry, Settle, Yorkshire.*

It is pretty evident that whatever the number of chords may be, the

mean cannot be greatly different from the chord of the quadrant or  $\sqrt{(50^2 + 50^2)} = 70.7106$ ; this dividing the given sum 429.7877, gives 6 nearly for the number of chords including the diameter, or the number of parts the semicircumference is divided into; therefore the chords are those of the arcs of  $16^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ$ , which are easily found to be respectively 27.8818, 50, 70.7106, 86.6025, 96.5919, and 100; the sum of all which is 429.7868, the number given nearly.

VII. QUESTION 990, *by* —.

I should be much obliged to any of your ingenious correspondents for a neat solution of the following set of equations, which have puzzled some of the best mathematicians; viz.  $x + 2\sqrt{(y - 2)} = 38$ ,  $x + \sqrt{(x - 2)} = 38$ ,  $5y + 2\sqrt{(x - 2)} = 38$ .

*Answered by the Rev. Mr. L. Evans.*

From the first equation,  $x = 38 - 2\sqrt{(y - 2)}$ , and from the third  $x = 363 - 95y + 6\frac{1}{2}y^2$ , therefore  $38 - 2\sqrt{(y - 2)} = 363 - 95y + 6\frac{1}{2}y^2$ , hence  $y^4 - 30.4y^3 + 335.04y^2 - 1580.902y + 2704.2048 = 0$ ; which corresponds exactly with the general form, in Dr. Hutton's Tracts, page 60, by which the four roots are easily found to be 10.1837, 9.8058, 5.3248, and 5.086; but the only value of  $y$  here to answer the original equations is 5.3248; whence  $x = 34.3531$ , and  $z = 15.2992$ .

*The same answered by Mr. Wm. Burdon, Acaster Malbis.*

Put  $\sqrt{(y - 2)} = v$ , and  $\sqrt{(x - 2)} = w$ , then will  $y = v^2 + 2$ , and  $x = w^2 + 2$ . These values substituted in the first and third equations, give  $w^2 + 2v = 36$ , and  $5v^2 + 2w = 28$ ; in the former  $w^2 = 36 - 2v$ , and in the latter  $w^2 = (14 - 2\frac{1}{2}v)^2$ , that is  $36 - 2v = (14 - 2\frac{1}{2}v)^2$ , or  $v^4 - 11\frac{1}{2}v^2 + \frac{8}{5}v = -25$ . Hence  $v = 1.823417$ , and  $w = 5.687984$ . Consequently,  $x = 34.353166$ ,  $y = 5.324849$ , and  $z = 4y - 6 = 15.299396$ .

VIII. QUESTION 991, *by Mr. Wm. Marriot, Neath, Glamorgan.*

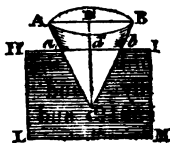
A cone whose diameter is twelve, and altitude ten inches, being put into a vessel filled with common rain water, with its base upward, was observed to sink to a point where the area of its section cut parallel to the base, was exactly eighty inches: required the weight of the cone?

*Answered by Mr. J. Rees, Bristol.*

The weight of the cone being, by hydrostatics, equal to the weight of the water displaced by immersion, and the water displaced being a

cone similar to the given one whose base is 80 square inches; it remains only to find its height to determine its bulk, and thence its weight.

Let *HIIM* be the vessel filled with water, *ABC* the cone, and *abc* the part immersed. By the question,  $80 \div .7854 = 101.85 = (ab)^2$ , and therefore  $\sqrt{101.85} = 10.0925$  inches  $= ab$ ; and by similar triangles, as *AB* : *DC* :: *ab* : *dc*, or as 12 : 10 :: 10.0925 : 8.4104; therefore the content of the part *abc* is *dc*  $\times \frac{1}{3}$  of 80 = 224.2746 cubic inches; this multiplied by .5787 the weight of a cubic inch of water, and divided by 16, gives 8.112 pounds for the weight of the whole cone *ABC*.



*The same answered by Mr. Wm. Marriot, of Neath.*

The area of a section of the cone at the surface of the water being 80 inches, the diameter of the section at that point will be  $\sqrt{(80 \div .78539)} = 10.09258$ ; and by similar triangles 12 : 10 :: 10.09258 : 8.41048 the height of that part of the cone immersed in the water, the content of which is  $80 \times 8.41048 \times \frac{1}{3} = 224.2746$  cubic inches; which multiplied by .036169lb. avoirdupois (the weight of a cubic inch of rain water) gives 8.111963788lb. the weight of the cone required.

*The same answered by Mr. Richard Embleton.*

As 12 : 10 or as 6 : 5 ::  $\sqrt{(80 \div .7854)} : 8.41$  inches, the depth of the cone in the water. Then  $80 \times 8.41 \times \frac{1}{3} = 224.2666$  cubic inches, the part immersed, which must displace the like quantity of water; and if water be 1000 ounces to the cubic foot, we have, as 1728 : 224.266 :: 1000 oz. : 122.8 ounces nearly, the weight of the water, which must be equal to the weight of the whole cone. But had the cone been 88.3 ounces heavier, it would have sunk until its base became level with the surface of the water.

IX. QUESTION 992, *by a Bengal Officer.*

To find a theorem that will determine an infinite variety of square integers, in a series of sursolids, whose roots are the arithmetical series 1, 2, 3, 4, 5, 6, &c.

*Answered by Mr. Richard Elliott, of Liverpool.*

With respect to the meaning of this question, I am not certain if I take it according to the proposer's intention. I suppose it to be, to find how far the series of  $1^2 + 2^2 + 3^2 + \&c.$  must be carried that the sum may be a square number; in order to which, we have

$$\frac{n^6}{6} + \frac{n^4}{5} + \frac{5n^2}{12} - \frac{n^2}{12}, \text{ or } \frac{n^4}{6} + \frac{n^2}{2} + \frac{5n^2}{12} - \frac{1}{12} =$$

$$\frac{2n^4 + 6n^3 + 5n^2 - 1}{12} = (n+1)^2 \times \frac{1}{12} \times (2n^2 + 2n - 1) \text{ to be}$$

a square number; and consequently  $\frac{1}{12} \times (2n^2 + 2n - 1)$  = a square number; or  $24n^2 + 24n - 12$  = a square number. Put  $n = x + 1$ , then by substituting  $24x^2 + 72x + 36$  or  $6x^2 + 18x + 9$  = a square number = suppose  $(ax - 3)^2 = a^2x^2 - 6ax + 9$ , then will  $x = (6a + 18) \div (a^2 - 6)$ ; where if  $a$  be taken = 3, then  $x = 12$ , and  $n = 13$ ; if  $a = 2.5$ , then  $x = 132$ , and  $n = 133$ ; if  $a = 2.45$ , then  $x = 13480$ , and  $n = 13481$ , and so on.

*The same answered by the Proposer.*

It is proved at page 204 of Simpson's Algebra, that the sum of this series is  $\frac{n^6}{6} + \frac{n^3}{2} + \frac{5n^4 - n^3}{12}$ , where  $n$  represents any number of

terms; and this formula is easily reduced to  $\frac{n(n+1)^2 n^2}{4} \times$

$$\frac{2n^2 + 2n - 1}{3}; \text{ which is to be a square. The first factor of this}$$

being a square, therefore makes the other equal to a square, viz.  $\frac{1}{3}(2n^2 + 2n - 1) = a^2$ , then is  $n = \frac{1}{2}(-1 + \sqrt{6a^2 + 3})$ . Now when  $n$  is an integer, then  $6a^2 + 3$  is a square =  $9c^2$ , hence  $3c^2 - 2a^2 = 1$ . Again, when  $a$  is an integer greater than unity, it is also greater than  $c$ ; therefore take  $p + q = a$ , and  $p - q = c$ , and of course  $3c^2 - 2a^2 = p^2 - 10pq + q^2 = 1$ , consequently  $p = 5q + \sqrt{(24q^2 + 1)}$ , a general theorem; for when  $q = 0$ ,  $p = 1$ ; when  $q = 1$ ,  $p = 10$ , when  $q = 10$ ,  $p = 99$ , &c. Hence, by continually taking the last preceding value of  $p$  = the next succeeding value of  $q$ , the law of continuation is manifest: for

when  $q$   $\left\{ \begin{array}{l} 0; 1; 10; 99; 980; 9701; 96030 \end{array} \right\}$  &c.  
 $p$   $\left\{ \begin{array}{l} 1; 10; 99; 980; 9701; 96030; 950599 \end{array} \right\}$  &c.

3.  $(p - q) = 3c = 3; 27; 267; 2643; 26163; 258987; 2563707$ ; &c. From hence the number of terms  $n = \frac{1}{2}(-1 + \sqrt{6a^2 + 3}) = 1; 13; 133; 1321; 13081; 129493; 1281853; 12689041$ , &c.

N. B. This calculation, apparently tedious, is easily simplified; for the first positive value of  $q = 1$  in this, and all other similar theorems; the opposite value of  $p$  may always be taken as a common multiplier, thus  $p \cdot p - q =$  in this theorem  $10 \cdot 10 - 1 = 99 = r$ , the 3d value; then  $p \cdot r - p = 99 \cdot 10 - 10 = 980 = s$ , the 4th value; and  $p \cdot s - r = 9701$ , the 5th value; and so on.

X. QUESTION 993, by the Rev. Mr. John Hellins.

If  $1 + \frac{1}{2} + \frac{1}{3}$  be put = A,  $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = B$ ,  $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = C$ , &c. then shall the arc of  $90^\circ$  be = A -  $\frac{1}{4}B + \frac{1}{4^2}C$  - &c. Query the demonstration?

*Answered by Amicus.*

The given series is composed of the three following ones, viz.

$$\left. \begin{aligned} 1 - \frac{1}{5.4} + \frac{1}{9.4^2} - \frac{1}{11.4^3} \text{ \&c.} \\ \frac{1}{2} - \frac{1}{6.4} + \frac{1}{10.4^2} - \frac{1}{14.4^3} \text{ \&c.} \\ \frac{1}{3} - \frac{1}{14.4} + \frac{1}{22.4^2} - \frac{1}{30.4^3} \text{ \&c.} \end{aligned} \right\} = \left\{ \begin{aligned} \sqrt{2} \times \left( \frac{1}{2^{\frac{1}{2}}} - \frac{1}{3.2^{\frac{1}{2}}} + \frac{1}{9.2^{\frac{1}{2}}} - \frac{1}{19.2^{\frac{1}{2}}} \text{ \&c.} \right) \\ + \frac{1}{2} - \frac{1}{3.2^{\frac{1}{2}}} + \frac{1}{5.2^{\frac{1}{2}}} - \frac{1}{7.2^{\frac{1}{2}}} \text{ \&c.} \\ + \sqrt{2} \times \left( \frac{1}{3.2^{\frac{1}{2}}} - \frac{1}{7.2^{\frac{1}{2}}} + \frac{1}{11.2^{\frac{1}{2}}} \text{ \&c.} \right) \end{aligned} \right.$$

Now, the second of these is well known to express the arc whose tangent is  $\frac{1}{2}$  and radius 1. And in the two others, if for the  $\sqrt{\frac{1}{2}}$  we write  $x$ , they become the known fluents of  $\frac{x\sqrt{2}}{1+x^2}$  and  $\frac{x^2x\sqrt{2}}{1+x^2}$  or of

$$\sqrt{2} \times \frac{x + x^3}{1+x^4} = \frac{x\sqrt{\frac{1}{2}}}{1-x\sqrt{2}+x^2} + \frac{x\sqrt{\frac{1}{2}}}{1+x\sqrt{2}+x^2}, \text{ hence the}$$

arc whose sine is  $\frac{x\sqrt{\frac{1}{2}}}{\sqrt{(1-x\sqrt{2}+x^2)(1+x\sqrt{2}+x^2)}}$  to rad. 1 + arc whose sine is

$$\frac{x\sqrt{\frac{1}{2}}}{\sqrt{(1+x\sqrt{2}+x^2)}} = (\text{when } x = \sqrt{\frac{1}{2}}) \text{ the arc whose sine is } \sqrt{\frac{1}{2}},$$

tang. 1, rad. 1 + arc whose sine is  $\sqrt{\frac{1}{10}}$ , cos.  $\frac{3}{\sqrt{10}}$ , tang.  $\frac{1}{3}$  and radius 1.

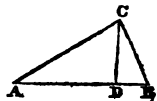
The tang. of the sum of these two arcs is  $\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$ , consequently the arc whose tang. is 2 = the sum of these two series, and the remaining one being equal that whose tang. is  $\frac{1}{2}$  = the complement of that whose tang. is 2, the whole series or sum of the three = a quadrant.

XI. QUESTION 994, by James Glenie, Esq.

On a given line as a base, to constitute a triangle such, that the sum of the squares of the other two sides, shall have to their difference, the ratio of 13 to 12; whilst the area of the triangle has to the square of the base a given ratio, suppose that of 1 to 12.

*Answered by Mr. Benjamin Gompertz.*

AB being the given base, take  $BD = \frac{1}{12}AB$ , then raise the perpendicular  $DC = 2DB$  or  $= \frac{1}{6}AB$ , join AC, BC, and ABC is the triangle required. For the area  $= \frac{1}{2}AB \times CD = \frac{1}{2}AB^2 : AB^2 :: 1 : 12$ ; also, since  $AC^2 = AD^2 + DC^2 = 121BD^2 + 4BD^2 = 125BD^2$ , and  $BC^2 = BD^2 + DC^2 = 5BD^2$ , hence  $AC^2 + CB^2 : AC^2 - CB^2 :: 130BD^2 : 120BD^2 :: 13 : 12$ .

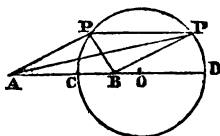


*The same answered by Mr. Jos. Mouldsdales, of Halton, Cheshire.*

*Analysis.* Put  $m = 13$ , and  $n = 12$ , then per question  $AC^2 + BC^2 : AC^2 - BC^2 :: m : n$ , and by composition and division  $2AC^2 : 2BC^2$  or  $AC^2 : BC^2 :: m + n : m - n :: 25 : 1$ , hence  $AC : BC :: 5 : 1$ , the given ratio of the sides. Again, by the question, the area, viz.  $AB \times \frac{1}{2}CD : AB^2 :: 1 : 12$ , or  $CD : AB :: 2 : 12 :: 1 : 6$ . Hence then are given the base, the perpendicular, and the ratio, of the sides, to construct the triangle; which is the 23d problem of Simpson's Algebra; where it is constructed.

*The same answered by Mr. W. Armstrong, Newcastle.*

Since the base is given, and also the ratio of the square of it to the area of the triangle, the perpendicular is therefore given. And since the sum of the squares, to the difference of the squares, of the sides, is a given ratio, viz.  $m$  to  $n$ , therefore the squares of those sides, and consequently the sides themselves, are in a given ratio, viz.  $\sqrt{m + n} : \sqrt{m - n}$ .



*Hence this Construction.* Take  $AB$  = the given base, which divide in  $c$  so that  $AC : CB :: \sqrt{m + n} : \sqrt{m - n}$  the ratio of the sides. Also take  $CO : CA :: CB : CA - CB$ . With radius  $CO$ , and centre  $O$ , describe the circle  $CP$ ; and parallel to  $AD$ , at the distance of the given perpendicular draw  $PP$ , cutting the circumference in  $P$  and  $P$ ; then join  $AP$  and  $BP$ , so shall  $ABP$  or  $ABP$  be the triangle sought.

For by lemma, page 334, Simpson's Algebra, 5th edit.  $AP : BP :: AC : BC :: \sqrt{m + n} : \sqrt{m - n}$ , or  $AP^2 : BP^2 :: m + n : m - n$ , or mixedly  $AP^2 + BP^2 : AP^2 - BP^2 :: 2m : 2n :: m : n$ .

*The same answered by Mr. Alex. Rowe, Reginnis.*

Let  $x$  and  $y$  represent the sides of the triangle; then by the question, as  $x^2 + y^2 : x^2 - y^2 :: 13 : 12$ , hence  $12x^2 + 12y^2 = 13x^2 - 13y^2$ , or  $x^2 = 25y^2$ , and  $x = 5y$ . Now put the base  $AB = b$ , the side  $BC$  (fig. 1)  $= y$ , and  $AC = 5y$ , and the perpendicular  $CD = 2z$ ; then the area  $ABC = bz$ , and hence as  $bz : b^2 :: 1 : 12$ , or  $z : b :: 1 : 12$ ; or  $2z = \frac{1}{6}b$  the perpendicular. Also as  $\sqrt{AC^2 - CD^2} = \sqrt{(25y^2 - \frac{1}{36}b^2)} = \sqrt{(y^2 - \frac{1}{36}b^2)} = BD$ ; therefore  $AD + DB = \sqrt{(25y^2 - \frac{1}{36}b^2)} + \sqrt{(y^2 - \frac{1}{36}b^2)} = b$ , the reduction of which equation gives  $y = \frac{1}{12}b\sqrt{5} = BC$ , and hence,  $5y = \frac{5}{12}b\sqrt{5} = AC$ .

## XII. QUESTION 995, by —.

Required the dimensions of a cone, which if suspended by its vertex will vibrate as often in a minute, as it has inches in altitude?

*Answered.*

This question is unlimited, as the solid content, or some other dimension, ought to have been given. By supplying some such condition, however, the solution has been properly given by several persons as below.

*The solution is thus given by Amicus.*

Let  $b$  = the radius of the cone's base, and  $a$  = its altitude in inches; then the distance of the centre of oscillation from the vertex  $= \frac{3}{2}a + \frac{1}{2}b^2 \div a^3$ , and the time in seconds of one vibration  $= 60 \div a$ ; hence as  $\sqrt{39.2} : 1 :: \sqrt{(\frac{3}{2}a + \frac{1}{2}b^2 \div a^3)} : 60 \div a$ , or  $a^3 + \frac{1}{2}b^2a = 176400$ ; hence when  $b$  is given, or given in terms of  $a$ ,  $a$  will be found. Thus if  $b = 2$ ,  $a = 56.09$  nearly. If  $b = a\sqrt{3}$ , or the cone be equilateral,  $a = \sqrt[3]{100800} = 46.53934$ , &c.

*The same answered by Mr. John Fennell, Penzance.*

Put  $s = 1728$  the solidity,  $p = 3.1416$ ,  $x$  = the altitude, and  $y$  = radius of the base in inches, also  $a = 39.2$  the length of the pendulum vibrating seconds. Then  $\frac{4}{3}pxy = s$ . Again (page 239, Emerson's Fluxions)  $(4xx + yy) \div 5x$  = distance of the centre of oscillation from its vertex. Hence,  $\sqrt{((4xx + yy) \div 5ax)}$  being = the time of one vibration,  $\sqrt{((4xx + yy) \div 5ax)} : 1 :: 60'' : x$ , or  $x\sqrt{((4xx + yy) \div 5ax)} = 60$ ; from hence and the foregoing equation, are found  $x = 56.0394$ , and  $y = 5.426$  inches.

*The same answered by Mr. Isaac Saul, Holland, near Wigan.*

The centre of oscillation in a cone, according to some authors, is  $\frac{4}{5}$  of the height from the vertex. Let  $a = 39.2$ ,  $b = 60$ , and  $x$  = the height of the cone; then as  $\frac{4}{5}x : b^3 :: a : x^2$  per question; therefore  $\frac{4}{5}x^3 = ab^3$ , and  $x = \sqrt[3]{\frac{5}{4}ab^3} = 56.0832$  inches.

## XIII. QUESTION 996, by Amicus.

Mr. T. Simpson, in his Algebra, shews how to resolve biquadratic equations, by reducing them to the difference of two complete squares, and says, the most considerable advantage of this method is, that the value of  $A$  in the cubic equation thence resulting, will be commensurate and rational, not only when all the roots of the given equation are commensurate, but when they are irrational, and even impossible. Is this advantage general for all biquadratics having rational coefficients, such as  $y^4 + \frac{8}{3}y^3 + 14y^2 - 12y - 51 = 0$ , or is it not so? A demonstration is required?

*Answered by Amicus.*

This would be a very considerable advantage indeed, were it but general, but, alas! it obtains but in few cases, comparatively speaking, as appears from the resolution of the equation  $y^4 + \frac{2}{3}y^3 + 14y^2 - 12y - 51 = 0$ ; for let it be  $(y^2 + \frac{1}{2}y + \Lambda)^2 - (ay + c)^2$ , and the equation resulting for  $\Lambda$  will be  $\Lambda^3 - 7\Lambda^2 + 31\Lambda - 91\frac{1}{2} = 0$ ; now let  $\Lambda = \frac{2}{3}x + \frac{1}{3}$ , and it will be transformed to  $x^3 + 33x - 151 = 0$ ; let  $x = 151 \div p$ , then  $p - 151 \div p^3 = 33$ , where, by my answer to question 904, *Diary 1790*, if  $x$  be rational,  $p$  must be an integer, but this is manifestly impossible; consequently  $x$  cannot be rational, therefore  $\Lambda$  is irrational, and the advantage not general. But by reducing the equation in this manner, it may always be known whether  $\Lambda$  be rational or not, and its value determined.

*The same answered by Mr. John Rykley, of Leeds.*

Here, according to Mr. Simpson's method, we have  $a = \frac{2}{3}$ ,  $b = 14$ ,  $c = -12$ ,  $d = -51$ ,  $k = 31$ ,  $l = \frac{250}{3}$ . Hence  $\Lambda^3 - 7\Lambda^2 + 31\Lambda - 91\frac{1}{2} = 0$ ; now put  $z + \frac{1}{3} = \Lambda$ , and the equation will be transformed to  $z^3 + 14\frac{2}{3}z = \frac{1208}{27}$ ; which being resolved, we have  $z = \frac{1}{3} \sqrt[3]{(604 + 300\sqrt{5})} - \frac{14\frac{2}{3}}{\sqrt{(604 + 300\sqrt{5})}} = 2.261696543$ , &c. and  $\Lambda = 4.595029876$ , &c.; the other two roots being impossible. Consequently the value of  $\Lambda$  will not be commensurate and rational in all cases.

*The same answered by Mr. John Surtees.*

Dr. Hutton says, at page 211 of his *Mathematical Dictionary*, vol. 1, that in any biquadratic equation having all its terms, if  $\frac{1}{8}$  of the square of the coefficient of the 4th term, be less than the product of the 3d and 5th, then that equation will have imaginary roots; which is exactly the case in this equation; and therefore Mr. Simpson's assertion fails.

XIV. QUESTION 997, by Capt. Wm. Mudge, *Royal Artillery*.

It having been surmized, that in the practice of artillery, the deflection of the shot from the line in which the gun is laid, chiefly arises from the motion of the gun during the time the shot is passing out of the piece; I desire to be informed what space an eighteen pounder will recoil or fly back, whilst the shot is passing out of the gun; supposing that its weight is 4800lb. that of the carriage 2400lb. the quantity of powder 8lb, the length of the cylinder 108 inches, that of the charge 13 inches, and the diameter of the bore 5.13 inches, admitting likewise, that the resistance arising from the friction between the platform and carriage, is = 3600lb.?

*Answered by Mr. Colin Campbell, of Kendal.*

Confined gunpowder when fired changes immediately into an elastic fluid of an uniform density, which expands itself in those directions where the resistance is least. Now in the problem the action of this fluid is exerted equally on the bottom of the bore of the gun and the ball during the passage of the latter through the cylinder; therefore the two bodies move in contrary directions, with velocities which are at all times in the inverse ratio of the quantities of matter moved. Let  $x$  be the space through which the gun recoils; then as the powder occupies 13 inches of the barrel, and the semidiameter of the ball 2.565 inches, the space moved through by the ball before it quits the piece  $= 108 - 13 - 2.565 - x = 92.435 - x$ ; and as the elastic fluid expands in both directions, the quantity which advances towards the mouth is to that which retires from it, as  $92.435 - x : x$ , consequently  $(8 \div 92.435) x =$  the quantity which moves with the gun, and  $8(92.435 - x) \div 92.435 =$  the quantity which moves with the ball; whence the quantities of matter moved in the

two directions are  $18 + \frac{92.435 - x}{92.435} \times 8$  and  $4800 + 2400 + 3600 + \frac{8x}{92.435}$ , or  $\frac{2403.31 - 8x}{92.435}$  and  $\frac{998298 + 8x}{92.435}$ . But when

the time and moving force are given, the spaces are inversely as the quantities of matter; therefore  $x : 92.435 - x :: 2403.31 - 8x : 998298 + 8x$ , and by composition,  $x : 92.435 :: 2403.31 - 8x : 1000701.31$ , and by division,  $x : 1 :: 2403.31 - 8x : 10826$ , therefore  $10826x = 2403.31 - 8x$ , or  $10834x = 2403.31$ , hence  $x = .2218$ , the recoil sought. *Note*, The ball is supposed to quit the gun when one hemisphere is out of the mouth.

PRIZE QUESTION, *by Terricola.*

Suppose the whole terraqueous globe, taken as a sphere, should be instantaneously turned into a uniform elastic æriform fluid, whose particles repel one another with a force which is to that with which those of air repel one another, as the density of the one to the density of the other, it will expand itself either to a finite or infinite extent, still preserving the form of a sphere. It is required to determine the force of gravitation tending towards the centre, and also the density, at any given distance from the centre; supposing the mean density of the earth to be 3825 times that of air at the surface of the earth.

*Answered by Amicus.*

To find the nature and properties of a perfectly elastic æriform fluid, whose particles are at rest; and only acted upon by their own gravity?

Let  $x$  = any variable distance from its centre,  $y$  = the density at that distance,  $p = 3.1416$ ,  $q$  = the quantity of matter in the sphere whose radius =  $x$ ; then, the gravity at the distance  $x$  being as  $q$  directly, and  $x$  inversely, must be as  $\frac{q}{x^2}$ : also the quantity of matter in  $\dot{x}$  must be  $y\dot{x}$ ; and consequently, the gravity being as the pressure and quantity of matter conjointly,  $-\dot{y} = \frac{cgy\dot{x}}{x^2}$ , where  $c$  is a constant quantity to be determined hereafter: and  $\dot{q} = 4pyx\dot{x}$ . From the first of these equations  $\frac{\dot{y}}{y} = -\frac{cq\dot{x}}{x^2}$ , and from the second, making  $\dot{q}$  constant,  $\frac{\dot{y}}{y} = -\frac{\ddot{x}}{\dot{x}} - \frac{2\dot{x}}{x}$ , consequently  $\frac{\ddot{x}}{\dot{x}} + \frac{2\dot{x}}{x} = \frac{cq\dot{x}}{x^2}$ , and  $cq = \frac{x\ddot{x}}{\dot{x}^2} + 2x$ , hence  $cq = 2x$ , for since  $\dot{q}$  is constant, when  $\ddot{q} = 0$ ,  $c\ddot{q} = 2\ddot{x}$ , and  $\ddot{x} = 0$ , therefore  $\frac{x\ddot{x}}{\dot{x}^2} = 0$ , and  $q = \frac{2x}{c}$ ,  $\dot{q} = \frac{2\dot{x}}{c}$ , and  $y = \frac{1}{2pcx}$ . Hence the quantity of matter in the globe whose

radius is  $x$ , is to the density at its surface ::  $2x : \frac{1}{2px^2} :: 4px^3 : 1 ::$

three times the content of the globe to unity. Therefore, when the density = 0,  $x$  is infinite, or the fluid is of infinite extent, and its quantity of matter greater than any finite magnitude. Consequently a globe of finite magnitude of this nature, cannot have its particles at rest. But to keep them together so that the pressure may balance the elasticity or spring, it is necessary that it be acted upon by some other force besides its own gravity, as the air is acted upon by the force of gravity of the earth besides its own, otherwise the particles, being once in motion, would fly off to infinity. In the perfectly elastic fluid above, since the density is as  $x^{-2}$ , it must be infinite at the centre where  $x = 0$ .

The extent and density at the surface of such a portion of the infinitely extended fluid, as is equal to any given finite body, as the earth, may be found thus. Let  $r$  = the radius of the earth, and  $d$  = its density;

then its quantity of matter =  $\frac{4}{3}pdr^3 = \frac{2x}{c}$ , and  $x = \frac{2}{3}pdr^3 =$  the

radius of such a globe. Moreover, per question, as  $\frac{1}{r^2} : \frac{1}{3825} :: \frac{1}{x^2}$

$:\frac{r^2}{3825x^2}$ , and if this =  $y$ , as it must be to render the fluid continuous,

$\frac{r^3}{3825x^3} = \frac{1}{2px^3}$ , and  $\frac{1}{c} = \frac{2pr^3}{3825}$ , which being substituted for  $\frac{1}{c}$

above, gives  $q = \frac{4pr^3}{3825}$ ,  $y = \frac{r^3}{3825x}$ , and the radius of the globe above  $= \frac{4}{3}pcdr^3 = 1275dr$ , or  $= 1275r$  when the density of the earth is unity.

*Corollary.* As the density of the earth  $= 1 : \frac{3q}{4px^3} =$  the mean density of such a globe  $:: \frac{4}{3}px^3 : \frac{4}{3}pr^3$ , or  $\frac{4}{3}pr^3 = q$ , and  $\frac{3q}{4px^3} = \frac{r^3}{x^3} = \frac{1}{1275}$ , = the mean density; and the density at its surface  $= \frac{r^3}{3825} \times \frac{1}{1275} = \frac{1}{3 \times 1275}$ , or its mean density  $= 3$  times that at its surface.

*The same by Clericus, of Southwold, from the Supplement.*

As the matter of the proposed ærial sphere is supposed to possess the well known properties of our atmospheric air, which consist of being subject to the common laws of gravitation, and to acquire a density in the direct proportion to the force impressed; it thence follows, that if  $A, B, C$ , be assumed three invariable quantities, the density of the proposed sphere, at  $x$  distance from its centre, will be  $Ax^{-2}$ ; the quantity of matter inclosed within that distance, will be  $Bx$ ; and the force of gravitation will be  $Cx^{-1}$ .

To prove which, and at the same time to determine the values of  $A, B, C$ , in given terms; let unity (1) represent both the force of gravity, and density of the air, at the earth's surface. Put  $r = 3977$  miles, the radius of the earth;  $m = 3825$  the mean density of the earth;  $h = 49$  miles nearly, the height of a homogeneous atmosphere, whose density is 1, and pressure at the earth's surface, equal to that of our atmosphere in its mean state. Also let  $p = 3.141592$ , &c. and  $E = \frac{4}{3}pr^3m$ , the quantity of air (density = 1,) which is equivalent to the quantity of matter in the earth. Then if  $D$  be the density at  $x$  distance from the centre;  $M$ , the quantity of air of the same kind with  $E$ , inclosed within that distance; and  $F$  the correspondent force of gravitation; we have, by Pneumatics,  $hD$  (the fluxion of the pressure)  $= F \times D \times -x$ . Now  $M = 4\pi x^3 \times Ax^{-2}$  ( $D$ )  $= Bx$ , therefore  $B = 4\pi A$ . Again,  $\frac{E}{r^3} : \frac{M}{x^3} (= \frac{B}{x}) = \frac{4\pi A}{x} :: 1 : F = \frac{4\pi r^3 A}{E x}$ . Whence  $hD = -\frac{4\pi r^3 A^2 x^{-3}}{E}$ , or  $D = \frac{2\pi r^3 A^2 x^{-2}}{hE}$ , which shews that  $D$

was rightly assumed. And therefore  $\frac{2pr^3A^2}{hE} = A$ , or  $A = \frac{hx}{2pr^3} = \frac{2hrm}{3}$ ,  $B = \frac{8phrm}{3}$ ; and  $C = \frac{4pr^3A}{E} = 2h$ . Whence density  $= \frac{2hrm}{3x^3}$ ; force of gravitation  $= \frac{2h}{x}$ ; and  $M = \frac{8phrmx}{3}$ .

Cor. 1. When  $D = 1$ ,  $x = \sqrt{\frac{2}{3}hrm} = 7438$  miles, the distance from the centre of the ærial sphere, where the density equals that of the air at the earth's surface.

Cor. 2. Let  $D = m$ ; then  $x = \sqrt{\frac{2}{3}hr} = 120$  miles, the distance where the said sphere hath its original density.

Cor. 3. When  $M = E$ , or  $\frac{8phrmx}{3} = \frac{4pr^3m}{3}$ ; then  $x = r^3 \div 2h = 1440000$  miles nearly, the radius of the whole ærial sphere, containing matter equal to the whole earth.

Cor. 4. At about 11 miles distance from the centre, the force of gravitation is equal to that at the earth's surface.

N. B.—The whole radius of the ærial sphere, and also its gravitating force throughout, continue the same, whatever may be the mean density ( $m$ ) of the earth.

Cor. 5. Hence the quantity of matter contained within any given distance of the centre, is to the quantity, supposing the density to be invariable, thence to the centre, as 3 to 1. For as  $D = \frac{2hmr}{3x^3}$ ; the quantity of matter in a sphere (radius = 1) of this density, is  $= \frac{4\pi x^3}{3} \times \frac{2hmr}{3x^3} = \frac{8\pi hmr}{9}$ , which is to  $M = \frac{8\pi hmr}{3}$  evidently :: 1 : 3.

Scholium. We hence understand that, when a body or collection of air forms itself into a sphere by the mutual attractive and repulsive forces of the particles among themselves, independent of any other force, it does not extend itself *ad infinitum*, as is the case with the atmosphere of the earth. And moreover, should there be, as very probably there is, some fixed limit to the density of air; still if that limit lies comparatively near the centre, the sphere will yet extend only to a finite distance, whilst the whole quantity of matter is finite. This appears from the above example, where, supposing the density of water, for instance, to be the fixed limit to that of air; this lies only at 258 miles distance from the centre; and therefore (Cor. 5) the quantity of matter thence to the centre is only  $\frac{86}{1440000}$ th part of the whole; which is inconsiderable.



is given by the question  $x + y = 80$ , and  $x^2 + 20y^2 =$  a minimum. The latter of these in fluxions gives  $3x^2\dot{x} + 40y\dot{y} = 0$ , and the former gives  $\dot{y} = -\dot{x}$ , which being substituted for it in the former, gives  $3x^2\dot{x} - 40y\dot{x} = 0$ ; dividing by  $\dot{x}$ , this is  $3x^2 - 40y = 0$ ; instead of  $y$  here write  $80 - x$ , which gives  $3x^2 + 40x = 3200$ . Hence, by completing the square and extracting the root, &c. we get  $x = 26\frac{2}{3}$ ; and consequently  $y = 53\frac{1}{3}$  the height of the two trees.

*The same by the Rev. Mr. J. Furnass.*

Let  $x =$  the oak's height, and  $80 - x =$  the elm's; then  $x^2 + 20 \times (80 - x)^2$  is a minimum; or  $x^2 + 20x^2 - 3200x =$  a minimum: this in fluxions gives  $3x^2\dot{x} + 40x\dot{x} - 3200\dot{x} = 0$ , or  $3x^2 + 40x = 3200$ ; hence  $x = 26\frac{2}{3}$  feet, the oak's height, and therefore  $80 - x = 53\frac{1}{3}$  feet, the elm's.

*The same by Mr. J. Hartley, Auditor's Office, Somerset Place.*

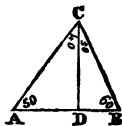
Let  $x$  represent the oak,  $y$  the elm, and  $a = 80$ , the sum of their heights; then will  $x + y = a$ , and  $y^2 = a^2 - 2ax + x^2$ . But  $x^2 + 20y^2 =$  a minimum; therefore  $x^2 + 20a^2 - 40ax + 20x^2 =$  a minimum. In fluxions and reduced,  $x^2 + \frac{40}{3}x = 1066\frac{2}{3}$ ; then, completing the square, and reducing  $x = 26\frac{2}{3}$ , and  $y = 53\frac{1}{3}$ , the height of each tree, as required.

III. QUESTION 1001, by Mr. Newton Bosworth, Peterborough.

Given the three angles of a triangle,  $50^\circ$ ,  $60^\circ$ ,  $70^\circ$ , and the area 12 acres; to find the sides of the triangle?

*Answered by Mr. John Cavill, of Beighton.*

Suppose ABC be the triangle proposed. Let fall the perpendicular CD, then we have the two vertical angles ACD, BCD given  $= 40^\circ$  and  $30^\circ$  respectively, being the complements to the given ones A and B at the base. Now calling the perpendicular CD unity or 1, AB will be the sum of the tangents of  $40^\circ$  and  $30^\circ$ , the half of which sum is therefore the area of such a triangle, which being put  $= t$ ; then as  $1 : 4t :: 120 \text{ chains} : 480t = AB^2 = 339.992$ ; hence  $AB = 18.438 \text{ chains}$ ,  $AC = 16.992$ , and  $BC = 15.031$ .



*The same by Mr. R. Dover, Teacher of Mathematics, Carlisle.*

The proportion for finding the base is, in Emerson's Trigonometry, page 112, thus: As cos. dif. — cos. sum of the angles at the base:

is to the sine of the vertical angle . . . . . ::

so is the area of the triangle: to the square of the base.

Now, put  $a$  for the cos. dif. and  $b$  for the cos. sum of the angles at the

*Questions proposed in 1796, and answered*

**I. QUESTION 999, by Mr. John**

Suppose I am out at sea in a calm day, as even with the surface of the water, how eye being 5 feet and a half above the was 7970 miles?

*Answered by Mr. John*

By Doctor Hutton's Mensuration:  
 $\therefore AB : BE = .00104166638$ ,  
 problem 19, section 1, it is 64  
 $= 16692.3634$ , the number

*The same b*

Put  $a = 5\frac{1}{2}$  feet,  
 $b = 3985$  miles,  
 $.00104$  miles the  
 er saw; then  
 quantity requ

First

also 5'

near

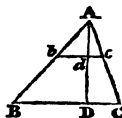
for

1' Given the three perpendiculars of a plane triangle, viz. from the three angular points to the opposite sides equal to 10, 11, 12; to determine the sides and the area of the triangle.

*Answered by Mr. Joseph Gittins, London.*

Because the double area of a triangle is equal to the rectangle under the base and perpendicular, or any side is equal to the double area divided by the perpendicular upon it from the opposite angle; therefore the three sides will be reciprocally as the three perpendiculars, that is as  $\frac{1}{10}$ ,  $\frac{1}{11}$ ,  $\frac{1}{12}$ , or as the numbers 1,  $\frac{11}{10}$ ,  $\frac{12}{10}$ , or  $\frac{6}{5}$ . Hence, having constructed upon  $bc = 1$ , the similar triangle  $bAc$  with these three numbers; demit the perpendicular  $Ad$  which produce till  $AD$  be equal to 10, the least given perpendicular; then through  $D$  draw  $ac$  parallel to  $bc$  meeting  $Ab$  and  $Ac$  produced in  $B$  and  $C$ ; forming the required triangle.

*Calculation.* Find the area ( $.35569974$ ) of the similar triangle



MATHEMATICAL QUESTIONS.

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$\triangle abc$ , by the 29th property of plane triangles in Dr. Hutton's Mathematical Dictionary, vol. 2, page 616; then as  $ad^2 : ad'^2 :: .35569974 : 70.281168$ , the area of the triangle  $abc$ ; the double of which divided severally by 10, 11, 12, gives 14.0568, 12.7789, 11.7140 for the three sides as required.

*The same answered by the Rev. Mr. Ewbank, Thornton Steward.*

Since in every triangle the sides are reciprocally as the perpendiculars upon them which are given equal to 10, 11, 12, the reciprocals of which are  $\frac{1}{10}, \frac{1}{11}, \frac{1}{12}$ , which are proportional to 66, 60, 55; therefore assume  $x$ , and let the three sides be denoted by  $66x$ ,  $60x$ , and  $55x$ ; then by the rule for finding the area from the three sides as above, the area will be  $\frac{1}{2}x^2 \sqrt{(181.49 \cdot 61 \cdot 71)}$ ; and by multiplying the base  $66x$  by its perpendicular 10, half the product  $330x$  is the area also; these two being put equal, gives  $x = 4 \times 330 \div \sqrt{(181.49 \cdot 61 \cdot 71)} = .21298$ ; which being multiplied severally by 66, 60, 55, gives 14.0568, 12.7789, 11.71399, for the three sides, and  $330x = 70.28399$  the area.

*The same by Miss Maria Middleton, of Eden, near Durham.*

Put  $x$  = the area of the triangle; then will the sides be  $\frac{2}{10}x, \frac{2}{11}x, \frac{2}{12}x$ ; from hence the area is  $\sqrt{(181x \cdot 49x \cdot 61x \cdot 71x \div 660^2)} = x$ ; hence  $x$  is found  $= \sqrt{(660^2 \div 38411639)} = 70.284$ , and the three sides are 11.714, 12.7789, and 14.0568.

V. QUESTION 1003, by Miss Nancy Mason, of Clapham.

A father dying left a square field to be divided among his five sons, the field containing just 30 acres, and to be divided in such a manner that the oldest son may have 8 acres, the second son 7, the third son 6, the fourth son 5, and the fifth, or youngest son, 4 acres. Now the fences are to be so made that the oldest son's share shall be a narrow piece of equal breadth all round the field, leaving the remaining four shares in form of a square; and in like manner for each of the other shares, leaving always the remainders in form of squares, one within another, till the share of the youngest be the innermost square of all, equal to 4 acres. It is required to divide the field geometrically, and to calculate the sides of all the fences?

*Answered by Mr. William Burdon, Acaster-Malbis.*

On any straight line  $AF$  take  $AB = 4, BC = 5, CD = 6, DE = 7$ , and  $EF = 8$ . Upon  $AF$  describe a semicircle, and raise the perpendiculars  $BG, CH, DK$ , and  $EL$ ; and draw  $AG, AH, AK, AL$ . Take  $ac = \sqrt{300} = 17.32051$ , the side of the given square, and draw a line  $Af$ , making



$\triangle bc$ , by the 29th property of plane triangles in Dr. Hutton's Mathematical Dictionary, vol. 2, page 616; then as  $Ad' : AD' :: .35569974 : 70.281168$ , the area of the triangle  $ABC$ ; the double of which divided severally by 10, 11, 12, gives 14.0568, 12.7789, 11.7140 for the three sides as required.

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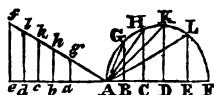
V. QUESTION 1003, by Miss Nancy Mason, of Clapham.

A father dying left a square field to be divided among his five sons, the field containing just 30 acres, and to be divided in such a manner that the oldest son may have 8 acres, the second son 7, the third son 6, the fourth son 5, and the fifth, or youngest son, 4 acres. Now the fences are to be so made that the oldest son's share shall be a narrow piece of equal breadth all round the field, leaving the remaining four shares in form of a square; and in like manner for each of the other shares, leaving always the remainders in form of squares, one within another, till the share of the youngest be the innermost square of all, equal to 4 acres. It is required to divide the field geometrically, and to calculate the sides of all the fences?

*Answered by Mr. William Burdon, Acaster-Malbis.*

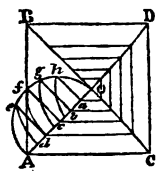
On any straight line  $AF$  take  $AB = 4$ ,  $BC = 5$ ,  $CD = 6$ ,  $DE = 7$ , and  $EF = 8$ . Upon  $AF$  describe a semicircle, and raise the perpendiculars  $BG$ ,  $CH$ ,  $DK$ , and  $EL$ ; and draw  $AG$ ,  $AH$ ,  $AK$ ,  $AL$ . Take  $Ac = \sqrt{300} = 17.32051$ , the side of the given square, and draw a line  $Af$ , making

any angle with it ; on which take  $Ag=AG$ ,  $Ah=AH$ ,  $Ak=AK$ ,  $Al=AL$ , and  $Af=AF$  ; join  $fe$ , parallel to which draw  $ld$ ,  $kc$ ,  $hb$ ,  $ga$  ; so shall  $Aa$ ,  $Ab$ ,  $Ac$ ,  $Ad$ , be the sides of the several squares within the field. For  $Ac^2 : AG^2 :: Af^2 : AG^2 :: AF^2 : AG^2$  ( $AF \times AB :: AF : AB$  ; and in like manner for the rest. The calculation gives  $Aa = 6.32455$ ,  $Ab = 9.48683$ ,  $Ac = 12.24745$ , and  $Ad = 14.83239$ , the sides of the squares.



*The same answered by Mr. Richard Embleton, Stony Hills.*

Let  $ABCD$  be the square,  $o$  the intersection of the diagonals, and let  $oa$ ,  $ob$ ,  $oc$ ,  $od$  be in proportion to 4, 5, 6, 7, 8. On  $AO$  describe a semicircle, and erect the perpendiculars  $de$ ,  $cf$ ,  $bg$ ,  $ah$  : then with the radii  $oe$ ,  $of$ ,  $og$ ,  $oh$ , let circles be described, intersecting the diagonals ; join the corresponding intersections, and the thing is done. For the demonstration, see question 9, page 6, vol. 1. Then  $\sqrt{220} = 14.832$  ;  $\sqrt{150} = 12.247$  ;  $\sqrt{90} = 9.487$  ;  $\sqrt{40} = 6.324$ , the sides of the inner squares.



VI. QUESTION 1004, by Mr. Alex. Rowe.

What plane triangle is that, the natural tangents of whose angles are whole numbers, the radius being unity ?

*Answered by Mr. R. Elliott, Liverpool.*

Let  $x$ ,  $y$ ,  $z$  be the natural tangents of the three angles of any plane triangle, to the radius 1 ; then, by Emerson's Trigonometry,  $1 - xy : 1 :: x + y : -z$ , or  $x + y = xyz - z$ , where  $x$ ,  $y$ ,  $z$  are to be whole numbers ; and as the least value of  $z$  is 1, we have  $x + y = xy - 1$  in that case, from which  $y = (x + 1) \div (x - 1) = 1 + 2 \div (x - 1)$ , where it is evident that  $x$  cannot be greater than 3, and consequently  $y$  not less than 2. Hence it appears, that the triangle, whose natural tangents are whole numbers, is when they are 1, 2, 3, and the angles are  $45^\circ$ , and  $60^\circ 26'$ , and  $71^\circ 34'$ .

*The same by Mr. Christ. Holme, Botcherby, near Carlisle.*

Let  $x$ ,  $y$ ,  $z$  be the tangents of the angles of the triangle ; then, by trigonometry,  $1 - xy : 1 :: x + y : -z$  ; hence  $x + y = xyz - z$ , or  $x + y + z = xyz$ , that is the sum of the tangents of all the three angles of any plane triangle, is equal to the continual product of the same, the radius being 1. Now, it is evident, that no other numbers beside 1, 2, 3, and  $-1$ ,  $-2$ ,  $-3$ , will answer this condition ; therefore 1, 2, 3, are the tangents, and  $45^\circ$ ,  $63^\circ 26' 6''$ , and  $71^\circ 33' 54''$ , the angles required.

*The same answered by Mr. John Haycock, of Wars.*

From a table of natural tangents, it is evident by inspection, that the whole numbers must be 1, 2, 3, which answer to  $45^\circ$ ,  $63^\circ 26' 6''$ , and  $71^\circ 33' 54''$ , the angles of the triangle sought.

VII. QUESTION 1005, by Mr. Thomas Hornby, of Wombledon.

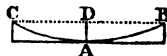
It is the common practice in land-surveying, for the leader and follower of the chain, at each chain's length, to suspend it from the tops of the arrows; now supposing the arrows to be a yard long, above the earth's surface, and fixed perpendicularly to the horizon on horizontal ground, how long ought the chain to be in this case, to give exactly statute measure; allowing it to be stretched till its centre of gravity descend the lowest possible, or the middle point just to touch the ground?

*Answered.*

This question is answered by our correspondents in a great variety of ways, some of them considering the chain as forming two sides of a triangle upon the straight-lined base, others as a regular curve, either a circle, or a parabola, or a catenary, any of which must be very near the truth, though none of them quite exact, not even the catenary, as the links are not indefinitely small, or the chain perfectly flexible, which it ought to be, to form that curve. Specimens of some of those methods are as below:

*The answer by Mr. John Ryley, of Leeds.*

It is evident that a chain suspended in the manner described in the question, will form a catenary very nearly. Therefore, if  $AD = x$ ,  $BD = y$ ,  $AB = z$ , and the tension of the chain at the point A in the horizontal direction  $= a$ , then, under the word *catenary*, in Dr.



Hutton's Dictionary, we have the following equations, viz.

$$a = \frac{z^2 - x^2}{2x} = \frac{z^2 - 1}{2}, \text{ and } y = a \times \text{hyp. log. } \frac{z + x}{z - x}; \text{ hence}$$

$$\frac{z^2 - 1}{2} \times \text{hyp. log. } \frac{z + 1}{z - 1} = 11; \text{ which equation being resolved,}$$

we get  $z = 11.06032$ ; therefore the length of the whole curve CAB is  $= 22.12046$  yards, or 22 yards  $4\frac{1}{2}$  inches, very nearly.

*The same by Mr. John Bruce, Schoolmaster, Newcastle.*

Let CAB represent the curve formed by the chain when suspended from the tops of the arrows, AD its axis, and CD an ordinate; also let the constant tension at A be denoted by  $a$ , and put  $a = AD = 3$  feet,

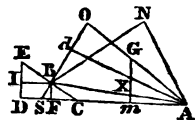
$db$  will represent the transverse diameter of the surface of the liquor remaining, Now, by trigonometry, we find  $bn = 1.409$ ,  $df = 1.054$ , and  $ve = 1.496$ . And because of similar triangles  $\sqrt{(ab \times df)} = 1.2552$  is the conjugate to the transverse  $db$ , and therefore  $db \times 1.2552 \times .7854 \times \frac{1}{3} ve = .689796$  is the content of the oblique cone  $bev$ . Again,  $1.5' \times .7854 \times \frac{1}{3} = 1.1781$  is the content of the whole glass; the half of which, or  $.589$  is equal to  $dvb$ . Then, similar cones being as the cubes of their heights, we have  $\sqrt[3]{.689796} : \sqrt[3]{.589} :: 1.496 : 1.4192 = ve$ ; and therefore  $.589 \times 3 \div 1.4192 = 1.247$  is the area of the surface required.

XII. QUESTION 1010, by Mr. Colin Campbell, Kendal.

If one end of a block  $AB$  rest upon a horizontal plane  $ACD$ , and the other be supported by a wedge  $CDE$  in form of a right-angled triangle; the length and weight of the block, as also the base  $CD$ , and perpendicular  $DE$  of the wedge being given. Query what force must be applied to the back of the wedge parallel to the horizon to keep the block from falling, when a given part  $CF$  of the base of the wedge is introduced underneath it, and the extremity  $A$  fixed so as to be prevented from sliding?

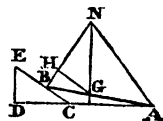
*Answered by Amicus.*

Through  $g$  the centre of gravity of the block  $BNA$  draw  $AO$ ; let fall  $em$  perpendicular to the horizon  $CA$ ,  $BO$  perpendicular to  $CE$  at  $B$ , and produce it to the horizon at  $s$ ;  $Ad$  perpendicular to  $BO$ ,  $BF$  parallel to  $em$ , and  $IB$  to  $SA$ . Then, by Simpson's Fluxions, art. 450,  $(Am \div Ad) \times w =$  the force which acting at  $B$  in direction  $OB$ , would sustain the block; which resolved into the directions  $BF$  and  $BI$ , becomes  $(BF \cdot Am \div BS \cdot Ad) \times w$  and  $(SF \cdot Am \div BS \cdot Ad) \times w$ , the former is destroyed by the horizontal plane  $SA$ , the latter therefore is that required  $= (sd \cdot Am \div SA \cdot Ad) \times w = (mx \div SA) \times w$ .



*The same answered by Mr. W. Armstrong, Newcastle.*

From  $g$  the centre of gravity of the block draw the indefinite line  $GN$  perpendicular to the horizontal plane  $ACD$ ; make  $BN$  perpendicular to  $CE$  to intersect  $GN$  in  $N$ , and join  $NA$ , parallel to which draw  $GH$  meeting  $BN$  in  $H$ . Then will  $GN : NH :: w$  (the weight of the block) :  $(HN \div GN) \times w$  the force acting perpendicularly on the side  $CE$  of the wedge; and therefore, by the principles of that mechanical power,  $CE : DE :: (HN \div GN)w : (DE \cdot HN \div CE \cdot GN)w = (DE \cdot CD \div CE^2) w$ , the force required acting perpendicularly against  $DE$ , because the triangles  $GUN$  and  $CDE$  are similar.



XIII. QUESTION 1011, *by Mr. John Garnet.*

If an 80lb. weight with a velocity of 32 feet per second, striking against a spring, be found to bend it 2 inches; required the elastic force of the spring so bent, with the velocity lost and the time in describing each 8th part of an inch?

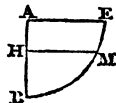
*Answered by Mr. Colin Campbell, of Kendal.*

Let  $r$  = the retarding force of the spring when the motion of the weight is just destroyed,  $x$  = any variable distance through which the spring is bent in the time  $t$ ,  $v$  = the co-temporary velocity of the weight, and  $g = 193$  inches =  $16\frac{1}{2}$  feet. Then, the action of the spring being in a constant ratio to the distance  $x$ , or (Hutton's Dictionary, vol. II, page 488) as the compressing force, we have  $2 : f :: x : \frac{1}{2}fx$  = the retarding force at  $x$ ; and therefore (Hutton's Conic Sections and Select Exercises, theorem 10, page 169)  $v\dot{x} = -2g \times \frac{1}{2}fx = -gfx$ ; the correct fluent of this gives  $c^2 - v^2 = gfx^2$ , where  $c$  is the velocity of the weight when it first strikes the spring, = 32 feet = 384 inches; hence  $v = \sqrt{c^2 - gfx^2}$ . To find the greatest elastic force of the spring when bent, which is when  $v = 0$ , and  $x = 2$ ; take  $v = 0$ , then  $0 = c^2 - gfx^2 = c^2 - 4gf$ , therefore  $f = c^2 \div 4g = 191$ ; consequently 80f = 15280lb. is the weight equivalent to the spring's force at the two inch bend. Also substituting the value of  $f$  instead of it in the value of  $v$ , gives  $v = \sqrt{c^2 - \frac{1}{2}c^2x^2} = \frac{1}{2}c\sqrt{4 - x^2} = 192\sqrt{4 - x^2}$  for the general value of  $v$ . And substituting  $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}$ , &c. successively for  $x$  in the value of  $v$ , these values become  $24\sqrt{16^2 - 1^2}$ ,  $24\sqrt{16^2 - 2^2}$ ,  $24\sqrt{16^2 - 3^2}$ , &c. the differences of which give 7507, 22618, 37978, 53837, &c. for the several velocities lost in describing every 8th part of an inch.

Again, for the time,  $t = \frac{x}{v} = \frac{2}{c} \cdot \frac{x}{\sqrt{4 - x^2}}$ , the fluent of which gives  $t = \frac{2}{c} \times \text{arc whose sine is } \frac{1}{2}x \text{ to radius 1.}$  Then, by taking  $x$  successively equal to  $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}$ , &c. the differences of the results will be the times of describing every 8th part of an inch, which comes out .000326, .000327, .000329, .000334, &c. of a second; and the whole time is .0081812 of a second.

*The same by Amicus. (Suppl.)*

By many experiments made with springs it appears that the bending force is always as the space bent through. With radius  $AB = 2$  inches, the space bent through in the present case, describe a quadrant  $BME$ ; and let  $AH$  be any part of this space, where the velocity at  $H$  is =  $v$ , let  $f$  = the bending force at  $B$ ; then  $AB : f :: AH : \text{the bending force at } H$ , which  $\propto AH = -v\dot{v}$ ; but —



$AM \times A'H = HM \times H'M$ , consequently  $\frac{1}{2}f \times HM^2 = v^2$ ; but when  $HM = 2$ ,  $v = 32$  feet  $= 384$  inches, and  $2f = 384^2$ , or  $f = 192 \times 384 = 73728$  inches per second. Moreover  $v = HM \times 192$ , the elementary time  $A'H$  divided by  $v$  is the arch  $E'M \div 192 AE$ , and the time of bending through  $AH = \frac{EM}{192 AE}$ , and through  $AB = \frac{EMB}{192 AE} = \frac{1.5708 \times 2}{192 AE}$

$= 0.008181$  the whole time. Whence that of describing every 8th of an inch  $=$  that of passing through the arches whose sines are  $\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$ , &c. to radius 1 becomes known, &c.

Since the bending or resisting force  $\frac{1}{2}f \times AH$  is variable, the fluent of  $\frac{1}{2}f \cdot AH \cdot A'M$ , where  $AH = 2 = AB$ , or  $\frac{1}{2}f \cdot AB^2$ , must be the whole action of the spring, compared with that of gravity in describing the same right line  $AB$ , will be as  $\frac{1}{2}f \cdot AB$  to  $2 \times 193$ , or  $\frac{1}{2}f$  to  $2 \times 193$  the measure of gravity, or as 96 to 1 nearly. It would therefore require the gravity of a weight  $= 96 \times 80 = 7680$ lb. proceeding from rest at  $A$ , to bend the spring through the same distance  $AB$ .

*The same answered by Mr. J. Nicholson, Newcastle. (Suppt.)*

Let  $f$  = the force, or body  $\times$  velocity,  $b$  = the body,  $a$  = the distance the spring is bent,  $s = 32$  the velocity,  $g = 16 \frac{1}{12}$ ,  $x$  = any variable distance the spring is bent by the force, and  $v$  = the velocity at  $x$ . Then, by Dr. Hutton's Mathematical and Philosophical Dictionary, vol. 2, page 488, the intensity being as the compressing force, we have  $a : f :: x : fx \div a$  = the force at  $x$ ; but the velocity generated or destroyed in any given time being as the force and time directly and

body inversely, we get  $\frac{2gfx}{abv} = -v$ , or  $\frac{2gfx}{ab} = -v^2$ , the cor-

rect fluent of which gives  $\frac{2gfa}{ab} = s^2 - v^2$ . Now, taking here  $x = a$ ,

and  $v = 0$ , this becomes  $\frac{2gfa}{b} = s^2$ ; hence  $f = \frac{bs^2}{2ga}$  the general

value of  $f$ , which in the present case is  $= 191b = 15280$  pounds. Also the general value of  $f$  substituted in the general equation for  $v$ ,

gives  $v = \frac{s}{a} \sqrt{(a^2 - x^2)}$  the general value of  $v$ .

Hence, for the time,  $t = \frac{x}{v} = \frac{ax}{s\sqrt{(a^2 - x^2)}}$ ; the fluent of which

is  $t = \frac{a}{s} \times$  arc whose sine is  $\frac{x}{s}$  to radius 1; and the whole time of

describing the two inches is  $\frac{a}{s} \times$  the quadrantal arc to radius 1.

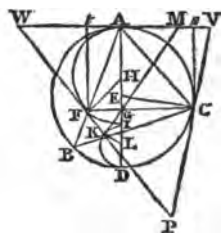
Mr. Nicholson then calculates and sets down all the times and the velocities lost for every 8th part of an inch.

XIV. QUESTION 1012, *by Amicus.*

In a certain annual publication, the 768th Diary question is reposed, and an answer given in the succeeding number by Mr. Brookes, of Leeds, making the triangle the same as the greatest under a given base and ratio of the sides, which is utterly wrong. It is therefore required to give a purely geometrical answer to the question, and more simple than that in the Diary for 1782?

*Answered by Amicus.*

**Construction.** Let  $E$  be the centre and  $AD$  the diameter of the given circle; take  $AI$  to  $AD$  in the given duplicate ratio of the sides, and on the diameters  $AI$ ,  $DI$ , and centres  $H$  and  $L$ , describe the semicircles  $AFI$  and  $IKD$ , on a tangent at  $A$  take  $AM = 3IL$ , and from  $M$  draw a tangent  $MK$  to the semicircle  $IKD$ ; and through  $G$ , where it cuts  $AD$ , draw  $FC$  perpendicular to  $AD$ ; through  $F$  draw  $AB$ , then join  $BC$ ,  $AC$ , and  $CAB$  is the required triangle.



**Demonstration.** By similar triangles,  $AI : AD :: AF : AB :: AI : AG = AF^2 : AD \cdot AG = AC^2 = AB \cdot AF$ , therefore  $AB^2 : AC^2 :: AC^2 : AF^2 :: AD : AI$  the given duplicate ratio per construction, and  $::$  triangle  $ABC : \text{triangle } AFC$ ; therefore when the triangle  $ABC$  is a maximum the triangle  $AFC$  is a maximum. Through  $F$  and  $C$  draw the tangents  $WF$ ,  $VP$ . Then by Simpson's Geometry on Maxima and Minima  $WF = VP$ , and  $VC = CF$ ; consequently  $VC = VA$ ,  $WF = WA$ ,  $WF + VC = WV = 2FC$ ; and drawing  $tw$  and  $sc$  parallel to  $AD$ ,  $tw + vs = FC$ , consequently  $WF + tw + VC + vs = 3FC$ . By similar triangles, as  $CE : EG + EC = AG :: CS = AG : vs + VC$ , and  $FG : FH + HG = AG :: AG : tw + WF$ ; consequently  $FG \cdot GC = FG + GC = FC :: AG^2 : WF + tw + VC + vs = 3FC$ , or  $AG^2 = 3FG \cdot GC$ ; but  $FG^2 \cdot GC^2 = AG^2 \cdot GI \cdot GD = AG^2 \cdot GK^2$ ; therefore  $AG^2 = 3FG \cdot GC = 3AG \cdot GK$ , or  $AG = 3GK$ , and  $AM = 3KL$ , as by construction.

It is manifest, from the construction, that the problem is always possible, let the given ratio of the sides be what it will, and that the perpendicular will always fall within the triangle; whereas, according to Mr. Brookes, it would always fall without it.

Mr. George Sanderson gives the construction exactly in the same manner, and for the demonstration, he refers to his solution and scholium of question 800, in the Diary for 1783.

*The same answered by Mr. Wm. Armstrong, Newcastle.*

Suppose the thing done,  $ABC$  the required triangle; and  $AB$  to  $AC$  the



question is founded, and given by different correspondents, we shall insert here, and the Supplement, some specimens of them both.

*Answered by Amicus.*

This question is so indefinitely worded, that it may be taken in different senses : but if the first minute be supposed to commence when the man heard the first crack, and to end when he heard the 70th, at the instant the coach passed him. Let  $y$  = the distance of the man and coach when the first crack was given at  $P$ ,  $m = 4$  the man's velocity,  $s = 1142 \times 20 \times 60 \div 1760 = 778\frac{1}{4}$  the velocity of sound,  $d = s - m$ ,  $x$  = the velocity of the coach, and  $69a$  = the distance gone by the man in the minute before the coach came up to him. Then will the distance of the man from  $P$ , when he heard the first crack, be  $sy \div d$ , and betwixt giving the first crack and the man's hearing it, the coach had gone the distance  $xy \div d$ , the man's distance from  $P$  when he heard the 70th was  $sy \div d + 69a$ , therefore the coach had gone in one minute the distance

$$\frac{sy}{d} + 69a - \frac{xy}{d} = 69a + \frac{s-x}{d} \cdot y, \text{ and } m : x :: 69a : 69a + \frac{s-x}{d}y, \text{ hence } y = 69a \times \frac{d}{m} \times \frac{x-m}{s-x}.$$

In the second minute the coach at the 68th crack had gone  $68a + 68sy \div 69d$ , and the man when he heard it had gone  $69d$ , therefore the distance gone over by sound betwixt giving and hearing the 68th crack was  $\frac{68sy}{69} - a = \frac{68asx - 69ams + amx}{ms - mx}$ ; the man had gone

when the crack was given  $\frac{68msy}{69dx} + \frac{68am}{x}$ , which being taken from

$69a$ , gives the distance he had gone after it was given before he heard it: and as this quantity =  $\frac{as - 69ax + 68am}{s-x} : \frac{68asx - 69ams + amx}{mx - ms}$

$\therefore m : s$ , hence  $x = s \times \frac{s + 137m}{137s + m} = 9 \frac{8817594}{12907939}$  miles per hour, the velocity required.

*The same by Mr. Colin Campbell, of Kendal. (Supp.)*

Let  $x$  = the speed of the coach per minute,  $y$  = the circumference of the wheel,  $1142 \times 60 \div 1760 \times 3 = 12\frac{1}{4}$  miles =  $a$ , the velocity of sound per minute, and  $\frac{4}{60} = \frac{1}{15} = b$ . Then  $x : 1 :: y : y \div x$  = the time in which a revolution of the wheel was performed, and  $1 : b :: y \div x : by \div x$  = the distance walked in that time; so that the coach approaches to or recedes from him  $y - by \div x$  in every rotation of the wheel; and consequently he and the coach were distant  $69 \cdot (y - by \div x)$  when the first of the 70 cracks was made, by hypot-

$\frac{1}{2}AB^2$ ; hence  $AC^2 = 4BC^2$ , and  $AC = 2BC$ ; consequently  $AC + BC : AC - BC :: 3 : 1$ , the ratio of the sum of the sides to their difference; and  $AC^3 + BC^3 : AC^3 - BC^3 :: 9 : 7$ , the ratio of the sum of the cubes to the difference of the cubes; therefore the former ratio is to the latter ratio; or  $\frac{3}{1} : \frac{9}{7} :: 7 : 3$ .

*The same answered by Mr. John Surtees, of Sunderland.*

Draw  $AB = 3$ ,  $BD = 1$ , and  $DC$  (perpendicular to  $AB$ )  $= 2$ ; join  $AC$  and  $BC$ , then shall  $ABC$  be the triangle sought. For, let  $x$  and  $y$  denote the two sides; then, by the question,  $(x + y) \div (x - y) : (x^2 + y^2) \div (x^2 - y^2) :: 7 : 3$ ; hence  $x = 2y$ , so that  $AC = 2BC$ ; and from the other ratio the base and perpendicular are as 3 to 2. Again, let  $y = BC$ ,  $2y = AC$ ,  $3z = AB$ , and  $2z = DC$ ; then  $\sqrt{(4y^2 - 4z^2)} = \sqrt{(y^2 - 4z^2)} = 3z$ ; hence  $y = z\sqrt{5}$ ; so that when  $AB = 3$ ,  $AC = 2\sqrt{5}$ ; and  $BC = \sqrt{5}$ , consequently  $BD = 1$ , and  $CD = 2$ .

X. QUESTION 1008, by Mr. J. T. Connor, Kensington Academy.

A very fine thread of 60 yards being wrapped quite round a pole of one foot diameter, it is required to find the distance that a person must walk to unfold it quite off the pole, or to find the length of the spiral described by the end of the thread in unwinding?

*Answered by Mr. William Bardon.*

This question is similar to the 431st Diary question, and by either of the theorems in the answers to it, the distance travelled is found to be 6 miles 240 yards.

*The same answered by Mr. J. Facer, Adderbury School.*

Let  $y = 60$  yards  $= 180$  feet, the length of the thread: and  $a = \frac{1}{2}$ , the radius of the pole; then by art. 139, Simpson's Fluxions, we have  $(y^2 - a^2) \div 2a = 32399\frac{1}{2}$  feet; or 6 miles  $239\frac{1}{2}$  yards, the distance travelled, or length of the spiral sought.

*The same answered by Mr. Alexander Rowe, Reginnis.*

Put  $a = \frac{1}{2}$  foot, the semidiameter of the pole;  $b =$  the thickness of the tape or thread wrapped upon it,  $n =$  the whole number of rounds  $= 60 \times 3 \div 3.14159$ , &c.  $= 57.29578$ ; and  $p = 3.14159$ , &c.  $\times 2 = 6.2832$ . Then, by the Ladies Diary question 431, for the year 1757, we have this theorem (being the result of the process there) viz,  $n^2 p^2 \times (\frac{1}{2}a + \frac{1}{2}nb) = \frac{1}{2}an^2 p^2$  when  $b = 0 = 32400$  feet  $= 6$  miles 240 yards, as required.

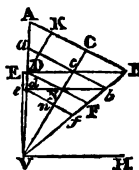
XI. QUESTION 1009, by the Rev. Mr. L. Evans.

A gentleman having drunk half a conical glass of wine, the diameter

of which being  $1\frac{1}{2}$  inch, and depth 2 inches, inclined its axis to an angle of  $65^\circ$  with the plane of the horizon. Query the area of the surface of the wine in that position?

*Answered by the Rev. Mr. Evans, the Proposer.*

Let  $ABV$  represent the cone, with its axis  $vc$  making an angle of  $65^\circ$  with the horizon  $vh$ ;  $bd$  the surface of the wine, and  $bd^*$  parallel to it; also  $ab$  and  $df$  parallel to  $AB$ . In the triangle  $cvb$ , we have  $cb$  and  $cv$ , to find  $\angle cvb$  or  $cva = 20^\circ 33' 21.7''$  and the complement of  $cva$  is the  $\angle a$  or  $d = 69^\circ 26' 38.3''$ . In the triangle  $ADB$ , are given  $AB$  and the angles  $A$  and  $B$ , to find the side  $DB = 1.4087$ . The complement of the  $\angle cvh$  is  $\angle cve = \angle kbd = 25^\circ$ ; hence  $DK = .5953$ , and  $vc - DK = vn = 1.40465$ ; then, by similar triangles,  $vc : AB :: vn : DF = 1.0534$ ; by the property of the cone  $\sqrt{(AB \times DF)} = 1257$  the conjugate to the transverse  $BD$ ; also  $\sqrt{(cv^2 + cb^2)} = bv = 2.136$ , and  $\angle ebv = bv h = 44^\circ 26' 38.3''$ ; hence  $ve = 1.4956$ ; then the solidity of the cone  $vdb = .6934 : db^3 ::$  the solidity of the cone  $vab = .589$ , being half the cone  $vab : db^3$ , therefore  $db = 1.3341$ . Again,  $DB : AB :: db : ab = 1.4206$ , and  $DB : DF :: db : df = .99774$ , then likewise by conics  $\sqrt{(ab \times df)} = 1.1905$  the conjugate to  $db$ , and  $db \times 1.1905 \times .7854 = 1.2475$  the area required.



*The same answered by Mr. Henry Hunter, Shillbottle, Alnwick.*

Let  $ABV$  be the conical glass,  $dbv$  the quantity of wine left; draw  $BE$  parallel to  $bd$ , and  $VE$  perpendicular to  $BE$ . Then, since the cone  $ABV$  is given, all the angles are given, and the slant side  $AV$ ; also in the triangle  $ABD$ , all the angles are given, and the side  $AB$ , to find  $AD$ ; and in the triangle  $BDF$ , are given all the angles and the side  $BD$ , to find  $DF$ . Then by conics  $\sqrt{(AB \times DF)} =$  the conjugate of the transverse  $BD$ , and hence the area of that ellipse is  $BD \times \sqrt{(AB \times DF)} \times .7854$ . Again, by similar triangles,  $AB : BV :: DF : DV$ ; then in the triangle  $VED$ , are given all the angles and the side  $VD$ , to find  $VE$ ; whence the solidity of the oblique elliptical cone  $DBV$  is  $= DB \times \sqrt{(AB \times DF)} \times VE \times .2618$ ; then by similar solids, the cone  $DBV : \text{cone } dbv :: db^3 : bd^3$ , and by similar ellipses  $DB^2 : db^2 ::$  the former ellipsis to the ellipsis on  $db$ , which comes out  $1.247565$ , the area required.

*The same by Mr. Richard Embleton, Stony Hills.*

Draw  $vh$  for the plane of the horizon, and  $cv$  making the given angle of  $65^\circ$  with it; upon which as an axis let the given cone  $ABV$  be constructed. And as the liquor will always (in whatever position the glass may stand) be parallel to the plane of the horizon, let  $bd$  and  $bd^*$  be drawn parallel to it; also  $ab$ ,  $df$ ,  $df^*$  parallel to  $AB$ . Then the oblique cones  $BVD$  and  $bvd$  will be similar, and their bases ellipses, and

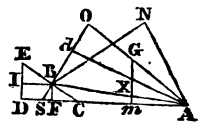
$db$  will represent the transverse diameter of the surface of the liquor remaining, Now, by trigonometry, we find  $bn = 1.409$ ,  $df = 1.054$ , and  $ve = 1.496$ . And because of similar triangles  $\sqrt{(AB \times df)} = 1.2552$  is the conjugate to the transverse  $db$ , and therefore  $db \times 1.2552 \times .7854 \times \frac{1}{3} ve = .689796$  is the content of the oblique cone  $dbv$ . Again,  $1.5^3 \times .7854 \times \frac{1}{3} = 1.1781$  is the content of the whole glass; the half of which, or  $.589$  is equal to  $dvb$ . Then, similar cones being as the cubes of their heights, we have  $\sqrt[3]{.689796} : \sqrt[3]{.589} :: 1.496 : 1.4192 = ve$ ; and therefore  $.589 \times 3 \div 1.4192 = 1.247$  is the area of the surface required.

XII. QUESTION 1010, by Mr. Colin Campbell, Kendal.

If one end of a block  $AB$  rest upon a horizontal plane  $ACD$ , and the other be supported by a wedge  $CDE$  in form of a right-angled triangle; the length and weight of the block, as also the base  $CD$ , and perpendicular  $DE$  of the wedge being given. Query what force must be applied to the back of the wedge parallel to the horizon to keep the block from falling, when a given part  $CF$  of the base of the wedge is introduced underneath it, and the extremity  $A$  fixed so as to be prevented from sliding?

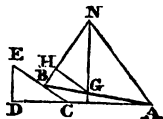
*Answered by Amicus.*

Through  $G$  the centre of gravity of the block  $BA$  draw  $AO$ ; let fall  $em$  perpendicular to the horizon  $CA$ ,  $BO$  perpendicular to  $CE$  at  $B$ , and produce it to the horizon at  $s$ ;  $Ad$  perpendicular to  $BO$ ,  $BF$  parallel to  $em$ , and  $IB$  to  $SA$ . Then, by Simpson's Fluxions, art. 450,  $(Am \div Ad) \times w =$  the force which acting at  $B$  in direction  $OB$ , would sustain the block; which resolved into the directions  $BF$  and  $BI$ , becomes  $(BF \cdot Am \div BS \cdot Ad) \times w$  and  $(SF \cdot Am \div BS \cdot Ad) \times w$ , the former is destroyed by the horizontal plane  $SA$ , the latter therefore is that required  $= (sd \cdot Am \div SA \cdot Ad) \times w = (mx \div SA) \times w$ .



*The same answered by Mr. W. Armstrong, Newcastle.*

From  $G$  the centre of gravity of the block draw the indefinite line  $GN$  perpendicular to the horizontal plane  $ACD$ ; make  $BN$  perpendicular to  $CE$  to intersect  $GN$  in  $N$ , and join  $NA$ , parallel to which draw  $GH$  meeting  $BN$  in  $H$ . Then will  $GN : NH :: w$  (the weight of the block) :  $(HN \div GN) \times w$  the force acting perpendicularly on the side  $CE$  of the wedge; and therefore, by the principles of that mechanical power,  $CE : DE :: (HN \div GN)w : (DE \cdot HN \div CE \cdot GN)w = (DE \cdot CD \div CE^2) w$ , the force required acting perpendicularly against  $DE$ , because the triangles  $GUN$  and  $CDE$  are similar.



## XIII. QUESTION 1011, by Mr. John Garnet.

If an 80lb. weight with a velocity of 32 feet per second, striking against a spring, be found to bend it 2 inches; required the elastic force of the spring so bent, with the velocity lost and the time in describing each 8th part of an inch?

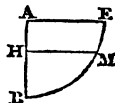
*Answered by Mr. Colin Campbell, of Kendul.*

Let  $r$  = the retardive force of the spring when the motion of the weight is just destroyed,  $x$  = any variable distance through which the spring is bent in the time  $t$ ,  $v$  = the co-temporary velocity of the weight, and  $g = 193$  inches =  $16\frac{1}{2}$  feet. Then, the action of the spring being in a constant ratio to the distance  $x$ , or (Hutton's Dictionary, vol. II, page 488) as the compressing force, we have  $2 : f :: x : \frac{1}{2}fx$  = the retardive force at  $x$ ; and therefore (Hutton's Conic Sections and Select Exercises, theorem 10, page 169)  $v\dot{v} = -2g \times \frac{1}{2}fx\dot{x} = -gfx\dot{x}$ ; the correct fluent of this gives  $c^2 - v^2 = gfx^2$ , where  $c$  is the velocity of the weight when it first strikes the spring, = 32 feet = 384 inches; hence  $v = \sqrt{c^2 - gfx^2}$ . To find the greatest elastic force of the spring when bent, which is when  $v = 0$ , and  $x = 2$ ; take  $v = 0$ , then  $0 = c^2 - gfx^2 = c^2 - 4gf$ , therefore  $f = c^2 \div 4g = 191$ ; consequently 80f = 15280lb. is the weight equivalent to the spring's force at the two inch bend. Also substituting the value of  $f$  instead of it in the value of  $v$ , gives  $v = \sqrt{c^2 - \frac{1}{2}c^2x^2} = \frac{1}{2}c\sqrt{4 - x^2} = 192\sqrt{4 - x^2}$  for the general value of  $v$ . And substituting  $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}$ , &c. successively for  $x$  in the value of  $v$ , these values become  $24\sqrt{16^2 - 1^2}$ ,  $24\sqrt{16^2 - 2^2}$ ,  $24\sqrt{16^2 - 3^2}$ , &c. the differences of which give .7507, 2.2618, 3.7978, 5.3837, &c. for the several velocities lost in describing every 8th part of an inch.

Again, for the time,  $t = \frac{x}{v} = \frac{2}{c} \cdot \frac{x}{\sqrt{4 - x^2}}$ , the fluent of which gives  $t = \frac{2}{c} \times \text{arc whose sine is } \frac{1}{2}x \text{ to radius 1}$ . Then, by taking  $x$  successively equal to  $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}$ , &c. the differences of the results will be the times of describing every 8th part of an inch, which comes out .000326, .000327, .000329, .000334, &c. of a second; and the whole time is .0081812 of a second.

*The same by Amicus. (Suppl.)*

By many experiments made with springs it appears that the bending force is always as the space bent through. With radius AB = 2 inches, the space bent through in the present case, describe a quadrant BME; and let AH be any part of this space, where the velocity at H is =  $v$ , let  $f$  = the bending force at B; then AB :  $f ::$  AH : the bending force at H, which  $\times A'H = -v\dot{v}$ ; but —



$AM \times A'H = HM \times H'M$ , consequently  $\frac{1}{2}f \times HM^2 = v^2$ ; but when  $HM = 2$ ,  $v = 32$  feet = 384 inches, and  $2f = 384^2$ , or  $f = 192 \times 384 = 73728$  inches per second. Moreover  $v = HM \times 192$ , the elementary time  $A'H$  divided by  $v =$  the arch  $E'M \div 192 AE$ , and the time of bending through  $AH = \frac{EM}{192 AE}$ , and through  $AB = \frac{EMB}{192 AE} = \frac{1.5708 \times 2}{192 AE} = 0.008181$  the whole time. Whence that of describing every 8th of an inch = that of passing through the arches whose sines are  $\frac{1}{16}$ ,  $\frac{1}{8}$ ,  $\frac{1}{4}$ , &c. to radius 1 becomes known, &c.

Since the bending or resisting force  $\frac{1}{2}f \times AH$  is variable, the fluent of  $\frac{1}{2}f \cdot AH \cdot A'M$ , where  $AH = 2 = AB$ , or  $\frac{1}{2}f \cdot AB^2$ , must be the whole action of the spring, compared with that of gravity in describing the same right line  $AB$ , will be as  $\frac{1}{2}f \cdot AB$  to  $2 \times 193$ , or  $\frac{1}{2}f$  to  $2 \times 193$  the measure of gravity, or as 96 to 1 nearly. It would therefore require the gravity of a weight =  $96 \times 80 = 7680$ lb. proceeding from rest at  $A$ , to bend the spring through the same distance  $AB$ .

*The same answered by Mr. J. Nicholson, Newcastle. (Suppt.)*

Let  $f =$  the force, or body  $\times$  velocity,  $b =$  the body,  $a =$  the distance the spring is bent,  $s = 32$  the velocity,  $g = 16 \frac{1}{12}$ ,  $x =$  any variable distance the spring is bent by the force, and  $v =$  the velocity at  $x$ . Then, by Dr. Hutton's Mathematical and Philosophical Dictionary, vol. 2, page 488, the intensity being as the compressing force, we have  $a : f :: x : fx \div a =$  the force at  $x$ ; but the velocity generated or destroyed in any given time being as the force and time directly and body inversely, we get  $\frac{2gfx}{abv} = -v$ , or  $\frac{2gfx}{ab} = -v^2$ , the correct fluent of which gives  $\frac{2gfx^2}{ab} = s^2 - v^2$ . Now, taking here  $x = a$ , and  $v = 0$ , this becomes  $\frac{2gfa}{b} = s^2$ ; hence  $f = \frac{bs^2}{2ga}$  the general value of  $f$ , which in the present case is  $= 191b = 15280$  pounds. Also the general value of  $f$  substituted in the general equation for  $v$ , gives  $v = \frac{s}{a} \sqrt{(a^2 - x^2)}$  the general value of  $v$ .

Hence, for the time,  $t = \frac{x}{v} = \frac{ax}{s\sqrt{(a^2 - x^2)}}$ ; the fluent of which is  $t = \frac{a}{s} \times$  arc whose sine is  $\frac{x}{s}$  to radius 1; and the whole time of describing the two inches is  $\frac{a}{s} \times$  the quadrantal arc to radius 1.

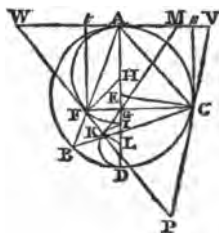
Mr. Nicholson then calculates and sets down all the times and the velocities lost for every 8th part of an inch.

## XIV. QUESTION 1012, by Amicus.

In a certain annual publication, the 768th Diary question is reposed, and an answer given in the succeeding number by Mr. Brookes, of Leeds, making the triangle the same as the greatest under a given base and ratio of the sides, which is utterly wrong. It is therefore required to give a purely geometrical answer to the question, and more simple than that in the Diary for 1782?

*Answered by Amicus.*

**Construction.** Let  $E$  be the centre and  $AD$  the diameter of the given circle; take  $AI$  to  $AD$  in the given duplicate ratio of the sides, and on the diameters  $AI$ ,  $DI$ , and centres  $H$  and  $L$ , describe the semicircles  $AFI$  and  $IKD$ , on a tangent at  $A$  take  $AM = 3IL$ , and from  $M$  draw a tangent  $MX$  to the semicircle  $IKD$ ; and through  $G$ , where it cuts  $AD$ , draw  $FC$  perpendicular to  $AD$ ; through  $F$  draw  $AB$ , then join  $BC$ ,  $AC$ , and  $CAB$  is the required triangle.



**Demonstration.** By similar triangles,  $AI : AD :: AF : AB :: AI : AG = AF^2 : AD \cdot AG = AC^2 = AB \cdot AF$ , therefore  $AB^2 : AC^2 :: AC^2 : AF^2 :: AD : AI$  the given duplicate ratio per construction, and  $::$  triangle  $ABC : \text{triangle } AFC$ ; therefore when the triangle  $ABC$  is a maximum the triangle  $AFC$  is a maximum. Through  $F$  and  $c$  draw the tangents  $WF$ ,  $VP$ . Then by Simpson's Geometry on Maxima and Minima  $WF = FA$ , and  $VC = CF$ ; consequently  $VC = VA$ ,  $WF = WA$ ,  $WF + VC = WV = 2FC$ ; and drawing  $tW$  and  $sc$  parallel to  $AD$ ,  $tW + vs = FC$ , consequently  $WF + tW + VC + vs = 3FC$ . By similar triangles, as  $cs : EG + EC = AG :: cs = AG : vs + VC$ , and  $FG : FH + HG = AG :: AG : tW + WF$ ; consequently  $FG \cdot GC : FG + GC = FC :: AG^2 : WF + tW + VC + vs = 3FC$ , or  $AG^2 = 3FG \cdot GC$ ; but  $FG^2 \cdot GC^2 = AG^2 \cdot GI \cdot GD = AG^2 \cdot GK^2$ ; therefore  $AG^2 = 3FG \cdot GC = 3AG \cdot GK$ , or  $AG = 3GK$ , and  $AM = 3KL$ , as by construction.

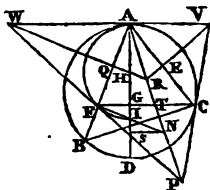
It is manifest, from the construction, that the problem is always possible, let the given ratio of the sides be what it will, and that the perpendicular will always fall within the triangle; whereas, according to Mr. Brookes, it would always fall without it.

Mr. George Sanderson gives the construction exactly in the same manner, and for the demonstration, he refers to his solution and scholium of question 800, in the Diary for 1783.

*The same answered by Mr. Wm. Armstrong, Newcastle.*

Suppose the thing done,  $ABC$  the required triangle; and  $AB$  to  $AC$  the

given ratio of the sides. From  $c$ , perpendicular to  $AD$  the diameter of the circumscribing circle, drop  $CG$ , which produce to meet  $AB$  in  $F$ ; then will the triangles  $ABC$  and  $AFC$  be similar, and having the side  $AC$  common, the triangle  $AFC$  will be a maximum when  $ABC$  is; and if a semicircle whose centre  $H$  is in  $AD$  be described through the points  $A, F$ , the problem will be the same as, to find a point  $G$  in  $AI$  through which a right line  $FC$  being drawn perpendicular to  $AI$ , and the points  $AF, AC$  joined, the triangle  $AFC$  may be a maximum, for it may easily be shewn that any triangle  $AFC$  inscribed between the peripheries  $AFI, ADC$ ; whose base  $FC$  is perpendicular to  $AD$ , will have its sides  $AC, AF$  in the given ratio of  $AB$  and  $AC$ . Now, from prop. 8, and Cor. Simpson on the Maxima and Minima, it will readily appear, that if tangents  $WAV, PFW, PCV$  be so drawn that the distances  $PW, PV$  be bisected in the points of contact  $F$  and  $C$ , the triangle  $AFC$  will be greater than any other whose base is parallel to  $WV$  which can be inscribed in the triangle  $WVF$ , and much more then must it be greater than any other inscribed between the peripheries  $AFI, ACD$ . To determine the point  $F$  from whence those tangents must be drawn, let  $WH$  be joined, and produced to meet  $AF$  in  $R$ ; then since  $AW = WF$ ,  $WH$  will bisect  $AF$  at right angles in  $Q$ , and therefore be parallel to  $FI$ , which suppose produced to cut  $AF$  in  $N$ ; and by similar triangles we shall have  $AR = RN = NP$ , and because  $AT = TP$ ,  $RT = TN$ , and  $AT = 3TN$ . From  $N$  demit  $NS$  perpendicular to  $AD$ , and join  $VR$ ; then since  $PW = 2WA$ , and  $PV = 2AV$ ,  $AP$  will bisect the angle  $WPF$ , and because  $WR$  bisects  $\angle W$ ,  $VR$  will bisect  $\angle V$  and the right line  $AC$  at right angles in  $E$ , and therefore  $R$  will be the centre of the circle which circumscribes the triangle  $AFC$ , which circle will also pass through the points  $N$  and  $S$ , and consequently  $AG \cdot GS = FG \cdot GC$ , or  $FG^2 : GS^2 :: AG^2 : AG \cdot GD = GC^2 :: AG : GD :: AG \cdot GI = GF^2 : GD \cdot GI = GS^2$ , or  $GS = \frac{1}{3}AG$  = a tangent drawn from  $G$ , to a circle described on the diameter  $ID$ , and therefore the construction as in the Diary for 1782.



PRIZE QUESTION, by Mr. John Rodham, *Richmond, Yorkshire.*

Walking uniformly at the rate of four miles an hour, on a straight road, on which was a coach travelling in the same direction at an uniform rate; one of its wheels giving a crack at every revolution, I counted or heard 70 of these cracks a minute before it passed me, and only 68 afterwards. Required, by a direct investigation, how many miles the coach travelled an hour; the motion of sound being considered?

*Answered*

There being two different hypotheses upon which the solution of this

question is founded, and given by different correspondents, we shall insert here, and the Supplement, some specimens of them both.

*Answered by Amicus.*

This question is so indefinitely worded, that it may be taken in different senses : but if the first minute be supposed to commence when the man heard the first crack, and to end when he heard the 70th, at the instant the coach passed him. Let  $y$  = the distance of the man and coach when the first crack was given at  $r$ ,  $m = 4$  the man's velocity,  $s = 1142 \times 20 \times 60 \div 1760 = 778\frac{1}{11}$  the velocity of sound,  $d = s - m$ ,  $x$  = the velocity of the coach, and  $69a$  = the distance gone by the man in the minute before the coach came up to him. Then will the distance of the man from  $r$ , when he heard the first crack, be  $sy \div d$ , and betwixt giving the first crack and the man's hearing it, the coach had gone the distance  $xy \div d$ , the man's distance from  $r$  when he heard the 70th was  $sy \div d + 69a$ , therefore the coach had gone in one minute the distance

$$\frac{sy}{d} + 69a - \frac{xy}{d} = 69a + \frac{s-x}{d} \cdot y, \text{ and } m : x :: 69a : 69a + \frac{s-x}{d}y, \text{ hence } y = 69a \times \frac{d}{m} \times \frac{x-m}{s-x}.$$

In the second minute the coach at the 68th crack had gone  $68a + 68sy \div 69d$ , and the man when he heard it had gone  $69d$ , therefore the distance gone over by sound betwixt giving and hearing the 68th crack was  $\frac{68sy}{69} - a = \frac{68asx - 69ams + amx}{ms - mx}$ ; the man had gone

when the crack was given  $\frac{68msy}{69dx} + \frac{68am}{x}$ , which being taken from  $69a$ , gives the distance he had gone after it was given before he heard it: and as this quantity =  $\frac{as - 69ax + 68am}{s - x} : \frac{68asx - 69ams + amx}{mx - ms}$

$:: m : s$ , hence  $x = s \times \frac{s + 137m}{137s + m} = 9 \frac{8817594}{12907939}$  miles per hour, the velocity required.

*The same by Mr. Colin Campbell, of Kendal. (Suppl.)*

Let  $x$  = the speed of the coach per minute,  $y$  = the circumference of the wheel,  $1142 \times 60 \div 1760 \times 3 = 124\frac{1}{11}$  miles =  $a$ , the velocity of sound per minute, and  $\frac{4}{60} = \frac{1}{15} = b$ . Then  $x : 1 :: y : y \div x = b :: y \div x : by \div x$  = the distance walked in that time; so that the coach approaches to or recedes from him  $y - by \div x$  in every rotation of the wheel; and consequently he and the coach were distant  $69 \cdot (y - by \div x)$  when the first of the 70 cracks was made, by hypot-

thesis. Now  $a - b : 1 :: 69 \cdot \left(y - \frac{by}{x}\right) : \frac{xy - by}{a - b} \times \frac{69}{x} =$   
the time taken up by the sound of the said crack in reaching his ear,  
and  $a + b : 1 :: 68 \cdot \left(y - \frac{by}{x}\right) : \frac{xy - by}{a + b} \times \frac{68}{x} =$  the time  
that elapsed before the sound of the last, or 68th crack was heard after  
its being made. Therefore, by the question,  $\frac{69y}{x} - \frac{xy - by}{a - b} \times$   
 $\frac{69}{x} = 1$ , and  $\frac{68y}{x} + \frac{xy - by}{a + b} \times \frac{68}{x} = 1$ ; hence, by resolving  
these equations we find  $x = \frac{a + 137b}{b + 137a} \times a = .161385$ : and there-  
fore the coach travelled  $6x = 9.6831$  miles an hour.

*The same answered by Mr. Geo. Stevenson. (Suppl.)*

The first minute beginning when the sound of the first crack meets the ear, and the second minute ends when the sound of the last 68 cracks meets the same. Now put  $x =$  the circumference of the wheel;  $b = 1142$  feet, the velocity of sound;  $a = 352$  feet, the space gone over by the man in 1 minute: then as  $b : 1'' :: 69x - a :$   
 $(69x - a) \div b$  the time the sound of the first crack flies to the man's ear at first; and as  $b : 1 :: 68x - a : (68x - a) \div b$  the time sound is in flying to the ear after 68 cracks; then  $69x : 68x :: 60 + (69x - a) \div b : 60 - (68x - a) \div b$ , a proportion which gives this equation  $69x \times \left(60 - \frac{68x - a}{b}\right) = 68x \times \left(60 + \frac{69x - a}{b}\right)$ ,  
which reduced gives  $x = 12.44075$ ; and hence the rate of travelling by the coach is easily found  $= 9.683114$  miles per hour.

*The same answered by Viator. (Suppl.)*

This question seems ambiguous; for it does not appear from the enunciation, whether the walker counted 70 cracks *in the last* minute before the coach overtook him, and 68 *in the next* minute after; or, *at the rate* of 70 before, and 68 afterwards. In the former case it admits of innumerable answers; but the limits may be determined thus: Suppose the coach overtook him exactly at the time of the 70th crack; then as all the motions are uniform, he will walk  $\frac{152}{8}$  feet in the interval between his hearing two cracks *before* the coach passed him; and  $\frac{352}{8}$  feet *afterwards*. Let  $x$  be the distance the person walked while the sound of the 69th crack was overtaking him;  $r = 1142 \div 5\frac{1}{2}$  (the velocity of sound divided by the rate of walking);  $a = 352$  the distance walked per minute;  $b = 68$ ;  $c = 69$ . Then  $rx + a \div c =$  the distance run by the coach between the 69th and 70th

cracks: and  $\left(\frac{a}{b} - \frac{a}{c} - x\right) \times r + \frac{a}{b}$  = the distance it run between the 70th and 71st cracks; therefore  $rx + \frac{a}{c} = \left(\frac{a}{b} - \frac{a}{c} - x\right) \times r + \frac{a}{b}$ ; hence  $x = \frac{rac - rab + ac - ab}{2rbc}$ ; consequently, while the person walked  $\frac{rac - rab + ac - ab}{2rbc} + \frac{a}{c}$ , the coach ran  $\frac{rac - rab + ac - ab}{2rbc} \times r + \frac{a}{c}$ ; which expressions (putting  $s = c + b$  and  $c - b = d$ ) are as  $s + d \div r$  and  $s + dr$ . Hence,  $s + d \div r : s + d \times r :: 4 \text{ miles} : 9\frac{5817504}{14907939}$  miles, the least number per hour for the coach.

If the coach is supposed to have passed the *moment* before the 71st crack; then  $b = 67$ , and  $c = 70$ , and the rate will be  $21\frac{206816}{3302989}$  miles, the greatest limit.

But if he counted *at the rate* of 70 cracks before, and 68 afterwards: the question is limited; for, in that case,  $b = 67$ ,  $c = 69$  and  $s = 136$ . Therefore  $s + d \div r : s + d \times r :: 4m : 15\frac{3379164}{4107164}$  miles, the distance per hour.

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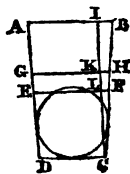
*Questions proposed in 1797, and answered in 1798.*

I. QUESTION 1014, by Mr. T. Hind, of Reading.

Having a pail filled six inches from the bottom with water, into which I immersed a globe of unknown dimensions, and observed the water to rise three inches higher in the vessel. Now the pail being sixteen inches perpendicular in depth, seven inches the bottom diameter, and nine inches the top; it is required to determine the diameter of the globe?

*Answered by Mr. A. Roulliers, of London.*

Let ABCD represent the pail, EF the surface of the water previous to the immersion of the globe; GH its surface after immersion; then will the solidity of the frustum GF be equal to that of the globe. Now, by similar triangles, as CI (16) : IB (1) :: CL (6) : LF ( $\frac{3}{8}$ ), hence EF =  $7\frac{1}{4}$ . And CI (16) : IB (1) :: CK (9) : KH ( $\frac{9}{16}$ ), hence GH =  $8\frac{1}{8}$ . Then, by mensuration,  $(GH^2 + GH \cdot EF + EF^2) \times KL \times .2618 = 148.47$  = the content of the frustum GF, or of the globe. Consequently  $\sqrt[3]{(148.47 \div .5236)} = 6.5698$  inches is the diameter of the globe required.



*The same by Miss Maria Middleton, of Eden.*

Here are given  $AB = 9$  inches,  $CD = 7$ ,  $CI = 16$ ,  $CL = 6$ , and  $KL = 3$ . Now, by similar triangles, as  $CI : IB :: CK : KH(\frac{9}{16}) :: CL : LF(\frac{6}{16})$ . Hence  $GH = 8.125$ , and  $EF = 7.75$ . Then, by mensuration,  $(GH \cdot EF + \frac{1}{2}(GH - EF)^2) \times KL \times .7854 = 148.4774$ , the content of the frustum  $GF$ , or of the globe; whose diameter will therefore be  $\sqrt[3]{(148.4774 \div .5236)} = 6.569$  inches.

*The same answered by Mr. W. Virgo, of Thornbury.*

In the vertical section  $ABCD$ , let  $AB = 9$ ,  $CD = 7$ ,  $EF$  the surface of the water before immersion, and  $GH$  the same after. Draw  $CI (= 16)$  perpendicular to  $CD$ . Then  $CI : IB :: CL : LF (= \frac{3}{8}) :: CK : KH = \frac{9}{16}$ ; consequently  $EF = 7\frac{1}{2}$ , and  $GH = 8\frac{1}{2}$ .

Then, by the Compendious Measurer, page 131, rule 3,  $(EF^2 + GH^2 + EF \cdot GH) \times 3 \times .2618 = 148.47736$  inches nearly, and (by page 137, ib.)  $\sqrt[3]{(148.47736 \div .5236)} = 6.57$  inches nearly, is the diameter.

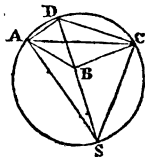
II. QUESTION 1015, by Mr. Wm. Marriot, of Neath.

Required a solution, both by geometry and trigonometry, to question 58, page 135, Dr. Hutton's Conic Sections.

*Question.* In a garrison there are three remarkable objects  $A, B, C$ , the distances of which from one to another are known to be  $AB$  213,  $AC$  424, and  $BC$  262 yards; I am desirous of knowing my position and distance at a place or station  $s$ , from whence I observed the angle  $ASB$   $13^\circ 30'$ , and the angle  $CSB$   $29^\circ 50'$ .

*Answered by the Rev. Mr. L. Evans, Froxfield.*

*Construction.* Make the triangle  $ABC$  with the three given lines; then draw the two lines  $AD$  and  $CD$  so as to make the angle  $DAC = 29^\circ 50'$ , and the angle  $DCA = 13^\circ 30'$ ; and through the three points  $A, D, C$  draw a circle; lastly, through the two points  $D$  and  $B$  draw the straight line  $DBS$  meeting the circle in  $s$ , which will be the place of the observer required. For, (joining  $AS, CS$ ) the angle  $ASD$  is equal to the angle  $ACD$ , and the angle  $CSB$  is equal to the angle  $DAC$ , being angles on the same segment.



*Calculation.* In the triangle  $ADC$ , are given the base  $AC$ , and the adjacent angles; to find  $AD = 144.2361$ , and  $CD = 307.3712$ . In each of the triangles  $DAB$  and  $DCB$ , are given two sides and an included angle, to find the angles  $ADB (= ACS) = 78^\circ 37' 13''.92$ , and  $BDC (= SAC) = 58^\circ 2' 46''.06$ . The angles  $CAB$  and  $ACB$  being found  $= 29^\circ 57' 6''.25$  and  $23^\circ 56' 50''.2$  respectively, we have the angle  $BAS = 28^\circ 5' 39''.82$ , and the angle  $BCS = 54^\circ 40' 23''.73$ ; hence  $BS = 429.6814$ ;  $CS = 524.2365$ , and  $AS = 605.7122$ .

*The same by Mr. J. Shirreff, at Mr. Green's Academy, Deytford.*

Make the triangle ABC as given; then upon AC describe the segment of a circle capable of the sum of the given angles at s ( $43^{\circ} 20'$ ), and upon AB describe another segment to contain the angle ASB ( $13^{\circ} 30'$ ), intersecting each other at s, the station required.

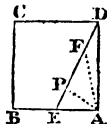
*Calculation.* In the triangle ABC, the three sides being given, the angles are therefore found by trigonometry, viz.  $\angle BAC = 29^{\circ} 57'$ , the  $\angle ACB = 23^{\circ} 58'$ , and the  $\angle ABC = 126^{\circ} 5'$ . In the triangle ADC are given the side AC and all the angles, to find the sides AD = 144.24, and CD = 307.4. In the triangle BCD are given two sides and the contained angle, to find the angle CBD =  $84^{\circ} 30'$ , and the angle BDC =  $58^{\circ} 2'$ ; therefore the angle CBS =  $95^{\circ} 30'$ , and the angle SCB =  $54^{\circ} 40'$ . In the triangle SBC are given the side BC and all the angles, to find BS = 429.5, and CS = 524. Lastly, in the triangle ABS are given AB, BS, and all the angles, to find the side AS = 605.5.

### III. QUESTION 1016, by Mr. Jonathan Horn, of Briscoe.

Being employed to survey a field, which I was told was an exact geometrical square, but by reason of a river which runs through part of it, I could only measure from the west corner on the south side nine chains, or BE, and EF eighteen chains in a line from E towards D, and there took an angle EFA of  $28^{\circ} 30'$ , from hence the area of the field is required?

*Answered by Mr. John Surtees.*

Put  $a = 9 = BE$ ,  $2a = 18 = EF$ ,  $s =$  sine of the given  $\angle F$ , to radius 1, and  $x =$  the segment AE. Then as  $x : s :: 2a : 2as \div x = \sin. \angle EAF$ , and its cosine  $\sqrt{(1 - 4a^2s^2 \div x^2)}$  is the sine of  $\angle DAF$ ; hence  $\sin. F : \sin. DAF :: AD : DF = (a + x) \sqrt{(1 - 4a^2s^2 \div x^2)} \div s$ ; also  $DF = DE - EF = \sqrt{(AE^2 + AD^2)} - EF = \sqrt{(a^2 + 2ax + 2x^2)} - x$ ; these two made equal, give  $x = 8.596877$ . Hence the side of the square AB is 17.596877 chains; and the area = 30 acres, 3 roods, 35 perches.



*The same answered by Amicus.*

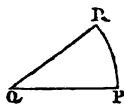
Let fall AF perpendicular to DE. Then, if AP be radius, AE and AD are the secant and cosecant of  $\angle PAE = ADE$ , and their difference = BE = 9; also EF = the tangent of  $\angle PAE$ , and PF that of  $\angle PAF = 61^{\circ} 30'$ ; hence as tang. of  $61^{\circ} 30' (= 1.8417709) + \text{tang. } \angle ADE : \text{the diff. between the secant and cosecant of } ADE :: 18 : 9 :: 2 : 1$ ; hence, by comparison with the tables, the  $\angle ADE = 26^{\circ} 24'$ , &c.

### IV. QUESTION 1017, by Mr. Wm. Burdon, Auster Malbis.

June 6, 1794, at six in the morning, the altitude of the sun was observed to be one third of the latitude: required the place of observation?

*Answered by Mr. Richard Nicholson, Teacher of the Mathematics, in the Rev. C. Vincent's Academy, Leeds.*

Let  $QR$  represent part of the horizon,  $Qr$  the sun's declination on the given day,  $PR$  his altitude at six o'clock, and the angle  $PQR =$  the required latitude. Put the sine of  $PQ = d$ , that of  $PR = x$ , then will the sine of three times the arch  $PR$  be  $= 3x - 4x^3$ , which, per question, is the sine of the angle  $PQR$ ; whence by spherics,  $d : \text{radius} :: 1 : x : 3x - 4x^3$ ; hence  $x = \sqrt{((3d - 1) \div 4d)}$   $= .3203499$  the sine of  $18^\circ 41' 2''$ ; the triple of which gives  $56^\circ 3' 6''$  for the required latitude.



*The same by Mr. Hugh Hutchinson, Schoolmaster, of Allendaletown.*

Let  $a = .3861744$  the natural sine of  $22^\circ 43'$ , the sun's declination for the given day, to radius 1, and  $x =$  sine of the altitude at 6. Then  $3x - 4x^3$  is the sine of the latitude; and, by spherics, as  $1 : a :: 3x - 4x^3 : x$ ; hence  $1 = 3a - 4ax^2$ , and  $x = \sqrt{((3a - 1) \div 4a)}$   $= .32035$  the sine of  $18^\circ 41'$  the alt. at 6; three times this gives  $56^\circ 3'$  for the latitude required.

*The same by Miss Maria Middleton, of Eden.*

Put  $a = .3861744$ , the sine of the sun's declination,  $22^\circ 43'$ , and  $x =$  the sine of the latitude; then will  $ax =$  sine of altitude at the hour of 6; and per question  $(3 - 3a^2x^2) \cdot ax - a^3x^3 = x$ , or  $x = \sqrt{((3a - 1) \div 4a^3)}$   $= .8295471$ , the sine of  $56^\circ 3'$ .

V. QUESTION 1018, *by Miss Nancy Mason, of Clapham.*

There is a vessel in form of the frustum of a cone, whose top diameter is six inches, bottom diameter four, and depth eight inches, inside dimensions, which is made of standard silver one-sixth of an inch thick. Required the value of the vessel when full of red wine, allowing it to be worth three shillings the quart, and the silver five shillings per ounce avoirdupois?

*Answered by Mr. L. W. Dillwyn, Brighthelmstone.*

We find the solid content of the inside, the diameter being 6 and 4, altitude 8, by the proper rule for conic frustums, to be  $159.1744$  inches, equal to  $2.756$  wine quarts, which at 3 shillings each amounts to  $8s. 3\frac{1}{2}d.$  nearly the value of the wine.

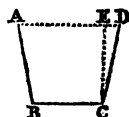
Again, supposing  $\frac{1}{2}$  added to each diameter, and  $\frac{1}{2}$  to the depth, which seems to have been the proposer's intention, the outside diameters will be  $6\frac{1}{2}$ ,  $4\frac{1}{2}$ , and the depth  $8\frac{1}{2}$ , from which dimensions, by the same rule, the outside content, or that of both wine and silver toge-

ther, will be 184.5835 inches. From this taking 159.1744 the content of the wine, leaves 25.4091 inches for the content of the silver; which, by the rules for specific gravity, is equal  $149\frac{1}{2}$  ounces; which, at 5s. the ounce, amounts to 37l. 5s.  $7\frac{1}{2}d.$  Therefore the two together, or 37l. 13s.  $10\frac{1}{4}d.$  is the whole value of both silver and wine.

*The same by Mr. John Craggs, of Hilton.*

By the Mensuration, page 156, the quantity of wine is equal to  $(6^3 + 4^3 + 6 \times 4) \times 8 \times .26179938 = 159.174023$  cubic inches; which at 3 shil. for  $57\frac{1}{2}$  inches, amounts to 8s. 3.22d. the value of the wine.

Now, supposing the thickness of the metal to be measured perpendicular to the slant side, as  $CE : CD :: \frac{1}{8} : .16773$ , this multiplied by 2 gives .33546, therefore 6.33546 is the outside diameter of the top. Also as  $CE : ED :: \frac{1}{8} : .02083$ , the double of which is .04166, and this taken from 4.33546, leaves 4.2938 for the outside diameter of the bottom; and the depth is  $8\frac{1}{2}$ ; therefore by the same rule, the content of the metal and wine together is 183.3957248. Hence the difference of the contents is  $24.22170176 =$  the quantity of silver. Then  $24.22170176 \times 16 \times 5 \times .380787 = 737.86473 = 36l. 17s. 10.376d.$  the value of the silver. Hence the value of the cup and wine is 37l. 6s.  $1\frac{1}{2}d.$

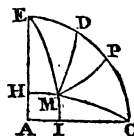


VI. QUESTION 1019, by Mr. John Liddell, of Thornton School.

If the inclination of the lunar orbit to the ecliptic be five degree twenty minutes; and the ascending node in the first point of Cancer; what is the latitude and longitude of the moon, when she is equidistant from the poles of the equinoctial and ecliptic?

*Answered by Mr. John Liddell.*

Let CIA represent the ecliptic, E its pole, P the pole of the equinoctial, c the ascending node, or first point of Cancer, CMH the moon's orbit making an angle  $\angle ACH = 5^\circ 20'$ , M the place of the moon, which, by the question, is equidistant from E and P. From M let fall the perpendicular MD; then because  $ME = MP$ , by spherics  $DE = DP = 11^\circ 44'$ ; or half the distance of the two poles. But since CE is a quadrant, CP is given  $= 66^\circ 32'$ , and consequently  $CD = CP + PD = 78^\circ 16'$ . In the right-angled spheric triangle DCM, are given DC and the angle C, to find CM, then in the triangle ICM will be given MC and the angle ICM, to find IM and IC, the moon's latitude and longitude. Thus, by cases 4 and 2, page 155, of Dr. Hutton's Tables, as



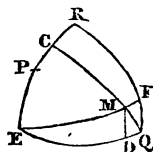
Radius : cot. CD :: cos.  $\angle DCM$  : cot. CM  $= 88^\circ 53' 30''$

Radius : sin. CM :: sin.  $\angle ICM$  : sin. IM  $= 5\ 19\ 56$ , the latitude.

Radius : tan. CM :: cos.  $\angle ICM$  : tan. IC  $= 88\ 53\ 12$ , the longitude from Cancer. Here the declination and latitude are equal.

*The same answered by Mr. Richard Elliott, Liverpool.*

Let  $P$  represent the pole of the equinoctial,  $R$  the pole of the ecliptic, and  $EF$  a quadrant of the moon's orbit intersecting the ecliptic in  $E$ , the ascending node. Then as  $R$  is the pole of  $EQ$ , the sides  $ER$ ,  $RQ$ , and  $EQ$  will be quadrants; and if  $PR$  be bisected in  $C$ , by the quadrant  $QC$ , then  $M$ , the point of intersection with the lunar orbit, will be the moon's place when equidistant from  $P$  and  $R$ . Moreover, if  $MD$  be drawn perpendicular to  $EQ$ , it will be the moon's latitude, and  $DQ$  her longitude. Now, in the right-angled triangle  $MRQ$ , are given  $RQ = 5^\circ 20'$ , and the angle  $Q$  ( $CR$ )  $= 11^\circ 44'$ , to find the side  $MQ = 5^\circ 26' 50''$ . Again, in the right-angled triangle,  $DMQ$ , we have the side  $MQ$ , and the angle  $MDQ$  ( $90^\circ - 11^\circ 44'$ )  $= 78^\circ 16'$ , to find  $MD = 5^\circ 19' 59''$  the latitude, and  $DQ = 1^\circ 6' 40''$  the longitude.



*The same answered by Amicus.*

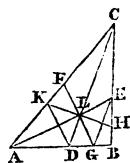
Bisect the arc of the great circle between the given poles of the equinoctial and ecliptic, with another great circle perpendicular thereto, which will pass through the point of the lunar path required.

#### VII. QUESTION 1020, by Mr. Alex. Rowe, *Reginnis*.

In a right-angled triangle, given the lengths of the two lines bisecting the two acute angles, and terminating in the opposite sides; to determine the triangle?

*Answered by Mr. William Burdon, Acaster Malbis.*

Let  $ABC$  be the right-angled triangle,  $AE$ ,  $CD$  the two given lines bisecting the two acute angles and intersecting one another in  $L$ . Draw  $FLG$  perpendicular to  $AE$ , and  $KLH$  perpendicular to  $CD$ ; and join  $EG$ ,  $KD$ . Now it appears by the 19th property of triangles in Dr. Hutton's Dictionary, volume 2, page 616, that each of the angles at  $L$  is equal to half a right angle, and consequently that  $LF = LG = LE$ , and  $LK = LH = LD$ . Put  $AE = a$ ,  $CD = b$ ,  $LF = LG = LE = x$ , and  $LK = LH = LD = y$ ; so shall  $AL = a - x$ ,  $CL = b - y$ ,  $EG = x\sqrt{2}$ , and  $DK = y\sqrt{2}$ . By similar triangles,  $AL : LD :: AE : EG$ , and  $CL : LF :: CD : DK$ , which give these two equations, viz.  $ay = (a - x) \cdot x\sqrt{2}$ , and  $bx = (b - y) \cdot y\sqrt{2}$ ; from the former  $y = (a - x) \cdot x\sqrt{2} \div a$ , which substituted in the latter, gives  $(a - x)^2 \cdot 2x\sqrt{2} = (a - 2x) \cdot ab$ , a cubic equation by which  $x$  will be found.



*The same answered by Mr. J. Hartley, London.*

Make  $AE = a$ ,  $CD = b$ ,  $s = \text{sine of } \angle ALD = \angle CLB = 45^\circ$ ,  $y =$

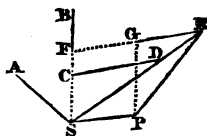
tang.  $\angle EAB$  or  $\angle ACB$ ,  $1 \div x =$  its cosine, and  $y \div x$  of course  $=$  its sine; also  $(1 - y^2) \div 2y =$  tang.  $\angle ACB$ , and  $(s + sy) \div x = \sin. D = \cos. DCB$  or  $DCA$ . Now, by trigonometry, as  $1 : a :: \frac{1}{x} : \frac{a}{x} = AB$ , and  $1 : b :: \frac{s + sy}{x} : \frac{bs + bsy}{x} = CB$ . Hence tang.  $\angle ACB = \frac{a}{bs} \times \frac{1}{1 + y} = \frac{1 - y^2}{2y}$ . From whence, by a cubic equation, the angles are found, and consequently the sides.

VIII. QUESTION 1021, by Mr. John Rodham, *Richmond*.

A privateer observes a sloop bearing SW by W. distant seven miles, the wind blowing from the WNW : the sloop makes good her way five points from the wind five miles an hour; the privateer runs nine miles an hour, but cannot sail closer than six points of the wind : how far on each tack must she run to come up with the sloop ?

*Answered by Mr. John Rodham, the Proposer.*

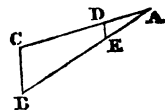
**Construction.** Let  $P$  and  $S$  be the places of the privateer and sloop, As the direction of the wind; draw  $SB$  for the course of the sloop, and from any point therein, as  $C$ , draw  $CD$  to make an angle with  $AS = 6$  points, and make  $CD : CS :: 9 : 5$ ; produce  $SD$  to meet a line drawn through  $P$  perpendicular to  $AS$  in  $E$ ; through  $E$  to meet  $SB$  in  $F$  draw  $EF$  parallel to  $DC$ , also to meet  $EP$  draw  $PG$  making the angle  $EPG = FEP$ ; and  $PG$  and  $GF$  are the courses of the privateer; and  $F$  the place where he falls in with the sloop.



**Demonstration.** Since, by construction,  $PG$  and  $GF$  make equal angles with  $PE$  perpendicular to  $AS$ , they also make equal angles with  $AS$ , which, by construction, is 6 points, the privateer's course from the wind. Also  $GP = GE$ , and  $FG + GP = FG + GE = FE$ ; and the triangles  $EPS$  and  $DCS$  are similar, therefore  $(EF =) FG + GP : SF :: CD : CS$ , the given ratio of sailing by construction.

*The same answered by Mr. John Fildes, Liverpool.*

It is clear, from the conditions of the question, that the privateer will sail S. W. on the first tack, and N. on the second; and that the sloop will sail S. W. by W. making a right line with her bearing from the privateer at the commencement of the chase. Therefore, in the annexed figure, let  $A$  represent the place the privateer starts from,  $B$  where she tacks, and  $C$  where she overtakes



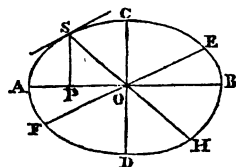
the sloop; then, in  $AC$ , if  $AD$  be taken  $= 7$ , and  $DE$  drawn parallel to  $BC$  meeting  $AB$  in  $E$ ; it will be, as  $\sin. \angle AED = 4$  points :  $AD = 7 :: \sin. \angle ADE = 11$  points :  $AE = 8.231 :: \sin. \angle DAE = 1$  point :  $DE = 1.931$ . Next, put  $CD = x$ ; then  $AD = 7 : AE + DE = 10.162 :: AC = x + 7 : \frac{1}{4}(10.162x + 71.134) = AB + BC$ ; therefore  $\frac{1}{4}x = (10.162x + 71.134) \div 7 \times 9$ ; from which equation is found  $x = 29.177$ ; consequently  $AD = 7 : AE = 8.231 :: AC = 36.177 : AB = 42.538$  miles, the distance the privateer sails on the first tack; and  $AD = 7 : DE = 1.931 :: AC = 36.177 : BC = 9.979$  miles, the distance on the second tack.

IX. QUESTION 1022, by Mr. O. G. Gregory, *Yaxley, Hunts.*

The axes of an ellipse are 12 and 8. What is the difference between its area, and the area of an osculatory circle to the curve, at the point where an ordinate rightly applied to the semi-transverse touches the ellipse?

*Answered by Mr. John Ryley, of Leeds.*

Let  $ADBC$  be the given ellipse;  $rs$  an ordinate rightly applied to the middle of the semi-transverse axis (which I take to be the meaning of the question, for it is not clearly expressed); draw the diameter  $SH$ , and its conjugate  $EF$  parallel to the tangent at  $s$ . Then, by the property of the ellipse,  $AB^2 : CD^2 :: AP : PB : PS^2 = 12$ ; and (by Euclid 47, 1,)  $os^2 = \sqrt{21}$ ; but, by the property of the curve,  $AB^2 + CD^2 = SH^2 + EF^2$  (Hutton's Conic Sections, prop. 17,) therefore  $ro = \sqrt{31}$ .



Now, by rule 2, page 349, Dr. Hutton's Dictionary, vol. 1,  $ro^3 \div AO \cdot CO = \frac{1}{4} \sqrt{31}$  is the radius of the osculatory circle corresponding to the given point  $s$ , the area of which is therefore  $= 162.48508 = 31^3 \times .78539$ , &c.  $\div 24$ ; from which deduct  $75.3984 = 12 \times 8 \times .78539$ , &c. the area of the ellipse, and the remainder  $87.08668$  is the required difference of the areas.

X. QUESTION 1023, by James Glenie, *Esq.*

On a given right line as a base, to constitute a triangle such, that the ratio of the other two sides, to the ratio of the sum of their squares to the difference of their squares, shall be equal to the ratio of 40 to 51; whilst the area of the triangle has a given ratio to the square of the base; suppose that of 15 to 32.

*Answered by Amicus.*

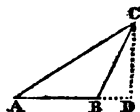
By the question the ratio of the sides, is to the ratio of the sum of

their squares to the difference of their squares, as  $40 : 51 :: 5 \times 8 : 3 \times 17 :: \frac{5}{3} : \frac{17}{8} = \frac{34}{16} = \frac{25+9}{25-9} = \left( \left( \frac{5}{3} \right)^2 + 1 \right) \div \left( \left( \frac{5}{3} \right)^2 - 1 \right)$ , the ratio of the sides is therefore that of 5 to 3; and since the base and area are given, we have the base, perpendicular, and ratio of the sides, to construct the triangle.

*The same answered by Mr. Henry Hunter, of Alnwick.*

In the given base AB produced take BD =  $\frac{1}{16}$  of AB, on which erect the perpendicular =  $\frac{1}{16}$  of AB; then join AC and BC, so will ABC be the required triangle.

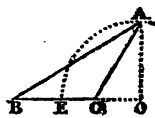
For since, by construction, DC =  $\frac{1}{16}$  AB, therefore the area AB  $\times \frac{1}{2}$  DC, or AB  $\times \frac{1}{32}$  AB : AB<sup>2</sup> :: 15 : 32, the given ratio of the area to the square of the base. Again AC<sup>2</sup> = AD<sup>2</sup> + DC<sup>2</sup> =  $\left( \frac{25}{16} \text{ AB} \right)^2 + \left( \frac{1}{16} \text{ AB} \right)^2 = \frac{625}{256} \text{ AB}^2$  and BC<sup>2</sup> =  $\left( \frac{9}{16} \text{ AB} \right)^2 + \left( \frac{1}{16} \text{ AB} \right)^2 = \frac{81}{256} \text{ AB}^2$ , therefore AC<sup>2</sup> : BC<sup>2</sup> :: 850 : 306 :: 25 : 9, and AC : BC :: 5 : 3, the ratio of the two sides; also AC<sup>2</sup> + BC<sup>2</sup> : AC<sup>2</sup> - BC<sup>2</sup> :: 25 + 9 : 25 - 9 :: 34 : 16 :: 17 : 8; hence  $\frac{4}{3} : \frac{17}{8} :: 40 : 51$ , the given ratio by the question.



*The same answered by James Glenie, Esq. the Proposer.*

Divide the given base BC, so that BE be to EC as 5 to 3, and take EO a fourth proportional to BE — EC, EC, and BE; erect OA perpendicular and equal to EO, so shall A be the vertex of the triangle ABC required.

For, BA : AC :: BE : EC :: 5 : 3, therefore BA<sup>2</sup> : AC<sup>2</sup> :: 25 : 9, and BA<sup>2</sup> + AC<sup>2</sup> : BA<sup>2</sup> - AC<sup>2</sup> :: 34 : 16 :: 17 : 8; but the ratio of 5 to 3, to the ratio of 17 to 8, is equal to the ratio of 40 to 51, the given ratio.



#### XI. QUESTION 1024, by the Rev. Mr. L. Evans.

By a certain author it is said that the fluxion of

$z^x$  is  $= x^x + x^x + x^x z^{x-1} - z^x$ ; query an investigation?

*Answered by Mr. Wm. Burdon, Acaster Malbis.*

Let  $z$  and  $x$  by flowing become  $z + z'$  and  $x + x'$ , then will the new or cotemporaneous value of  $z^x$  be  $(z + z')^{x+x'} = z^{x+x'} + xz^{x+x'-1} + x'z^x z^{x-1} + (x + x') \times \frac{z^{x-1} - 1}{2} \times x'z^x z^{x-2}$

&c. by the binomial theorem. But  $x'z^x z^{x-2}$ , &c. being nothing, or indefinitely small in respect to the other terms, may be omitted, therefore  $(z + z')^{x+x'}$  is  $= z^{x+x'} + xz^x z^{x-1}$ , and the incre-

ment of  $z^x = (z + z')^x + x' - z^x = z^x + x' + xz'z^{x-1} - z^x$ . Consequently (by scholium, page 631, Dr. Hutton's Dictionary, vol. 1) the fluxion of  $z^x$  is  $z^x + x' + xz'z^{x-1} - z^x$ , as in the question.

*The same by Mr. Thomas Coultherd, Frosterly.*

Let  $z'$  and  $x'$  be the increments of the variable quantities  $z$  and  $x$ , and the given expression so increased becomes  $(z + z')^x + x'$ , which expanded, by the binomial theorem, is  $z^x + x' + (x + x') \times z^{x-1} z' + \&c. = z^x + x' + xz'z^{x-1} + \&c.$  And because  $z'$  is supposed indefinitely small, every power of it above the first will be nothing, and consequently the remaining part of the above series will vanish. Therefore  $z^x + x' + xz'z^{x-1} - z^x$  is the increment of the given quantity; in which let fluxional characters be written, and it becomes  $z^x + \dot{x} + xz\dot{z}z^{x-1} - z^x$ , as was to be proved.

*The same by Mr. John Craggs, of Hilton.*

Let  $z$  be augmented to  $z + \dot{z}$ , and  $x$  to  $x + \dot{x}$ , the fluxion is  $(z + \dot{z})^x + \dot{x} - z^x$ . The former part of this raised by the binomial theorem is

$$z^{x+\dot{x}} + (x + \dot{x})z\dot{z}z^{x-1} + \frac{x + \dot{x}}{1} \times \frac{x + \dot{x} - 1}{2} z^2 \dot{z}^2 z^{x-2}, \&c.$$

Now by rejecting the insignificant quantities, we get only  $z^{x+\dot{x}} + xz\dot{z}z^{x-1} - z^x$ , as required.

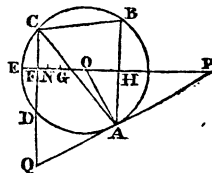
### XII. QUESTION 1025, by Amicus.

In a given circle to inscribe a triangle, having its vertex in a given point of the periphery, its base parallel to a line given in position, and the solid under its three sides a maximum?

*Answered by Amicus, the Proposer.*

Since the rectangle under the two sides, is equal to that under the perpendicular of the triangle and the given diameter of the circle, the area of the triangle must also be a maximum; hence this construction.

Through the given point  $c$ , and parallel to the line given in position, draw  $cq$ , cutting the circle in  $D$ ; bisect  $cd$  with the radius  $eo$  in  $F$ , which produce, and bisect  $fo$  in  $G$ , and  $eo$  in  $N$ ; then take, by Simpson's Geometry, 5, 18, the rectangle  $oh, gh =$

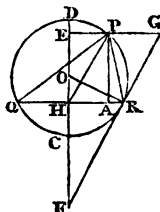


OE . EN, and draw AHB perpendicular to EH, and the triangle ACB is that required.

For, drawing the tangent QP at A, cutting CQ in Q and EH produced in P, then since OH . GH = OE . EN,  $OE^2 = OA^2 = 2GH . OH$ ; but OH : OA :: OA : OP, therefore  $2GH = OP = FO + 2HO = FH + HO$ , and FH = OP - HO = HP, consequently AQ = AP; and the triangle is a maximum, by Simpson's Geometry, page 201.

*The same by Mr. Rd. Nicholson, Leeds.*

**Lemma.** Let PQR represent the triangle & the given point in the circumference of the circumscribing circle, CQDPR, whose diameter CD is perpendicular to the base QR, and consequently to the line given in position; therefore the point C is given; upon QR and CD demit the perpendiculars PA and PE respectively. Now, the solid QP . PR . QR is to be a maximum, or its equal CD . PA . QR; but CD is given, therefore PA . QR must be a maximum or its 4th part, viz. the triangle HPR, which is the greatest when a tangent FG is drawn to meet CD and EP produced, in F and G, and bisected in the point of contact R by a well-known property in Simpson's Geometry, on the maxima and minima of Geometrical Quantities.

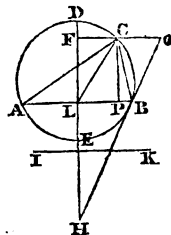


**Construction.** Take OH, by problem 6, Simpson's Exercises, so that  $(\frac{1}{2}OE + OH) . OH$  may be equal to half the square of the radius OC; through H, and parallel to the position line, draw the base QR; and join PQ, PR.

**Demonstration.** Join OR, which, by the nature of the circle, is perpendicular to FG. Then, by right-angled triangles,  $FO . OH = OR^2$ , but by the construction  $(\frac{1}{2}OE + OH) . OH = \frac{1}{2}OC^2 = \frac{1}{2}OR^2$  the double of which is =  $(OE + 2OH) . OH = OR^2$ ; hence by equality  $(OE + 2OH) . OH = FO . OH$ , and therefore  $OE + 2OH = FO$ ; but  $OE + 2OH = EH + HO$ , and  $FO = FH + HO$ , therefore  $EH = FH$ , and consequently  $QR = RF$  by reason of the parallels.

*The same by Mr. John Ryley, of Leeds.*

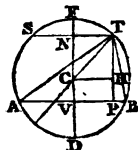
**Analysis.** Suppose the thing done; AEBD the given circle; c the given point; DE the diameter drawn perpendicular to IK the line given in position; and ABC the required triangle. Then, by the question, AB . AC . BC is to be a maximum; but AC . BC = DE . CP, and therefore AB . CP is a maximum, because DE is constant, or LB . CP is a maximum; but by the solution to question 1012 Diary, or by Simpson on the Maxima and Minima, the triangle LCB, or ACB, is a maximum, when the tangent GBH is bisected by the point B: hence the triangle may readily be



determined, for  $cr$  is given ; and  $(DE - DF) \cdot DF = rc^2$ , whence the method of constructing the problem is manifest.

*The same by Mr. Joseph Mouldsdaie, of Halton.*

Let  $r$  be the given point,  $rs$  the line given by position, and parallel to the base  $AB$  of the required triangle  $ATB$  ;  $FD$  a diameter perpendicular to the base  $AB$  and to  $ST$  ;  $c$  the centre of the circle,  $TP$  perpendicular to  $AB$ , and  $CH$  perpendicular to  $TP$ . Now, by the property of the circle,  $TP \cdot FD = AT \cdot TB$ , therefore  $AT \cdot TB \cdot AB = TP \cdot FD \cdot AB = a$  maximum, or  $TP \cdot AB$  a maximum, because  $FD$  is given. But  $CH$  and  $TH$  are also given ; therefore put  $TH = a$ , the radius  $CT$  or  $CD = r$ , and  $HP = CV = x$  ; then by the circle,  $AV = VB = \sqrt{(r+x) \cdot (r-x)} = \sqrt{(r^2 - x^2)}$ , and  $TP = a + x$  ; therefore  $(a+x) \sqrt{(r^2 - x^2)}$  is a maximum, which being put into fluxions, &c. gives  $2x' + ax' = r^2$ , and hence  $x = \frac{1}{4} \sqrt{(8r^2 + a^2)} - \frac{1}{4}a$ .



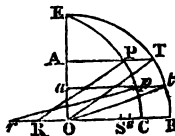
**XIII. QUESTION 1026, by Colonel Edw. Williams, Royal Artillery.**

If  $L$  denote the length of a degree perpendicular to the meridian ; in the latitude whose sine and cosine are  $s$  and  $c$  ; and  $l, s, c$ , those corresponding to another degree, remote from the former ; then, supposing the earth to be an ellipsoid, the ratio of the axes will be that of  $\sqrt{(l^2c^2 - L^2c^2)}$  to  $\sqrt{(L^2s^2 - l^2s^2)}$ . Required a demonstration ?

*Note.* This theorem is published in the Philosophical Transactions of 1775, but without a demonstration.

*Answered by Amicus.*

The ellipsoid hypothesis is, that the degrees of longitude, are as the distances  $ps = ao$  of the places from the polar radius  $oc$  ; and the length of a degree of longitude there, divided by the co-sine of the latitude. That is,  $LC = N$ , and  $lc = n$ , are the lengths of a degree of longitude at  $p$  and  $p$ . Hence the ratio becomes  $\sqrt{(n^2 - N^2)} : \sqrt{(s^2N^2 \div c^2 - s^2n^2 \div c^2)} ::$  therefore  $\sqrt{(ao^2 - ao^2)} : \sqrt{(s^2 \cdot ao^2 \div c^2 - s^2 \cdot ao^2 \div c^2)} = \sqrt{(rs^2 - rs^2)} = \sqrt{(ap^2 \cdot oe^4 \div oc^4 - ap^2 \cdot oe^4 \div oc^4)} = (oe \div oc) \sqrt{(at^2 - at^2)} = (oe \div oc) \sqrt{(oe^2 - ao^2 - oe^2 + ao^2)} = (oe \div oc) \sqrt{(ao^2 - ao^2)} :: oc : oe$ .



*The same answered by Colonel Williams, the Proposer.*

Let the semiaxes be  $r$  and  $t$ , and  $r = 57^\circ 29'$ , &c. the degrees in radius ; then, by the nature of the ellipse, the radii of curvature will be

$rL = T^2 \div \sqrt{(T^2c^2 + s^2t^2)}$  and  $rl = t^2 \div \sqrt{(T^2c^2 + s^2t^2)}$ ; from which equations  $T : t :: \sqrt{(L^2s^2 - T^2s^2)} : \sqrt{(T^2c^2 - L^2c^2)}$ .

*The same by Mr. John Ryley, of Leeds.*

If  $L$  = the length of a degree in the greater latitude,  $s$  and  $c$  = the sine and cosine of the same, also  $l$ ,  $s$ ,  $c$ , those corresponding to the less latitude; then it is evident, by what is shewn at page 403, Dr. Hutton's Dictionary, vol. 1, that the ratio of the lesser axe to the greater, will be that of  $\sqrt{(T^2c^2 - L^2c^2)} : \sqrt{(L^2s^2 - T^2s^2)}$ .

XIV. QUESTION 1027, by Mr. George Sanderson, London.

To determine the density of the air at the depth of 1000 miles below the earth's surface; supposing a perforation made to that depth, the diameter of the earth 7957 $\frac{1}{2}$  miles, the density of air at the earth's surface to that of water, as 1 $\frac{1}{2}$  to 1000; and the pressure of the atmosphere on a square inch of the earth's surface 14 $\frac{1}{2}$  pounds, or 236 ounces.

*Answered by Amicus.*

If  $r$  denote the earth's radius = 3978.875,  $x$  = the distance from the centre,  $z$  = the air's density there, and  $d$  = the density thereof at the earth's surface; applying the data of the question to problem 16

of Emerson's Fluxions,  $\frac{z}{d}$  = the number whose logarithm is

$$\frac{r^2 - x^2}{2 \times 2.302585 \times 2pr} = \frac{(3978.875 + 2978.875) \times 1000 \times 5280}{2 \times 2.302585 \times 3978.875 \times 28320}, \text{ or}$$

its equal the number whose logarithm is  $27831000 \div 12.35 \times 31831$  or 70.7964615, that is,  $z \div d = 6258374$ , &c. to 71 places of figures, the answer.

*Corol.* At the earth's centre, where  $x = 0$ ,  $z \div d = 12246$ , &c. to 162 places of figures, which, though not infinite, as in the prize question for 1795, is amazingly great.

*The same answered by Mr. John Surtees, of Sunderland.*

Let  $r$  = the radius of the earth in feet,  $d$  = the density of the air at the surface,  $v$  = the density at  $x$  distance from the centre,  $n = 28320$  feet = the height of the atmosphere at a mean density  $d$ , as per question, and  $l = 2.302585$ . Then, allowing the earth to be a true sphere, and equally dense throughout, (by Emerson's Fluxions, page 389),  $v = d \times$  number belonging to log.  $(r^2 - x^2) \div 2rnl$ ; and when  $r - x = 1000$  miles,  $v = d \times$  number of log. 70.7954.

But if we allow gravity to be the same at all distances, then  $v = d \times$  number of log.  $(r - x) \div nl = d \times$  number of log. 80.9704.

And if we allow it to be inversely as the square of the distance from

the centre, then  $D = d \times \text{number of log. } (r^2 - rx) \div \ln r = d \times \text{number of log. } 108.1519.$

PRIZE QUESTION, by Capt. Wm. Mudge, Royal Artillery.

If a hollow cylinder, as AB, be filled with water, and made to turn horizontally about an axis BD at one end, with a given velocity  $v$ : Query, in what time will the cylinder be emptied of half the water through a hole in the end A; on supposition that the length of the cylinder is  $l$ , internal diameter  $d$ , that of the hole  $h$ ; and supposing also the hinder surface of the water always to continue at right angles to the cylinder AB?

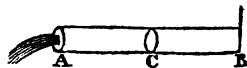
Answered by Amicus.

When the cylinder is emptied to  $y$  distance on its axis, from the centre of motion, let  $c$  = the given celerity of the outer end A,  $g = 32.2$  the force of gravity, and  $p = 3.14159$ . Then, the velocity of the effluent water, must be equal to that due to an height  $x$  above the centre of the orifice in a vessel at rest  $= \frac{c^2}{2g} - \frac{c^2 y^2}{2g l^2}$ , as may be easily collected from my solution to the 952d Diary question. Hence, as  $\sqrt{g}$ :  $g :: \frac{c}{l\sqrt{2g}} \times \sqrt{(l^2 - y^2)} : \frac{c}{l\sqrt{2}} \sqrt{(l^2 - y^2)} =$  the velocity measured by quantity, not celerity. Hence the celerity along the axis of the cylinder  $= \frac{h^2 c}{d^2 l \sqrt{2}} \sqrt{(l^2 - y^2)}$  and  $\frac{d^2 \sqrt{2}}{h^2 c} \times \frac{l y}{\sqrt{(l^2 - y^2)}}$  is the fluxion of the time, whose fluent  $\frac{d^2 \sqrt{2}}{h^2 c} \times \text{arc whose sine is } y$  and radius  $l$ ; when  $y = \frac{1}{2}l$ , and the arc  $= 30^\circ$ , it is  $\frac{p l d^2 \sqrt{2}}{6 c h^2} =$  the time required.

N. B. No notice is here taken of the force or velocity due to the height of the radius of the cylinder above the centre of the orifice, because, on the hypothesis in the question, of the hinder surface being perpendicular to the cylinder's axis, that force cannot be supposed to act.

The same answered by Captain Mudge, the Proposer.

Let  $ca$ , any variable distance,  $= x$ ;  $2g = 32\frac{1}{2}$  the force of gravity; and  $u$  = the velocity of the water at  $c$  in the direction  $ca$ . Now, if the whole cylinder was filled with water, and the end A revolved as supposed in the question, the force of the cylinder on the axis B would be  $\frac{1}{2}v^2$ ; this is well known. But



the water would also press against the bottom of the cylinder with the same force; consequently as the part included between  $cb$  is empty, the force of the water remaining in the cylinder between  $a$  and  $c$ , on the bottom at  $a$ , is equal to the difference of the forces of the cylinders  $bc$  and  $ca$ . Hence, that force, compared with gravity, is  $(v^2 \div 2l^3) \times (l^2 - x^2)$ .

Now, if the cylinder was suspended perpendicularly, it is clear, the velocity of the issuing water would be  $2\sqrt{g(l-x)}$ . And since with any other force, provided gravity be not supposed to act, the velocity of the water would be as that force, it is plain  $\frac{v^2}{2l^2\sqrt{g}} \times (l^2 - x^2) \times \sqrt{l-x}$  is the velocity of the water, issuing through the hole at  $a$  when the surface is at  $c$ ; hence  $\frac{hv^2}{2dl^2\sqrt{g}} \cdot (l^2 - x^2) \cdot \sqrt{l-x} = u$ , therefore  $\frac{2dl^2\sqrt{g} \cdot \dot{x}}{hv^2(l^2 - x^2) \cdot \sqrt{l-x}} = \dot{t}$ , the fluxion of the time.

Now to find the fluent of  $\frac{2dl^2\sqrt{g} \cdot \dot{x}}{hv^2(l^2 - x^2)\sqrt{l-x}}$ , put  $y = l-x$ , then will  $\frac{2dl^2\sqrt{g} \cdot \dot{x}}{hv^2(l^2 - x^2)\sqrt{l-x}} = \frac{2dl^2\dot{y}}{hv^2y^{\frac{3}{2}} \cdot (y-2l)}$ ; the fluent of which is  $\frac{2dl^2\sqrt{g}}{hv^2} \times \left( \frac{1}{l\sqrt{y}} - \frac{\sqrt{y}}{l\sqrt{(2l-y)}} \cdot \sqrt{\left( \frac{1}{2l} + \frac{1}{\sqrt{y}} \right)} \right) = \frac{2dl^2\sqrt{g}}{hv^2} \times \left( \frac{1}{l\sqrt{(l-x)}} - \frac{\sqrt{(l-x)}}{l\sqrt{(l+x)}} \cdot \frac{1}{2l} + \frac{1}{\sqrt{(l-x)}} \right) = t$ , and corrected  $\frac{2dl^2}{hv^2} \times \frac{\sqrt{(l+v)} - \sqrt{(l-v)}}{\sqrt{(l+v)}} \times \left( \frac{1}{2l^2} + \frac{1}{l\sqrt{(l-x)}} \right) = t$ , or the time; for when  $x = 0$ ,  $t = 0$ . Hence, when  $x = \frac{1}{2}l$ ,  $t = \frac{d}{hv^2} \cdot \frac{\sqrt{3}-1}{\sqrt{3}} \cdot (1 + 2\sqrt{(2l)})$ .

*Corol.* From the general expression of the time, it is obvious all the water will never run out; for when  $x = l$ ,  $t = \frac{1}{0}$ , or infinite. This must needs be.

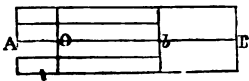
*Answered by Mr. John Surtees, Sunderland. (Suppt.)*

Let  $ac = x$  any variable distance from the end  $a$ ; also let  $v$  = the velocity of the end  $a$  per second,  $g = 16\frac{1}{2}$  feet,  $a$  = the weight of a cubic foot of water, and  $c = 3.1416$ . Then  $\frac{2lc}{v}$  = the time of one revolution, and  $(v^2b^2ca \div 16g^3l) \times (2lx - x^2)$  = the centrifugal force of the water  $ac$  in the hole  $h$ ; and by the laws of hydrostatics,  $(v^2 \div l\sqrt{2}) \times \sqrt{(2lx - x^2)}$  = the velocity of the issuing water. Also for the time,

we have  $\dot{t} = \frac{\dot{z}}{v} = \frac{d^2\sqrt{2}}{h^2v} \times \frac{-lx}{\sqrt{(2lx-x^2)}}$ , the fluent of which is  $t = \frac{d^2\sqrt{2}}{h^2v} \times \text{arch whose sine is } l-x \text{ to radius } l$ ; and when  $x = \frac{1}{2}l$ , that arch = 30 degrees, and then  $t = lcd^2 \div 3vh^2\sqrt{2}$  seconds, the time required.

*The same answered by Mr. John Howard, Mathematician, Newcastle-upon-Tyne. (Suppl.)*

Suppose  $ab$  the length of the fluid remaining at any position of exhaustion. And suppose a cylinder of equal density with the fluid, whose diameter is  $h$ , and length  $ab$ , inserted in the tube  $ab$ . Now it is plain, that all the particles without this tube  $ab$ , being resisted by the area of the base at  $a$  in the direction  $ab$ , can have no effect in increasing the velocity of the tube towards  $a$ . And hence, the whole quantity of the fluid acting at any position upon the hole  $h$ , will be truly measured by  $ab$ .



Put  $s = 16\frac{1}{2}$  feet,  $p = .7854$ ,  $n = \frac{760}{3}$  grains troy, the weight of a cubic inch of water,  $ab = z$ , and  $ab = l - z$ , also let  $ob$ , any variable part of the line  $ab = x$ . Then the cylinder  $ob = ph^2x$ , and its weight (supposing  $h$  and  $x$  taken in inches)  $nph^2x$ . Now  $l : z + x :: v : (vz + vx) \div l =$  the circular velocity of a particle at  $o$ , and  $(vz + vx)^2 \div l(z + x) = (v^2z + v^2x) \div l =$  its centrifugal force at  $o$ , which multiplied by  $nph^2x$ , the fluxion of the column  $ob$ , &c. gives  $(nph^2v^2 \div 2sl^2) \cdot (zx + x^2)$ , the fluxion of its pressure towards  $a$ ; and its fluent (supposing  $z$  constant)  $(nph^2v^2 \div 4sl^2) \cdot (2zx + x^2)$ , its whole pressure towards  $a$ . Which, when  $x = l - z$ , becomes  $(nph^2v^2 \div 4sl^2) \times (l^2 - z^2)$ , the measure of velocity with which the fluid issues through the hole  $h$ .

Again, by Dr. Hutton's treatise on the descent of fluids, in his Mathematical Miscellany, page 4, we have  $d^2 : h^2 :: \frac{nph^2v^2}{4sl^2} \cdot (l^2 - z^2) : \frac{nph^2v^2}{4d^2sl^2} \cdot (l^2 - z^2)$  the velocity per second of the exhausting surface at  $b$ .

Also  $\frac{nph^2v^2}{4d^2sl^2} \cdot (l^2 - z^2) : \dot{z} :: 1 : \frac{4d^2sl^2}{nph^2v^2 \cdot (l^2 - z^2)} \dot{z}$  the fluxion of the time of exhausting, and its fluent (by Art. 126, Simpson's Fluxions)  $\frac{2d^2sl}{4ph^2v^2} \times \text{hyp. log. } \frac{l+z}{l-z}$ , or when  $z = \frac{1}{2}l$ ,  $\frac{2d^2sl}{nph^2v^2} \times \text{hyp. log. of } 3$ , the time required.

*Questions proposed in 1798 and answered in 1799.*

## I. QUESTION 1029, by Mr. John Hawkes, of Finedon.

What two numbers are those, whose product, difference of their squares, and the ratio or quotient of their cubes, are all equal to each other?

*Answered by Mr. Wm. Davis, Schoolmaster, of Crowan.*

Put  $x$  = the greater number, and  $y$  = the less. Then  $xy = x^2 - y^2$  and  $xy = x^3 \div y^3$ , or  $x^2 = y^2$ , or  $x = y^2$ ; then by substitution, &c. we have  $y^2 - y = 1$ . By completing the square, &c. we find  $y = \frac{1}{2} + \sqrt{1\frac{1}{4}} = 1.61803$ . Consequently  $x = 2.61803$ .

*The same by Mr. John Eadon, jun. Sheffield.*

Let  $x$  = the greater, and  $y$  = the less number. Then, by the question  $xy = x^2 - y^2$ , and  $xy = x^3 \div y^3$ ; therefore  $y^2x = x^2$ , and  $y^2 = x^2$ , and  $y^2 = x$ . Put  $y^2$  for  $x$  in the first equation, and we get  $y^2 = y^2 - y^2$ , or  $y^2 - y = 1$ . Hence  $y = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ , and then  $x = y^2 = 1\frac{1}{2} + \frac{1}{2}\sqrt{5}$ , which are the two numbers sought.

For proof:  $xy = 2 + \sqrt{5}$ , and  $x^2 - y^2 = 2 + \sqrt{5}$ , and  $x^3 \div y^3 = 2 + \sqrt{5}$ .

*The same by Mr. John Ramsay, London.*

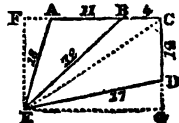
Suppose  $x$  the greater number, and  $y$  the less. Per quest.  $xy = x^2 - y^2 = x^3 \div y^3$ . By equating the two first quantities is got  $x = \frac{1}{2}y \times (1 \pm \sqrt{5})$ , and by equating the first and third  $x = y^2$ ; hence  $y = \frac{1}{2} \pm \frac{1}{2}\sqrt{5} = 1.618$ , &c. or  $- .618$ , &c.; and  $x = y^2 = \frac{3}{2} \pm \frac{1}{2}\sqrt{5} = 2.618$ , &c.; or  $.382$ , &c.

## II. QUESTION 1030, by Miss Sarah Cowen.

Asking lately the contents of a particular field, in form of a trapezium, I was answered that the contents were forgotten, but the dimensions in chains were as in the annexed figure; but as I cannot from hence compute the contents, shall be obliged to my friend, Lady D. to do it for me?

*Answered by Mr. Thos. Coultherd, Frosterly.*

Let CE be drawn, and produce CA to F, letting fall the perpendicular EF. Then  $AF = (BE^2 - BA^2 - AE^2) \div 2AB = 1.7727$ , and  $\sqrt{(AE^2 - AF^2)} = \sqrt{EF^2} = 17.9125$ ; also  $\sqrt{(CE^2 + EF^2)} = CE = 24.5394$  and  $\frac{1}{2}AC \times FE = 134.3437$  the area of the triangle ACE. Again, in the triangle CDE,



having the three sides given, by a like process is easily found, the perpendicular  $GE = 15.469589$ ; and thence  $\frac{1}{2}CD \times GE = 92.8175$ , the area. Consequently the sum of these two areas gives  $227.1612$  square chains, or  $22Ac. 2R. 34\frac{1}{2}P.$  for the area required.

*The same by Mr. J. Gee, Elswick, near Newcastle.*

In the triangle  $ABE$ , the three sides are given, to find the angle  $A = 95^\circ 39'$ . Hence, if the diagonal  $CE$  be drawn, we shall have two sides and the included angle of the triangle  $ACE$ , to find the said diagonal  $= 24.539$  chains. Then in each of the triangles  $ACE$ ,  $DCE$ , the three sides are known, whence the sum of their areas is easily found  $= 22Ac. 2R. 35P. =$  the content required.

*The same by Mr. R. Oliver, Assistant to the Rev. Mr. Cursham, Sutton, near Mansfield.*

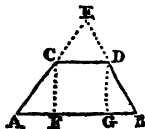
In the triangle  $ABE$ , all the three sides are given, to find the angle  $ABE = 54^\circ 30'$ , the supplement of which is  $125^\circ 30' =$  the angle  $EBC$ . If  $CE$  be drawn, we then have the sides  $EB$ ,  $BC$ , and the included angle, whence  $CE$  is easily found  $= 24.5$ . Hence we have the sides of all the triangles  $EAB$ ,  $EBC$ ,  $ECD$ , from which, (by rule 3, page 97, Hutton's large Mensuration, 2d edition, or by rule 2, page 96, of his Compendious Measurer,) their areas may be found, the sum of which comes out  $22Ac. 3R.$  nearly.

### III. QUESTION 1031, by Mr. Wm. Newby, Barningham.

A gentleman has a field in form of a trapezoid, the area of which is two acres three roods; the longest side, or base,  $AB$ , which is one of the two parallel sides, is 1432 links; also the angle at  $A$  is  $34^\circ 17'$  and that at  $B$   $54^\circ 18'$ : it is required to find what the four sides of the field will cost fencing, at  $6d.$  a rod?

*Answered by Mr. John Blackwell, Hungerford.*

Let  $ABCD$  represent the field; and continue the lines  $AC$ ,  $BD$  to the point  $E$ . Then, in the triangle  $ABE$ , are given all the angles and the base or side  $AB$ ; from which are found the other sides and area, viz.  $AE = 1163.26$ ,  $BE = 806.8717$ , area of  $ABE = 469157$ ; from this taking away the given area of the field, leaves 194157 the area of the triangle  $CDE$ . But as similar triangles have their like sides proportional to the square roots of their areas, we have, as  $\sqrt{ABE} :: \sqrt{CDE} ::$   
 $AB : CD = 921.238$   
 $AE : CE = 414.928$   
 $BE : DE = 287.806$   
 Then  $AE + BE - CE - DE = AC + BD = 702.734$ , which added to  $AB + CD$ , gives 3056 links  $= 122.24$  rods, amounting to  $3l. 1s. 1\frac{1}{2}d.$



*The same by Mr. Green, Academy, Deptford.*

Make the side AB, and the angles A and B, &c. as in the question, producing the sides AC, BD to meet at E. In the triangle ABE are given all the angles and the side AB, from which are found the side AE = 1163·259, BE = 806·872, and the area 469201·967; from which taking the given area of the trapezoid = 275000 square links, there remains the area of the triangle CDE = 104201·967. Then say, as triangle ABE : triangle CDE :: AB<sup>2</sup> : CD<sup>2</sup> = 848750·096, its square root is 921·276 = side CD. Hence, by similar triangles,

$$\text{as } AB : CD :: AE : CE = 748·382, \text{ and again,}$$

$$\text{as } AB : CD :: BE : DE = 519·094; \text{ then}$$

AE — CE = AC = 414·877, and BE — DE = BD = 287·778; hence AB + AC + CD + BD = 3055·931 links = 122·2372 rods, which at 6 pence each, come to 3*l.* 1*s.* 1½*d.* 69, the answer.

*The same by Mr. Joseph Mouldsley, of Runcorn.*

Put AB = 1432 links = *g*, the perpendicular CF or DG = *x*, sine of A = *a*, its co-sine = *b*, sine of B = *d*, its co-sine = *e*, and the given area = 275000 square links = *c*. Then, by trigonometry,

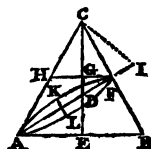
AF =  $bx \div a$ , BG =  $ex \div d$ , and FG =  $CD = g - 2mx$  (putting  $b \div a + e \div d = 2m$ ); hence the area is  $(g - mx) \times x = c$ ; this quadratic gives  $x = 233·7$  links. Hence the sides are AC = 415, CD = 921·2, BD = 287·8, their sum = 30·56 chains, which at 2*s.* per chain is 3*l.* 1*s.* 1*d.* 1·76*q.*

IV. QUESTION 1032, *by Mr. Richard Dover, Schoolmaster, Carlisle.*

There is an upright cone, in which a divisional plane is drawn from the extremity of the base to the opposite side, making an angle with the axis of 58°. Now that part of the middle divisional line, or transverse diameter of the elliptic section, included between the cone's axis and the extremity of the base, is given = 8 inches, and that part of the axis included between the vertex and its intersection with the said divisional line, is = 20 inches. Query the solidity of the whole cone, and the diameter of a globe whose solidity is to the solidity of the top part of the cone, which is above the dividing plane, as 1 to 6?

*Answered by Mr. J. Hartley, Auditor's Office.*

In the annexed figure, are given AD = 8, CD = 20, angle FAB = 32°, and the angle CDF = 58° by the question. Then, by trigonometry, ED = 4·24, AE = EB = 5·78, CE = 24·24, CB = CA = 25·08 inches; the angle CAE = CBE = 81° 30', and the angle AFB = 66° 30', the side FB = 7·83, and AF the transverse diameter = 14·62 inches. By mensuration, the so-



lidity of the cone = 1166.86; then by similar triangles, as  $CB : BE :: CF : FG = 4.66$ ; whence  $FB = 9.32$ , and  $\sqrt{(HF \times AB)}$  = the conjugate diameter of the ellipse = 11.24 =  $KL$ . Then, as radius :  $AC :: \sin. \angle CAF : CI = 15.44$  the perpendicular. Hence,  $AF \times KL \times .7854 \times \frac{1}{3}CI$  gives 663.388 for the solidity of the oblique cone  $CAFC$ ;  $\frac{1}{3}$  of this is 110.56 = the solidity of the globe; consequently its diameter will be  $\sqrt[3]{(110.56 \div .5236)} = 6$  inches nearly.

*The same by Mr. John Surtees, of Alstone.*

Let  $n = AD = 8$  inches,  $\frac{1}{2}n = CD = 20$ ,  $s$  and  $c$  = sine and co-sine of the angle  $D$  to radius 1, and  $a = .7854$ . Then  $AB = 2ns = D$ ,  $CB = \frac{1}{2}n(5 + 4c) = H$ , and  $HF = 10sn \div (5 + 2c) = d$ . Hence (by Hutton's Mensuration, page 173) the solidity of  $ACF' = \frac{1}{3}adH\sqrt{dd}$ ; and therefore the diameter of the globe =  $\sqrt[3]{(\frac{1}{3}adH\sqrt{dd})} = n\sqrt{(5 \div (5 + 4c))} \times \sqrt[3]{((5 + 2c) \times \frac{1}{3}s^2)} = 6.02629$  inches, as required.

*The same by Mr. Rob. Wilkinson, North Shields.*

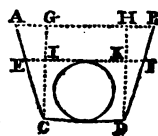
Let  $ABC$  represent the cone, and  $AF$  the dividing plane. Then  $AD = 8$ ,  $CD = 20$ , and the angle  $ADE = 58^\circ$ . Now radius :  $AD :: \sin. \angle DAE : DE$ , hence  $CE$  is known; radius :  $AD :: \cos. \angle DAE : AE$ , hence  $AB$  is known. Then  $.2618AB^3$ .  $CE$  is the solidity of the whole cone. And, by Hutton's Mensuration, cor. 2, page 228, 2d edition, the whole cone is to the top part  $CAF$ , as  $CE^{\frac{2}{3}}$  to  $CD^{\frac{2}{3}}$ , which gives the solidity of the top part, which call  $a$ . Hence  $\sqrt[3]{(a \div 3.1416)}$  is the diameter of the globe, = 6 inches nearly.

Y. QUESTION 1033, by Mr. Henry Armstrong, Bewcastle.

There is a vessel in the form of a frustum of a cone, standing on its lesser base, whose solidity is 8.67 feet, the depth 21 inches, its greater base diameter to that of the lesser, as 7 to 5, into which a globe had accidentally been put, whose solidity was  $2\frac{1}{2}$  times the measure of its surface. Required the lineal diameters of the above vessel and globe, and how many gallons of wine would be requisite just to cover the latter within the former?

*Answered by Mr. John Coultherd, Frosterly.*

By similar solids, as  $\sqrt{(7^2 + 5^2 + 7 \times 5)} \times .2618 : \sqrt{(8.67 \times 1728 \div 21)} :: 7 : 35 :: 5 : 25$ , so that 35 and 25 are the top and bottom diameters of the frustum. Again, if  $d$  denote the diameter of the globe, then  $3.1416d^2$  is its superficies, and  $3.1416d^3 \times \frac{1}{6}d$  is its solidity; therefore  $3.1416d^2 \times 2\frac{1}{2} = 3.1416d^3 \times \frac{1}{6}d$ , or  $2\frac{1}{2} = \frac{1}{6}d$ , and  $d = 2\frac{1}{2} \times 6 = 15$  the globe's



diameter, and consequently its solid content = 1767·146. Also, by similar triangles, as  $CG : CI :: AB - CD : EF - CD$ , that is, as  $21 : 15 :: 35 - 25 : 7\frac{1}{2}$ ; to this adding  $CD$ , gives  $EF = 32\frac{1}{2}$  the diameter at the surface of the wine. Then the solidity of the part  $CEFD$  is found = 9667·209; from which, taking the content of the globe 1767·146, leaves the quantity of the wine = 7900·063 cubic inches, or 34·2 wine gallons, as required.

*The same by Mr. D. Roberts, of St. Columb.*

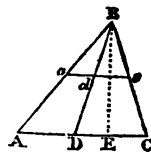
Put  $7x = AB$ ,  $5x = CD$ ,  $a = 21 = CG$ , and  $b = 2618$ . Then  $(49x^3 + 25x^3 + 35x^3) \times ab = 8\cdot67$  feet = 14981·76 cubic inches; which equation gives  $x = 5$ , and hence the diameters are 35 and 25. Now call the diameter of the globe  $d$ . Then is  $2bd^3$  the solidity, and  $12bd^2$  the surface, therefore  $2bd^3 = 2\frac{1}{2} \times 12bd^2$ , and  $d = 15 = CI$  or  $EK$ . Again, by similar triangles, as  $CG : AG :: CI : EI = 3\frac{1}{2}$ ; hence  $EF = 32\frac{1}{2}$ , and the content of  $CEFD = 9666\cdot9316$ , from which take the globe's content = 1767·15 leaves the content of the wine = 7899·78 inches, or 34·108 gal.

VI. QUESTION 1034, by Mr. James Sparrow, of Norwich.

In a plane triangle, given the angle at the vertex  $60^\circ$ , the length of the line bisecting it, and dividing the base into two parts, as 5 to 4, equal to 16; to find the sides and area?

*Answered by Mr. Wm. Baylis, Coventry.*

By Euclid 6, 3, the sides are proportional to the segments of the base made by the line bisecting the vertical angle; that is,  $5 : 4 :: AB : BC :: AD : DC$ . Now there are given  $AB = 5$ ,  $BC = 4$ , and  $\angle ABC = 60^\circ$ , to find the  $\angle A = \angle A = 49^\circ 6' 24''$ ; hence  $\angle C = 70^\circ 53' 36''$ . Then, in the triangle  $ABD$ ; are given all the angles and the side  $BD = 16$ , to find  $AB = 20\cdot7846$ , and  $AD = 10\cdot5830$ . And in the triangle  $BCD$ , are given all the angles and side  $BD$ , to find  $BC = 16\cdot6277$ , and  $DC = 8\cdot4664$ . Hence  $AC = 19\cdot0494$ ,  $BE = 15\cdot7117$ , and area = 149·647.



*The same by Mr. Wm. Burdon, of Acaster Malbis.*

**Construction.** Make the angle  $ABC = 60^\circ$ , and take  $AB : BC :: 5 : 4$  the given ratio of the segments of the base. Bisect the angle  $B$  with the line  $BD$ , which produce till  $BD = 16$ , the given length; then draw  $ABC$  parallel to  $adc$ , so shall  $ABC$  be the triangle required.

**Calculation.** In the triangle  $abc$ , are given the two sides  $ab$ ,  $bc$ , and the included angle  $B$ , to find the  $\angle a = 49^\circ 6' 24'' = \angle A$ ; hence



*Answered by Mr. John Bransby, Ipswich.*

Put  $x$  for the depth of rain in inches,  $a$  the area of the aperture,  $b$  the ounces in a cubic foot of water, and  $w$  the ounces of water in the gage. Then  $abx \div 1728 = w$ ; hence  $x = 1728w \div ab$ . Or, because  $b = 1000$  (see Hutton's Conics, page 138),  $x = 1.728w \div a$ . In words, multiply the ounces of water caught, by 1.728, and divide the product by the area of the aperture, for the depth of water fallen.

*The same by Mr. John Craggs, of Hilton.*

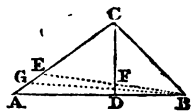
The quantity of rain that falls into any given vessel, must evidently be as the area of the orthographic projection of the vessel's aperture on a plane at right angles to the falling rain. Now when the aperture is a circle, its projection is an ellipsis, having its transverse axis equal to the diameter of the circle, and its conjugate is to the transverse, as the cosine of the inclination of the falling rain is to radius. Put  $w =$  weight of a cubic foot of water,  $d =$  diameter of the vessel,  $w =$  weight of water caught,  $a = .7854$ ,  $c = \cos.$  of inclination of the falling rain; then  $w \div ad^2w =$  depth of water in the vessel, also radius :  $d :: c : cd =$  conjugate axis, and  $acd^2 =$  area of the aperture; consequently  $w \div acd^2 =$  depth of water as required.

IX. QUESTION 1037, by Mr. Edward Warren.

When the vertical angle of any plane triangle is bisected; the sum of the sides is to their difference, as the tangent of the angle made by the bisecting line and base, is to the tangent of half the vertical angle. Required a demonstration?

*Answered by Mr. James Adams.*

Let ABC be any plane triangle, and CD the line bisecting the angle ACB. In CA take CE = CB, and draw BE. Then is the angle ACD or BCD half the vertical angle, CBE the complement of the  $\angle BCD =$  half the sum of A and B the angles at the base, and the angle ABE, or complement of  $\angle D$ , is half the difference of the angles A and B at the base. Now, by trigonometry, as  $AC + CB : AC - CB :: \text{tang. CBE} : \text{tang. ABE} :: \cotang. BCD : \cotang. D :: \text{tang. D} : \text{tang. BCD}$ , because the tangents and cotangents of arcs are reciprocally proportional.



*The same by Mr. Thomas Coultherd, Frosterly.*

The demonstration of this theorem is easily deduced from the 6th proposition in Emerson's Trigonometry. For, if CE be taken = CB, and CFD be drawn perpendicular to BE; also FE parallel to AB. Then will the  $\angle ECF = BCF$ , and  $GC = \frac{1}{2}$  the sum of AC and BC, also  $AG = \frac{1}{2}$

the difference of AC and BC. Hence, by similar triangles, as  $CG : GA :: CF : FD :: \text{tang. } \angle CBF : \text{tang. } \angle DBF$ . But the tangent of any angle is reciprocally as the cotangent of the same angle; therefore as  $AC + CB : AC - CB :: \cotang. DBF : \cotang. CBF :: \text{tang. } \angle D : \text{tang. } \angle BCF$ , the whole of each of the first terms being in the same ratio as their halves.

X. QUESTION 1038, *by James Glenie, Esq.*

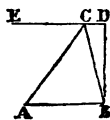
On a given right line as a base to constitute a triangle, such that the ratio of the squares on the other two sides, to the ratio of the sum of their cubes to the difference of their cubes, shall be equal to the ratio of 171 to 140, whilst the area of the triangle has to the square of the base a given ratio, suppose that of 1 to 2.

*Answered by Mr. Colin Campbell, of Kendal.*

Make BD equal and perpendicular to the given base AB, and draw the indefinite line DE parallel to it, then AC, BC being drawn to meet DE in c, so that  $AC : BC :: 3 : 2$ , by prop. 13, page 220, Simpson's Geometry, ABC will be the required triangle.

For, the area  $\frac{1}{2}AB \cdot BD = \frac{1}{2}AB^2 : AB^2 :: 1 : 2$ . And, because  $AC : BC :: 3 : 2$ ,  $AC^2 : BC^2 :: 9 : 4$ , and  $AC^3 : BC^3 :: 27 : 8$ , and therefore  $AC^3 + BC^3 : AC^3 - BC^3 :: 35 : 19$ .

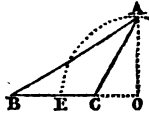
$$\text{Hence } \frac{AC^2}{BC^2} : \frac{AC^3 + BC^3}{AC^3 - BC^3} :: \frac{9}{4} : \frac{35}{19} :: 171 : 140.$$



*The same by Mr. John Rutherford, Lanchester School.*

Divide the given base BC, so that BE be to EC as 3 to 2, and take EO, a fourth proportional to BE — EC, EC, and BE, by Lemma, Prob. 21, page 334, Simpson's Algebra. Raise the perpendicular OA = OE; so shall A be the vertex of the triangle BCA required.

For then  $BA : AC :: BE : EC :: 3 : 2$ , also  $BA^2 : AC^2 :: 9 : 4$ , and  $BA^3 + AC^3 : BA^3 - AC^3 :: 35 : 19$ ; hence  $\frac{9}{4} : \frac{35}{19} :: 171 : 140$ , the given ratio.



XI. QUESTION 1039, *by Mr. Thomas Coultherd, of Frosterly.*

Being in the head of a dale, about seven miles from Kendal, remarkable for the height of the hills on each side, I observed that the road on which I was going, intersected at right angles a line joining the middle of the bases of two amazing high rocky hills; the angle of elevation of the one was  $50^\circ$ , and of the other  $60^\circ$ , taken at the point of intersection. I then proceeded forward in the same direction, up a gentle declivity, rising at an angle of  $10^\circ$ , for the space of 200 yards; and then

the angle of altitude of the first was  $48^{\circ} 10'$ , and that of the other  $58^{\circ}$ . I desire to know the perpendicular height of the hills from the bottom of the dale, the distance between their summits, and how far I was from the top of each at both stations?

*Answered by Mr. John Ramsay, London.*

Let  $r = \text{tang. of } 60^{\circ} \text{ or } 50^{\circ}$ , the angles of eleva. at the first station;  
 $t = \text{tang. of } 58^{\circ} \text{ or } 48^{\circ} 10'$ , the like angles at the 2d stat. to rad. 1.  
 $x = \text{dist. from 1st stat. to middle of either hill's base.}$

By trigonometry, as  $1 : 200 :: \sin. 10^{\circ} : 34.73 = c$ , height of 2d stat. above 1st; and as  $1 : 200 :: \cos. 10^{\circ} : 196.96 = b$ , horizontal distance of the two stations. Again, as  $1 : x :: r : rx = \text{perpendicular height of either hill above 1st station, and } 1 : t :: \sqrt{(b^2 + x^2)} : t\sqrt{(b^2 + x^2)}$  the same above the 2d station. Therefore  $rx = c + t\sqrt{(b^2 + x^2)}$ ; which equation reduced gives

$$x = \frac{rc \pm t\sqrt{(r^2 - t^2)} \times b^2 + c^2}{r^2 - t^2} = 629.373 \text{ and } 816.154 \text{ yards.}$$

From these values of  $x$  the following are found :

Distance between the hills' tops = 1445.527 yards,

Perpendicular height of the hills, 1090.106 and 972.654 yards,

Distance of their tops from 1st station 1269.71 and 1258.746,

Distance of ditto from the 2d station 1258.806 and 1244.477.

*The same by Mr. John Ryley, of Leeds.*

As the declivity of the road, and the distance between the two stations upon it, are given, the altitude of the second station above the first is found by trigonometry = 34.72964 yards, and their horizontal distance = 196.96154. Now put  $a = 34.72964$ ,  $b = 196.96154$ ,  $c = \text{cotang. of } 60^{\circ}$ ,  $t = \text{cotang. of } 58^{\circ}$ , and  $x = \text{the height of the hill to these two angles.}$  Then, by trigonometry  $cx = \text{the distance from the first station to the middle of the hill's base, and } t \cdot (x - a) = \text{the distance from the second station; hence, by Euclid 1, 47, } t^2 \cdot (x - a)^2 - c^2 x^2 = b^2$ ; from which quadratic  $x$  is found = 1090 yards.

In like manner, if  $x$  denote the height of the other hill, whose angles of elevation are  $50^{\circ}$  and  $48^{\circ} 10'$ ;  $c$  and  $t$  the cotangents of the said angles, also  $a$  and  $b$  as above: then will  $x = 973$  yards, the height of the lower hill,

Now from what is here found, and the 47th of Euclid 1, the distance between the summits is found, = 1445.7; the distance from the first station to the top of the higher hill 1258.6, and to the top of the lower 1270; also from the second station to the top of the higher hill 1242.5, and to the top of the lower 1259.3 yards.

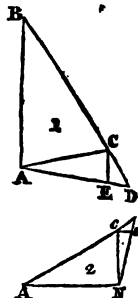
**XII. QUESTION 1040, by Mr. J. Rutherford, Schoolmaster, Lanchester.**

On Lammas day, in latitude  $54^{\circ} 40'$  north, I observed a tree, which I knew to be 20 yards in height, to cast its shadow along the declivity of a hill. Now, admitting the declivity to be a plane of an indefinite length, and to incline regularly from the east towards the west at an angle of  $20^{\circ}$ ; also the tree to stand perpendicular to the horizon, and exactly at the top of the said declivity; I desire to know how long the shadow was, supposing the time to be an hour before noon?

N.B. This was proposed in a monthly publication at Stockton, two years ago; but an erroneous solution was printed, being above seven yards too much, which is the reason of its being here re-proposed.

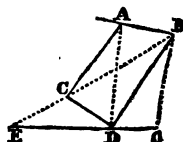
*Answered by Mr. John Bransby, Ipswich.*

From the given latitude, declination, and hour, the sun's altitude is found to be  $51^{\circ} 30' 23''$ , and his azimuth south  $22^{\circ} 16' 22''$  towards the east. Let  $\triangle ABC$  (fig. 1.) be a right-angled triangle, having its angle  $c =$  the sun's altitude, and  $AB = 20$  yards the height of the tree; thence will  $AC$  the length of the shadow on a horizontal plane be found  $= 15.89117$ . Let  $ac$  be continued to  $n$ , meeting  $AED$ , which is the plane of the declivity, and let  $ce$  be drawn perpendicular to the horizontal line  $ac$ : In fig. 2, let  $A$  be the bottom of the tree, and  $ACN$  a horizontal plane passing through that point; then  $AN$ , being the meridian line, and the angle  $CAN = 22^{\circ} 16' 22''$ , the azimuth,  $Ac$  will be the direction of the shadow, and  $= 15.89117$ , as above found; whence  $NC$  is found  $= 6.03457$ . Again, in the triangle  $cen$ , right-angled at  $c$ , having given  $nc$ , and the angle  $cne = 20^{\circ}$  the inclination of the declivity,  $ce$  is found  $= 2.196404$ , answering to  $CE$  in fig. 1. The angle  $CAE$  may be now found  $= 7^{\circ} 51' 44''$ ; whence  $D = 43^{\circ} 38' 39''$ ; and  $AD$  the length of the shadow is easily found  $= 18.036733$  yards.



*The same by Mr. Thomas Coultherd, Frosterly.*

The day of the month, the latitude and meridian distance being given, the sun's azimuth from the south is found  $22^{\circ} 16' 44''$ , and altitude  $51^{\circ} 26' 30''$ ; to which if there be added  $16'$  the semidiameter, and  $45''$  the refraction, the sum  $51^{\circ} 43' 15''$  will be the altitude of his upper limb. Now, in the figure, let  $B$  be the place of the tree,  $AB$  the edge of the horizontal plane on which it stands,  $ABDO$  the continuation of the same plane;  $CD$  at right angles with it,  $BD$  the direction of the shadow,  $AD$  an east and west line, the angle  $ADB$  the sun's



distance from the east, and angle  $nDO$  his altitude. Then, as  $\text{tang. } \angle nDO : \text{radius} :: \text{height of the tree} : DO = 15.783$ ; and as  $\text{radius} : DO :: \text{cosine } \angle ADB : AD = 6.2376$ ; also as  $\text{cosine } CAD : AD :: \text{sine } CAD : CD = 2.2703$ , and as  $DB : \text{radius} :: CD : \text{tang. } CBD = 8^\circ 11' 8''$ . Hence, in the triangle  $nBD$ , having the angle  $n$  last found, the angle  $EDB$  the supplement of  $ODB$ , consequently the angle  $E$  is known; then as  $\text{sine } E : DB :: \text{sine } EDB : EB = 17.987$  yards, the length of the shadow required.

XIII. QUESTION 1041, by Mr. John Ryley, of Leeds.

I should like to see a scientific solution to question 69, page 164, of Dr. Hutton's Conic Sections and Select Exercises, which is this: "If a glass tube, 36 inches long, close at top, be sunk perpendicularly into water, till its lower or open end be 30 inches below the surface of the water; how high will the water rise within the tube, the quicksilver in the common barometer at the same time standing at  $29\frac{1}{2}$  inches?" The answer being only there given, but not the solution.

*Answered by Mr. J. Gough, Kerkral.*

The density of the air is as its spring, which in the open tube is equal to a column of mercury of the same base and  $29\frac{1}{2}$  inches high; but in the immersed tube this weight is increased by a column of water  $30 - x$  inches high,  $x$  denoting the height of the water in the tube; but  $13600 : 1000 :: 30 - x : 2.205 - .0735x =$  a column of mercury of the same weight; and the whole pressure  $= 29.5 + 2.205 - .0735x = 31.705 - .0735x$ ; but when the matter is given, the magnitude is inversely as the density, or pressure in the present case, therefore  $31.705 - .0735x : 29.5 :: 36 : 36 - x$ ; hence  $x^2 - 467.36x = -1080$ , and  $x = 2.33$  inches, as required.

*The same by Mr. Thomas Hornby, Land Surveyor.*

At page 137 of Dr. Hutton's Conics, the specific gravity of quicksilver to that of water, is stated as 14 to 1. And since the heights retained above the level by the pressure of the atmosphere, are as their densities, we have by proportion  $1 : 14 :: 29.5 : 413$  inches  $= 34.416$  feet, the height at which water will stand when quicksilver stands at  $29.5$ . Therefore, to find what height water will rise in a tube 36 inches or 3 feet long, when sunk perpendicularly 30 inches or  $2\frac{1}{2}$  feet in water. Let  $x =$  the space occupied by water; then, will  $3 - x =$  the space occupied by air. But by the rule at page 390, vol. 1, Hutton's Dictionary, under the article Diving Bell, the space occupied by air, is to the space filled with water, as  $34.416$  feet, is to the depth of the surface of the water in the tube, below the common surface of it. That is,  $34.416 : 2.5 - x :: 3 - x : x$ ; consequently  $34.416x = 7.5 - 5.5x + x^2$ ; hence  $x$  is found  $= .1887875$  feet  $= 2.26545$  inches, the same as that in Hutton's Conics.

*The same by Miss Maria Middleton, Eden, near Durham.*

Let  $l = 36$  inches, the length of the tube,  $b = 30$  inches the part immersed,  $x =$  height of water in the tube, and  $f = 413$  inches, the height of a column of water equal to the pressure of the atmosphere, when the quicksilver stands at  $29\frac{1}{2}$  inches. Then, since the spaces occupied by the same quantity of air, are reciprocally as the compressing forces, it will be, as  $l - x : l :: f : lf \div (l - x) =$  force of the air in  $l - x$ ; hence  $lf \div (l - x) + x = b + f$ , and  $x = 2.2654115$  inches.

XIV. QUESTION 1042, by Mr. T. Milner, *Lartington Free School.*

I have seen a sheep leap from a bridge very high, into water, and swim out. Now, if a globe, whose weight is 112 pounds, and one foot in diameter, fall from an eminence ten yards high, how deep must the water be, just to destroy all the globe's velocity, supposing the density of air, water, and the globe to be as the numbers  $1\frac{1}{2}$ , 1000, and 10,000 respectively?

*Answered by the Rev. J. Furnass, Heddon on the Wall.*

The numbers given in this question do not seem to be rightly proposed; for first a globe of one foot in diameter, and density 10 times that of water, will weigh near three times 112 pounds, or the given weight; and, again, a globe that is heavier than water, will never lose all its velocity, but will continually descend. It may indeed lose all its force in the water, so as to come to move with a uniform velocity, when the velocity has increased so far that the resisting force has become equal to the motive force urging the body downward. And all these circumstances, with proper data, may be determined by Doctor Hutton's Select Exercises, page 227, 230, &c. or his Dictionary, vol. 2, page 361. Thus,

Put  $d =$  the diameter of the globe,  $n =$  its density,  $n = 1\frac{1}{2}$  the density of the air,  $s = 30$  feet the height of the eminence,  $g = 16\frac{1}{2}$  feet, and  $v =$  the velocity of the globe at the surface of the water. Now, to determine  $v$  in terms of  $s$ , put  $b = 3n \div 8nd$ ,  $c = 2.718281828$ , and  $ab = 2g$ ; then, by page 231, Select Exercises,  $v = \sqrt{(a - ac^{-2bs})} = 43.82$  nearly, or nearly the same as the velocity freely generated by gravity, and is the velocity with which the globe enters the water.

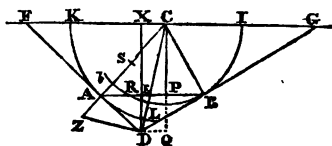
Now put  $x$  for any space moved in the water, and  $e = 43.82$ , the first velocity, the other letters being as above: then, by prob. 31, page 227, the general equation is  $bx = \log. (e \div v)$ , or  $x = (1 \div b) \times \log. (e \div v)$ , where the space  $x$  may be found answering to any given velocity  $v$ .

PRIZE QUESTION, *by Amicus.*

To construct a triangle, having two of the sides given, and such, that a perpendicular to the third side being drawn from its opposite angle, that perpendicular shall be the height of a prism, whose base is the triangle, and solidity a maximum?

*Answered by Amicus.*

Make  $CA$  and  $cb$  = the two given sides, and  $cs$  = their third proportional; produce  $CA$  to  $z$  till  $Az \cdot zs = \frac{1}{4} cb^2$ , erect  $AD$  perpendicular  $CA$ , meeting a semicircle described on the diameter  $cz$  in  $D$ ; draw  $CD$ , and with the radii  $cb$ ,  $CA$  describe two circles  $bBI$ ,  $KAL$ ; from  $D$  draw  $DB$  a tangent to  $bBI$  in  $B$ ; draw  $CB$ ,  $AB$ , and  $ACB$  is the triangle required.

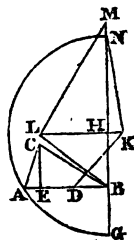


For, drawing through  $c$  a parallel to  $AB$ , meeting  $DA$ ,  $DB$  produced in  $F$  and  $G$ , and letting fall the perpendiculars  $CF$ ,  $DX$ ; by construction  $\frac{1}{4}cb^2 = AZ \cdot zs$ , and  $AC^2 - cb^2 = AS \cdot AC = AC \cdot zs - AC \cdot AZ = AC \cdot zs - AD^2 = DB^2 - AD^2$ ,  $DB^2 = AC \cdot zs$ ,  $AD^2 = AC \cdot AZ$ ,  $AD^2 = AC^2 \cdot zs \cdot AZ = \frac{1}{4}AC^2 \cdot cb^2$ , or  $DB \cdot AD = \frac{1}{2}AC \cdot CB = CD \cdot DR = \frac{1}{2}CD \cdot CF$ , therefore  $RX = 2DR$ . Now, if  $FDG$  be a given triangle, and it be required to inscribe another within it such that  $AB \times CF^2$  may be a maximum when  $c$  is a given point and  $AB$  parallel to  $FG$ , then since  $DX : FG :: DR : AB$ , the ratio of  $DR$  to  $AB$  being then given,  $DR \cdot RX^2$  is a maximum, therefore by Simpson's Geometry, page 208,  $RX = CF = 2DR$ , and when this is a maximum within the tangents  $DF$ ,  $DG$ , it must needs be such within the circles  $KAL$ ,  $bBI$ ; consequently  $CF \cdot AB$  and the prism in question is a maximum.

*The same answered by Mr. John Surtees, Alston.*

Let  $x = AB$  the base,  $m$  and  $n$  = the two sides  $BC$  and  $AC$ . Then  $m^2 ((x^2 + m^2 - n^2) \div 2x)^2 = CE^2$ , and by the question  $((m^2 - n^2) 2x^2 - x^4 - (m^2 - n^2)^2) \div x = a$  maximum, which put into fluxions and reduced, gives  $x^4 - \frac{2}{3}x^2 \times (m^2 + n^2) = \frac{1}{3} \times (m^2 - n^2)^2$ . Hence this construction:

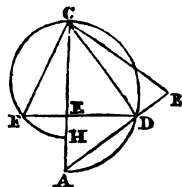
*Construction.* Take  $HM (= 3 \times (m - n))$  perpendicular to  $LH = \sqrt{(m^2 + n^2)}$ ,  $\angle BLH = \angle LMH$ ,  $BD$  parallel and equal to  $HK = \frac{1}{3}(m + n)$ ,  $KN = KD$ , and  $BG = m - n$ ; then a mean proportional between  $BG$  and  $BN$  will be the base of the triangle.



*The same by Mr. John Craggs, of Hilton.*

Suppose  $ACB$  to be the triangle,  $AC$  and  $BC$  the given sides, and take

$CH : CA :: CB^2 : EA^2$ , and on the diameters  $CA$  and  $CH$  describe the semicircles  $CDA$  and  $CFH$ ; and conceive the line  $DEF$  drawn perpendicular to  $CH$ , and join  $CF$ . Because  $CDA$  is a right angle, the semicircle passes through  $D$ , therefore  $CE \times CA = CD^2$ , and  $CE \times CH = CF^2$ , therefore  $CF^2 : CD^2 :: CH : CA :: CB^2 : CA^2$ , hence  $CF : CD :: CB : CA$ , and because the  $\angle CDF = \angle CAB$ , the triangles are similar, whence  $CA^2 : CD^2 :: CD \times AB : CE \times DF$ , therefore  $CA \times DF = CD \times AB$ , by multiplying by  $CD$  we get  $CA \times CD \times DF = CD^2 \times AB$ ; but  $CA$  is a constant and given quantity, therefore  $CD \times DF$  is a maximum. Put  $x = CE$ ,  $b = CA$ ,  $a = CH$ , then  $\sqrt{(bx - x^2)} = DE$ , and  $\sqrt{(ax - x^2)} = EF$ , also  $\sqrt{(bx)} = CD$ ; consequently  $\sqrt{(bx)} \times \sqrt{(bx - x^2)} + \sqrt{(bx)} \times \sqrt{(ax - x^2)}$  or  $\sqrt{(bx^2 - x^3)} + \sqrt{(ax^2 - x^3)}$  is a maximum, this, in fluxions, and reduced, gives  $3x^2 = (a + b) \times 4x - 4ab$ , an equation similar to Sanderson's solution to question 800 in the Diary. Hence the construction will be similar, and needless to repeat here.



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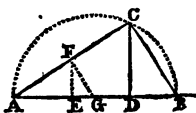
*Questions proposed in 1799, and answered in 1800.*

I. QUESTION 1044, by Mr. Robert Langdon, of Allow.

On a given right line, as a base, to construct a triangle such, that if a perpendicular be let fall on the base from the vertical angle, it may be a mean proportional between the segments of the base; the other two sides being in the ratio of 5 to 4?

*Answered by Mr. John Hawkes, Finedon.*

Divide the given base  $AB$  in  $D$ , so that  $AD : DB :: 25 : 16$ , on which erect the perpendicular  $DC =$  the mean proportional between them, or  $= 20$  of these parts. Join  $AC$  and  $BC$ ; so shall  $ABC$  be the required triangle. For  $AC : BC :: \sqrt{(AD^2 + CD^2)} : \sqrt{(BD^2 + CD^2)} :: \sqrt{(25^2 + 20^2)} : \sqrt{(16^2 + 20^2)} :: \sqrt{5^2} : \sqrt{4^2} :: 5 : 4$ , the given ratio of the sides.



*The same by Mr. Wm. Haydock, jun. Blackburn.*

On the given base  $AB$ , as a diameter, describe the semicircle  $ACB$ ; on  $AB$  take  $AE = 5$ , and raise the perpendicular  $EF = 4$ ; draw  $AFC$ ; then join  $BC$ , and let fall the perpendicular  $CD$ , and it is done.

For, the two right-angled triangles  $AEF$ ,  $ACB$  being equiangular, it will be, as  $AE : EF :: AC : CB :: 5 : 4$  the given ratio. Also the angle  $C$  being a right one, the perpendicular  $CD$  is a mean proportional between the segments  $AD$ ,  $DB$ , by Hutton's Geom. Theor. 87.

*The same by Mr. Henry Hunter, Alnwick.*

On the given base AB describe the semicircle ACB; make AD to DB as 25 to 16; erect the perpendicular DC to meet the circle in C; then join AB and BC, so shall ABC be the triangle required.

For, by the nature of the circle,  $CD^2 = AD \cdot DB$ ; and, by construction, as  $25 : 16 :: AD : DB :: AC^2 : BC^2$ , consequently  $AC : BC :: 5 : 4$ .

II. QUESTION 1045, by Mr. Geo. Boulby, of Ackworth.

The sum of the vibrations made by three pendulums in one minute is 252, and the ratios of the number of vibrations made by each, as 5, 7, 9; required the lengths of those pendulums, supposing the length of the seconds pendulum to be  $39\frac{1}{8}$  inches?

*Answered by the Rev. J. Furnass.*

First,  $5 + 7 + 9 = 21$ ; then as  $21 : 252 :: 5 : 60 :: 7 : 84 :: 9 : 108$ ; therefore 60, 84, 108 are the number of vibrations of each pendulum respectively. Now  $39\frac{1}{8}$  inches being the length of the pendulum corresponding to 60 seconds, we have, by Dr. Hutton's Dictionary, article Pendulum, as  $84^3 : 60^3 :: 39\frac{1}{8} : 19.96$  inches, the length corresponding to 84 vibrations: and  $108^3 : 60^3 :: 39\frac{1}{8} : 12.07$  inches, the length corresponding to 108 vibrations.

*The same by Mr. Charles Johnson.*

First,  $252 \div (5 + 7 + 9) = 12$ , which multiplied by 5, 7, 9 separately, give 60, 84, 108, the numbers of vibrations made by each pendulum in one minute. Hence, as  $80^3 : 60^3 :: 39\frac{1}{8} : 19.9628$ , and as  $108^3 : 60^3 :: 39\frac{1}{8} : 12.0756$  inches, their lengths as required.

*The same by Mr. Thomas Scott, of Wigton.*

As  $21 : 252 :: 5 : 60$ , the first number of vibrations;  
as  $21 : 252 :: 7 : 84$ , the second ditto;  
as  $21 : 252 :: 9 : 108$ , the third ditto.

Then, the lengths of pendulums being to one another reciprocally as the squares of the times of their several oscillations;

as  $60^3 : 60^3 :: 39\frac{1}{8} : 39\frac{1}{8}$  the length of the first,  
as  $84^3 : 60^3 :: 39\frac{1}{8} : 19\frac{3}{4}$  ditto the second,  
as  $108^3 : 60^3 :: 39\frac{1}{8} : 12\frac{1}{8}$  ditto the third.

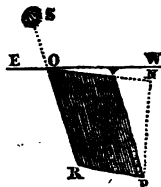
*The same by Mr. John Smith, of Alton Park.*

First,  $5 + 7 + 9 = 21$ . Then  $21 : 252 :: 5 : 60 :: 7 : 84 :: 9 : 108$ ; so that 60, 84, and 108 are the vibrations made by each pendulum. Then, the length of the first, or that which vibrates 60 times

edge of the entrance, and the base of the part enlightened to be in the same straight line with the outside of the house : the latitude of Fros-terly being  $54^{\circ} 56'$  ?

*Answered by Mr Thomas Coultherd.*

Let *EW* be an east and west line, *OVN* a part of the outside of the front wall, and *ORPV* the part enlightened. Put  $b = 6\frac{1}{2}$ , and  $c = 3\frac{1}{2} = ov$ . Then, from the data, the sun's azimuth from the south is found  $23^{\circ}33'34''$ , and the altitude of his centre  $56^{\circ}36'52''$ ; and when  $15'22''$ , the difference between the semidiameter and refraction, is deducted, the remainder  $56^{\circ}21'\frac{1}{2}$  will be the altitude of his lower limb, whose tangent call *a*; also the difference between the sun's distance from the east and the declination of the house, gives the angle  $RON = RVN = 47^{\circ}56'26''$ , the sine of which call *n*, then as  $a : b :: n : bn \div a = RN$ , and  $bnc \div a = 11.24$ , the number of square feet required.



*The same by Mr. Thomas Hopper, and Mr. James Thoubren, of Mr. Rutherford's School, Lanchester.*

The latitude, the declination and meridian distance being given, the sun's azimuth from the south, is found =  $25^{\circ} 33' 34''$ , and central altitude  $56^{\circ} 36' 52''$ , or  $56^{\circ} 20' 19''$  by correction for the sun's lower limb. Now, in the figure, the  $\angle VPN = 16^{\circ} 30' +$  azimuth from the south =  $42^{\circ} 3' 34''$ , and its complement  $47^{\circ} 56' 26'' = \angle PVN$ . Then as radius :  $6\frac{1}{2}$  :: cotang. sun's alt. : or or VP =  $4.3286$  the length of the shadow on the floor; also as radius : VP :: sine v : PN =  $3.2138$  the perpendicular length of the figure ORPV; hence PN  $\times$  OV =  $3.2138 \times 3\frac{1}{2} = 11.2483$  feet, the area enlightened, as required.

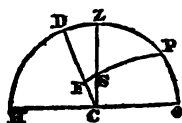
VI. QUESTION 1049, by Mr. Thomas Squire, of *Astwick*.

In latitude 36 degrees, the sun was observed to be due east, when the number of degrees from noon was double his altitude. Required the time of observation?

*Answered by the Rev. Mr. Ewbank, of Thornton Steward.*

In the adjacent figure, let p, z, and s represent the pole, the zenith, and the sun, respectively. Also, let  $a =$  the sine of  $54^\circ$ ,  $b =$  sine of  $36^\circ$ , and  $x = \text{tang. } \angle r$ . Then, by spherics,  $ax = \text{tang. of } zs \text{ the coaltitude}$ . But, by trigonometry, and the question,  $2ax \div (a'x' - 1) = x$ ; hence, by reduction,  $x = \sqrt{((2a + 1) \div aa)} = 2$ , the natural tangent of  $63^\circ 26'$ , &c. the number of degrees from noon; and consequently the

The diagram shows a horizontal baseline with point C at its center. A semi-circular arc is drawn above the baseline. Point p is located on the baseline to the left of C. Point z is located on the arc directly above C. Point s is located on the arc between p and z. Point D is located on the arc further to the left than p. Point R is located on the baseline to the right of C. Lines connect p to z, p to s, and z to s.



sun's altitude was  $31^{\circ} 43'$ . Let  $s$  = its sine; then, by spherics, the sine of the sun's declination =  $bs = \cdot 3090099$ , sine of  $18^{\circ}$  nearly. Hence the apparent time of observation was  $46\frac{1}{3}$  minutes past 7 o'clock in the morning of the 11th of May.

*The same by Mr. Kinnebrook, jun. Norwich.*

Let  $z$  be the zenith,  $p$  the pole, and  $s$  the sun's place. Then, in the right-angled spherical triangle  $szp$ , as sine  $zp$  : radius :: tang.  $zs$  : tang.  $zps$  = tang.  $zs \div$  sine  $zp$ . Now, per question, tang. of twice comp.  $zs$  or 2 radius square  $\div$  (tang.  $zs$  — cot.  $zs$ ) = tang.  $zs \div$  sine  $zp$ , hence tang.  $zs = \sqrt{(2 \sin. zp + 1)} = 1\cdot6180339$ , and  $zs = 58^{\circ} 16' 57''$ , also the angle  $p = 63^{\circ} 26' 6''$  answering to 7h. 46m. 16s. in the morning, and  $ps$  the co-declin. =  $72^{\circ}$ , which, if the observation was made in north latitude answers to the 11th of May, or 1st of August at 7h. 46m. 16s. morning.

VII. QUESTION 1050, by Mr. James Wilding, High-Ercall.

In the play ground belonging to the school at High-Ercall, is a remarkable fine beech tree, whose branches afford a pleasant shade in the summer season; around which tree I intend making a hexagonal or six-sided seat; for which purpose I have procured a deal plank  $16\frac{1}{2}$  feet long and 11 inches broad; I should be glad therefore to know, as a direction to my workman, the inner and the outer lengths of each side, so as to occasion the least loss in cutting?

*Answered by Mr. T. S. Evans.*

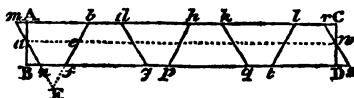
Put  $x$  = shortest side,  $t$  = tang. of  $30^{\circ}$ ; then  $x + 22t$  = longest side, and  $2x + 22t$  = 66 inches or  $\frac{1}{2}$  the length of the whole plank; therefore  $x = 26\cdot64915$  the shortest side, and  $39\cdot35085$  is the longer.

*Note.* This is a solution, taking the words of the question strictly; for then there will be no waste; the piece coming off one end must be put in wedge-ways.

*The same answered by Mr. O. G. Gregory, of Cambridge.*

This entertaining question may be answered in various ways; but perhaps the following practical method may be found as easy as any, and of service to the workman.

Suppose ABCD the plank of which the hexagonal seat is to be formed: let the line  $ax$ , along the middle, be divided into 6 equal parts, and on  $ax$ , one of those equal parts, let an equilateral triangle  $axx$  be constructed. Produce



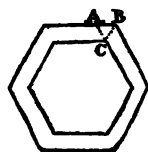
$ae$  and  $ea$  to  $b$  and  $m$ ; then make  $md$ ,  $dk$ ,  $kr$ , each equal to twice  $ae$ ; also set off  $bh$ ,  $hl$ ,  $ng$ ,  $gq$ ,  $qs$ ,  $fp$ ,  $pt$ , each equal to twice  $ae$ . Draw  $dg$ ,  $kq$ ,  $rs$ , which will evidently be parallel to  $mn$ ; also draw  $hp$ ,  $lt$ , which will be parallel to  $bf$ . Fill up the space  $ama$  with  $anb$ , and  $wds$  with  $wcr$ . Then the six pieces  $mbnf$ ,  $fgbd$ ,  $dhgp$ ,  $pghk$ ,  $klqt$ ,  $tslr$ , will be equal and similar to each other, and, when properly joined together, will form the hexagonal seat required.

To find the dimensions of each of these pieces, by calculation, it must first be considered, that  $aam$  is half an equilateral triangle, of which  $aa$  is given  $= 5\frac{1}{2}$  inches: then, as  $\sqrt{3} : 1 :: aa : am = \frac{1}{8}\sqrt{3} = 3.17542$  inches. To  $ae$  ( $= 33$  inches) add double  $am$ , the sum  $39.35084$  inches  $= mb = fg = dh$ , &c.; and, from  $ae$  subtract double  $am$ , the remainder  $26.64916$  inches  $= nf = bd = gp$ , &c.

The truth of the above method of cutting the plank, will be too obvious, after a little consideration, to need any formal demonstration.

*The same answered by Mr. James Mason, of Clapham.*

Let  $ac = 11$  inches, the breadth of the board. It is evident that  $ab$  must be  $= \frac{1}{2}bc$ ; therefore put  $x = ab$ , and  $2x = bc$ , then  $3x^2 = ac^2 = 11^2$ , or  $x^2 = 33.333$ , and  $x = 6.35$ ; hence  $16\frac{1}{2} \times \frac{1}{2} + 6.35 = 39.35$ , the outer length of each side, and  $16\frac{1}{2} \times \frac{1}{2} - 6.35 = 26.65$  inches the inner length of the same.



VIII. QUESTION 1051, by Mr. Thomas Hind, at Mr. Shepherd's Boarding School, Layton, Essex.

My clock, which ought to beat seconds, gains at the rate of 30 minutes per week; I should therefore be glad to know how many revolutions I must turn the nut of the screw part of the pendulum, downward, to reduce it to keep true time, supposing there be 40 rounds to an inch?

*Answered by Mr. J. Collins, Schoolmaster, Kensington.*

First,  $24 \times 7 \times 60 = 10080$  minutes in a week. Then, as  $10080 : 30 :: 60'' : \frac{1}{8}''$  gained per minute. Now the lengths of pendulums being inversely as the squares of their vibrations in the same time, therefore  $(60\frac{1}{8})^2 : 60^2 :: 39\frac{1}{4} : 38.893$  the length of the pendulum. Therefore  $39\frac{1}{4} - 38.893 = .232$  is the length to be added. Lastly, as 1 inch : 40 :: .232 : 9.28 rounds, or nearly  $9\frac{1}{4}$ .

*The same answered by Mr. W. Newby.*

As 7 days : 30 min. :: 1 min. : .1785 of a second, what the pendulum gained per minute. Now, as  $60.1785^2 : 60^2 :: 39\frac{1}{4} : 38.8932$  the length of the pendulum; which being subtracted from  $39.125$ , the re-

mainder .2318 is what the pendulum is too short ; which, at 40 rounds to an inch, will require 9.272 turns to make it keep true time.

IX. QUESTION 1052, *by Mr. J. Reffshir, Deptford.*

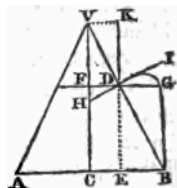
Admitting a right cone to be full of water, standing on a plane ; whereabouts in the side must a hole be bored, so that water may spout just to the circumference or edge of the cone's base ; supposing its axis 24 feet, and diameter of its base 20 feet ?

*Answered.*

Our Correspondents answer this question on two different principles, some on the supposition that the water spouts out in a direction parallel to the horizon, or perpendicular to the axis of the cone ; and others, that it issues in a direction perpendicular to the side of the cone. And although this seems the more accurate, yet they both bring out the same conclusion. We shall insert two or three instances of each of these methods.

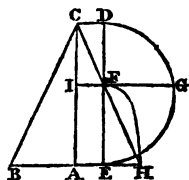
1. *By Mr. John Craggs, of Hilton.*

Let  $ABV$  be the cone, its axis  $VC$ , and suppose  $D$  the place of the required point in the side, and  $DE$  perpendicular to the base  $AB$ , also  $FDG$  parallel to the same, but  $HDI$  perpendicular to the side  $BV$ , which is the direction of the spouting fluid at  $D$ . Put  $x = VF$ ,  $s = \sin. \angle IDG$  or  $\angle EDB$  or  $\angle CVB = VB \div CB = 10 \div 26 = \frac{5}{13}$ , and  $c = \cos.$  of the same or  $\sin. \angle B = VB \div VC = \frac{12}{13}$ . By sim. tri.  $VC : VB :: VF : VD = \frac{12}{13}x$ ; consequently  $26 - \frac{12}{13}x = DB$ . Now, by mechanics (see Dr. Hutton's Mathematics, vol. 2, page 162,) the oblique range or distance  $DB$  or  $26 - \frac{12}{13}x = 4cx \div s^2$ ; hence is found  $x = 26 \div \left( \frac{4c}{s^2} + \frac{13}{12} \right) = \frac{600}{601} = .9983361$  of a foot  $= VF$ . Then, by similar triangles,  $VC : VB :: VF : VD = 1.0815307$  the distance from the vertex.



2. *By Mr. T. S. Evans.*

The principle of this question is, to find the position  $DE$ , so that  $FG$  may be equal  $\frac{1}{2}EH$ , supposing the water to issue out horizontally. By the property of the circle  $FG^2 = DF \cdot FE$ , and, by hydrostatics,  $EH = 2\sqrt{FG} = 2\sqrt{DF \cdot FE}$ . Put  $t = 2.4$  the tangent of the  $\angle H$ , and  $x = EH$ ; then  $tx = EF$ , and  $t(10 - x) = DF$ , also  $x^2 = 4tx(10t - tx)$ , or  $x = 4t^2(10 - x)$ ; hence  $x = 40t^2 \div (1 + 4t^2) = \frac{57.60}{601} = 9.584$  nearly. Hence  $CF$  the distance of the hole from the vertex is  $26 \div (1 + 4t^2) = \frac{650}{601} = 1.0815$ .



3. *By Mr. John Surtees, of Wearmouth.*

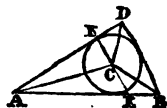
DCB being the parabolic curve described by the water (see the 1st fig.) in the direction DI; put  $m = VC = 24$ ,  $n = CB = 10$ ,  $p = VB = 26$ , and  $x = KD = VF$ . Then, by similar triangles,  $DB = p(m - x) \div m$ , also  $m'x \div p' = \frac{1}{4}$  the parameter of the parabola; then the distance on the inclined plane DB is  $4mpx \div n^2$ ; hence  $x = n^2m \div (4m^2 + n^2) = 600 \div 601 = .998336$  feet, and  $DV = 650 \div 601 = 1.08153078$  feet, as required.

X. QUESTION 1053, *by Mr. Wm. Burdon, Acaster Malbis.*

Two gentlemen bought a triangular estate, the sides of which are 2160, 3840, and 4750 links, which they have divided between them by a straight fence 1800 links long, drawn through the centre of its inscribed circle, and terminated by the two longest sides of the triangle; Query how much of the estate belongs to each person?

*Answered by Mr. David Kinnebrook, Jun. Norwich.*

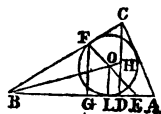
Let ABD represent the triangular estate, and c the centre of its inscribed circle; through which, by prob. 72, Simpson's Algebra, draw EF = the given fence; then will the triangle AEF, and trapezium EBDF be the parts required.



CALCUL. In the triangle ABD, are given all the sides, to find the  $\angle A = 26^\circ 31' = 2\angle DAC$ , and  $\angle D = 100^\circ 57' 2'' = 2\angle ADC$ ; also, in the triangle ACD, are given all the angles, and the side AD, to find  $AC = 3303.374$  links. Now, by the calcul. to prob. 72, Simpson's Algebra, as  $EF : AC :: \cos. \angle EAC$  or  $FAC : \text{tang. } 60^\circ 45' 24''$ , the half comp. of which is  $14^\circ 37' 18''$ ; again, as  $\text{tang. } 14^\circ 37' 18'' : \text{radius} :: \sin. \angle EAC : \cos. \frac{1}{2}(\angle F - \angle E) = 28^\circ 28'$ ; but  $\angle E + \angle F = 153^\circ 29'$ , hence  $\angle E = 48^\circ 16' 30''$ , and  $\angle F = 105^\circ 12' 30''$ , also as  $\sin. \angle E : AC :: \sin. \angle C : AE = 3890.5$  links; hence the area of the triangle  $AEF = AE \times EF \times \frac{1}{2} \sin. \angle E = 2613314$  links = 26 acres 0 roods 21 perches, and the trapezium EBDF = 14 acres 2 roods 13 perches.

*The same answered by Mr. John Hartley, Auditor's Office.*

Let ABC be the given triangle, EF the given dividing line, and o the centre of the inscribed circle; also the other lines and perpendiculars as in the figure. Then there are given  $EF = 18$  chains,  $AC = 21.6$ ,  $CB = 38.4$ , and  $AB = 47.5$ . Now, having the sides of the triangle ABC, the angles and perpendicular CD are found by trigonometry, viz.  $\angle A = 51^\circ 58'$ ,  $\angle B = 26^\circ 30'$ ,



$\angle c = 101^\circ 32'$ , and  $cd = 17.14$  chains, the area  $= 40$  ac. 2 r. 34 p. likewise the segments of the base  $AD = 13.14$ , and  $DB = 34.36$  chains.

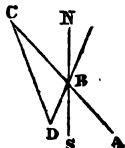
Dividing the area by half the sum of the three sides, gives the radius  $OL$  of the circle  $= 7.57$ , also  $BL = \frac{1}{2}AB + \frac{1}{2}BC - \frac{1}{2}AC = 32.15$ , hence  $AL = 15.35$ , and  $OB = \sqrt{(BL^2 + OL^2)} = 33.029$ , which line bisects the  $\angle B$ . Now in the triangle  $EFB$  are given  $EF$  and the line  $OB$ , which bisects the angle  $B$ ; then, by prop. 17, Simpson's Trigonometry, as  $OB : EF :: \sec. \frac{1}{2}\angle B : \text{tang. of an angle}$ , and as  $\text{tang. of half that angle} : \text{radius} :: \sin. \frac{1}{2}\angle B : \sin. \angle BOF = 61^\circ 30'$ , whence  $\angle BOE = 118^\circ 30'$ , and  $BEF = 48^\circ 15'$ . In the triangle  $BFE$ , are given the angles and one side  $EF$ , to find  $BE = 38.92$ . And in the right-angled triangle  $EGF$ , are given the angles and one side  $EF = 13.43$ ; from whence the area of  $BFE = 26$  acres 0 roods 21 perches; consequently the area of  $ACFE = 14$  acres 2 roods 13 perches, which are the required parts.

XI. QUESTION 1054, by Mr. Tho. Coultherd.

On Lammas Day, 1797, at 10 o'clock in the morning, in the latitude of  $54^\circ 40'$ , I observed a boy setting his kite up into the air with a cord of 80 fathoms. Now admitting the string when at its full stretch, to make an angle of 60 degrees with the plane of the horizon, the boy's hand to be 4 feet above the same, and the wind to blow from the south-south-west; I desire to know what distance the boy would be from the extremity of the shadow, which the kite would make on the ground when in a vertical position, and its top 3 feet above the cord, allowing the earth to be perfectly level?

*Answered by Mr. P. M. Laurent, assistant at the Rev. Mr. Whitaker's Academy, Southampton.*

The latitude, day, and hour being given, the sun's azimuth is found to be  $43^\circ 53' 48''$ , and the apparent altitude of his upper limb,  $46^\circ 57' 10''$ ; the length of the string, its angle with the plane of the horizon, &c. being given the perpendicular height of the top of the kite is  $= 422.69$  feet; now supposing  $NS$  to be a meridian line,  $B$  the point where the vertical line that would pass through the top of the kite meets the plane of the horizon,  $D$  the place where the boy stands,  $DB$  the direction of the wind; and  $AC$  a line making with  $NS$  an angle  $= 43^\circ 53' 48''$  the azimuth, the shadow will be projected in the direction of  $AC$ , the  $\angle DBS$ , by the question,  $= 22^\circ 30'$ ,  $DB$  is found by trig.  $= 239.99$  feet and if  $C$  be the extremity of the shadow, the vertical height and the sun's altitude being as above,  $BC$  is found  $= 394.85$  feet; the sides  $BC$  and  $BD$  being given in the triangle  $BCD$ , as well as the  $\angle B$ , the supplement of the  $\angle ABD$ , the side  $CD$ , for the required distance is found by trigonometry  $= 537.9$  feet  $= 179.3$  yards,



*The same answered by Mr. John Ramsay, London.*

In the triangle  $ABC$ , let  $A$  be the place of the boy,  $B$  where the kite was perpendicular,  $C$  where the extreme part of the shadow met the earth. By plane trig. the perp. height of the kite above the boy's hand is,  $415.692$  feet; this  $+ 4 + 3 = 422.692$  feet, its height from the ground; and  $AB = 240$  feet. By spherical trig. the sun's true alt. in lat.  $54^{\circ} 40'$  (at the time mentioned in the question.) was  $46^{\circ} 38' 3''$ , this  $+ 16'$  (= sun's semidiameter)  $+ 54''$  (= refraction)  $- 7''$  (= parallax) gives the apparent alt. of the sun's upper limb  $= 46^{\circ} 54' 50''$ , and the angle the sun's rays make with the meridian  $+ ss$  being 2 points from south  $= 22^{\circ} 30'$ , gives  $66^{\circ} 22' 20'' =$  the supp. of the  $\angle ABC$ ; then, by plane trigonometry  $AC$  is found  $= 538.485$  feet, the distance required.



XII. QUESTION 1055, by the Rev. Mr. Furnas, Heddon-on-the-Wall.

A gentleman has a circular plantation, in which are two walks, the one the chord of an arch of the fence, the other the versed sine or height of the same perpendicular from the middle of the chord, whose lengths are 4 chains and 1 chain respectively. Now the gentleman, wishing to have a ditch made round on the outside of the same, of 6 feet in breadth and  $4\frac{1}{2}$  feet in depth, the inside coinciding with the circumference of the plantation, has two proposals for this undertaking, the one at 2d. the solid yard, and the other at 6d. per yard running equitable circumference, or along in the middle of the ditch. It is required to shew which is most in favour of the owner?

*Answered by A + B.*

Without finding the size of the plantation, it is easy to conceive, that (since the ditch is to be 2 yards wide and  $1\frac{1}{2}$  yards deep) there will be 3 solid yards of ditching for every yard of the mean length; and as the price by the solid yard is  $\frac{1}{3}$  of that by the running yard, the expence will be the same either way.

*The same answered by Mr. J. Farrah, Tottenham.*

The owner may accept of either proposal, for they are exactly equal; because the breadth 2 yards  $\times 1\frac{1}{2}$  the depth, and by one length, gives 3 yards, which at 2 pence the yard is the same thing as 6 pence per yard in length. The whole expence will be 8l. 15s. 11d.

*The same by Mr. O. G. Gregory, Cambridge.*

The area of a circular ring being found by multiplying the circumference which runs through the middle of the ring, into the breadth of

the ring ; it is obvious that the solid content of the ditch will be found by multiplying the circumference along the middle of the ditch into the breadth of it, and that product into the depth of the ditch ; therefore in the present case, the circumference along the middle  $\times 2 \times 1\frac{1}{2}$  = the circumference along the middle  $\times 3$ , will be the solid content of the ditch. Hence the number of cubic yards in the ditch is equal to thrice the number of lineal yards in the equitable circumference, along the middle of the ditch ; and consequently, as the proposal by the solid yard is 2 pence, one third of 6 pence, the proposed price per lineal yard, the one proposal is no more in favour of the owner than the other.

If it were required to find the expence of making the ditch, it might be effected thus. A chord and versed sine of the circle being given, the diameter may be found by means of the following proportion : as 1 (the versed sine) : 2 (the half chord) :: 2 : 4 the suppl. versed sine, hence 5 chains or 110 yards, is the diameter of the circular plantation. To 110 add 2, the sum 112 multiplied into 3.141593 gives 351.8584 yards, for the running equitable circumference, which at 6d. per yard, amounts to 8l. 15s. 11d. the expence of making the ditch.

XIII. QUESTION 1056, *by Mr. Wm. Francis, jun.*

A cast-iron ball, of 4 inches in diameter, is put into a cylindrical copper vessel, open at top, the vessel and ball then together weighing 11lb. ; but the remainder of the vessel being then filled up with water, the whole was found to poise with 60lb. Now the inside diameter of the vessel being double its depth, all its dimensions may be hence found : Query how ?

*Answered by Mr. Wm. Francis, jun.*

First,  $.5236 \times 4^3 = 33.5104$  inches, the ball's content. And, according to Dr. Hutton, a ball of cast iron, 4 inches in diameter, weighs 9lb. therefore  $9 + 11 = 20$  the weight of the ball and vessel together, consequently  $60 - 20 = 40$ lb. or 480 ounces, is the weight of the water alone ; then as  $1000 : 1728 :: 480 : 1105.32$  cubic inches, is the content of the water ; therefore  $1105.32 + 33.5104 = 1139.4304$  is the content or capacity of the vessel ; hence  $\sqrt[3]{(1139.4304 \times 2 \div .7854)} = 14.2629$  inches is the vessel's diameter, and consequently  $7.13145$  its depth.

Now let  $x$  = the thickness of metal ; then  $14.2629^3 \times .7854x = 159.77417x$  is the solidity of the bottom, and  $(14.2629 + x) \times 3.1416 = 44.80832664 + 3.1416x$  is the mean or middle circumference, hence  $(44.80832664 + 3.1416x) \times (7.13145 + x) \times x = 319.548x + 67.2125x^2 + 3.1416x^3$  is the solidity of the upright sides, therefore  $159.77417x + 319.548x + 67.2125x^2 + 3.1416x^3$  is the whole solidity of the metal. Then, as  $9000 : 11 \times 16$  or  $176 :: 1728 : 33.792$  inches, the solidity by specific gravity ; consequently  $3.1416x^3$

+  $67\cdot2125x' + 479\cdot3225x = 33\cdot792$  or  $x^3 + 21\cdot39435x' + 152\cdot57273x + 362\cdot68828249 = 10\cdot7563 + 362\cdot68828249$  by dividing by  $3\cdot1416$  and completing the cube. Hence  $x + 7\cdot13145 = 7\cdot201268$  by evolution, and  $x = 0\cdot69818$ , the thickness of the metal.

*The same answered by Mr. James Gale, London.*

A cast iron ball of 4 inches diameter is 9lb. =  $33\cdot5104$  cubic inches; the copper in the vessel is 11lb. =  $33\cdot792$  cubic inches, which put =  $c$ ; there were also 40lb. of water in the vessel =  $1105\cdot92$  cubic inches; therefore  $1105\cdot92 + 33\cdot5104 = 1189\cdot4304$  is the inner content of the vessel, which call  $i$ ; also put  $n = \cdot7854$ , and  $d$  = the depth of the vessel; then  $2d$  = the inner diameter, and  $4d^2n$  = area of the bottom inside. Again  $4d^2n = i$ , hence  $d = \sqrt[3]{(i \div 4n)} = 7\cdot1315$  the inner depth, and conseq.  $2d = 14\cdot263$  is the diameter.

Put  $z$  = the thickness of the copper; then  $2d + 2z$  = the outer diameter of the bottom, and  $(2d + 2z)^2 \times n$ , or  $(d + z)^2 \times 4n$  = the area of the bottom on the outside, and  $(d + z)^2 \times 4n \times (d + z)$ , or  $(d + z)^3 \times 4n$  = the whole content =  $i + c$ , or  $(d + z)^3 = (i + c) \div 4n$ , and  $d + z = \sqrt[3]{((i + c) \div 4n)} = 7\cdot2013$  then  $7\cdot2013 - 7\cdot1315 = \cdot0698 = z$  the thickness of the copper.

XIV. QUESTION 1057, by Mr. John Sowerby, of Dudley.

If a grinding stone, 36 inches in diameter, and weighing 5 cwt. make 750 revolutions in one minute; what is the centrifugal force, or tendency it has to burst?

*Answered by Amicus.*

Since the stone makes 750 revolutions in a minute, the velocity at its circumference is  $3\cdot14159 \times 37\frac{1}{2}$  feet in a second; let this =  $c$ , also  $2g = 32\frac{1}{2}$  = gravity, the radius of the stone  $\frac{1}{2}$  =  $a$ ,  $p = 3\cdot14159$ ,  $w$  = the stone's weight,  $l$  = the thickness,  $r$  = any variable radius or distance from its centre,  $m = pr^2l$  = the mass of that part whose radius is  $r$ ; so shall the velocity at the distance  $r$  from the centre be  $cr \div a$ , and  $c^2r \div a^2$  = the centrifugal force at that distance, which multiplied by  $m \div 2g$ , gives  $(pc^2l \div 2ga^2) \times 2r^3$  the fluxion of the centrifugal motive force; whose fluent  $(pc^2l \div 2ga^2) \times \frac{2}{3}r^3 = (c^2r \div 2ga^2) \times \frac{2}{3}m$ , which when  $r$  becomes =  $a = \frac{1}{2}$ , gives  $c^2m \div 2ga^2$  or, expounding  $m$  by  $w$  the given weight of the stone,  $c^2w \div 2ga^2 = w \times 191\cdot761$ , that is nearly 192 times the weight of the body acts on the axis.

*The same by Mr. Wm. Eaton, jun. Sutton-on-the-Hill.*

Put  $d = \frac{2}{3}$  of 3 feet = 2 feet, the diameter of the circle of percussion,  $t = 60 \div 750 = \frac{2}{25} = \cdot08''$  the time of one revolution,  $q =$

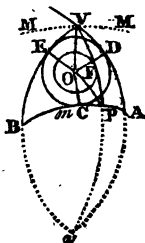
3.1459, and  $s = 16\frac{1}{2}$ ; then will  $q'd \div st' = 191.75$ , which multiplied by 5 cwt. gives 958.75 cwt. or 47.9375 ton, for the centrifugal force required.

PRIZE QUESTION, by Mr. John Howard, of Newcastle.

To construct the great circle triangle  $AVB$ , having given the vertical angle  $v$ , and the difference between each side and its adjacent segment of the base, made by a perpendicular let fall on it from the vertical angle, viz.  $AV - AP$  and  $BV - BP$ .

Answered by Amicus.

Suppose it done,  $AVB$  the triangle,  $VF$  the perpendicular,  $o$  the centre of the inscribed circle, touching the sides in  $c, d$ , and  $e$ . Bisect the base  $AB$  in  $m$ . Then, since  $AV - AP$  and  $BV - BP$  are given, the sum of these two differences, = the difference between the sum of the sides and base =  $2VD$ , is given. Also the difference of these differences =  $BP - AP + AV - BV = 2Pm - 2cm = 2CP$  is given, also  $OC = CD = CE$  and  $VO$  are given; hence in the right-angled spheric triangle  $FCO$ ,  $CF$  and  $CO$  are given, and consequently  $OF$ , the angle  $OPC$  and its complement  $OPV$  are given, and consequently the triangle  $VOP$  is given.



*Ergo Solutum.* Having on the legs of the given  $\angle v$ , taken  $VB = VE =$  half the sum of the two given differences, and constructed the triangle  $VPO$  according to the analysis, through  $P$ , perpendicular to  $VF$  draw  $AB$  and  $AVB$  is the triangle required.

The same answered by Mr. John Surtees. (Suppt.)

*Analysis.* Suppose the thing done,  $AB, VAD$ , and  $VBV$  circles of equal curvature and convexity,  $OC$  or  $OD$  or  $OE$  the radius of the inscribed circle, and  $VF$  perpendicular to  $AB$ . Then since  $BC = BE$ , and  $AC = AD$ ,  $VE = VD = VB - BC = AV - AC = \frac{1}{2}(AV - AP) + \frac{1}{2}(BV - BP)$ , and  $CF = \frac{1}{2}(AV - AP) - \frac{1}{2}(BV - BP)$ , are given.

Again, since the angle  $EVO$  is given, and the angle  $E$  a right angle,  $VO$  and  $OE$  or  $OD$  are also given. Hence the following

*Construction.* Having described the circles  $CDE$  and  $VMN$ , with the centre  $o$  and radii  $OD$  and  $OV$ , describe a circle  $BCA$  of equal curvature and convexity with  $VA$  and  $VB$  to touch  $CDE$  in any point  $c$ ; lay off  $CF = \frac{1}{2}(AV - AP) - \frac{1}{2}(BV - BP)$ , and erect the perpendicular  $FV$ , this will cut  $MM$  in  $v$  the vertex of the triangle required.

*Questions proposed in 1800, and answered in 1801.*

## I. QUESTION 1059, by Miss Sarah Cowen.

What is the length of that arch, whose tangent is equal to three times its sine, the radius of the circle being 10?

*Answered by Mr. James Adams, East Stonehouse.*

Since the ratio of the tangent to the sine, is as 3 to 1, and as the tangent : sine :: radius : cosine, we have as 3 : 1 :: 1 (radius) :  $\frac{1}{3}$  the cosine, hence  $\frac{2}{3}\sqrt{2} = 0.942809 =$  sine, to which we are to find the number of degrees, and the length of the arc. Now we might as readily obtain  $2\sqrt{2}$  for the tangent, or 3 for the secant; but the angle appearing to be large, about  $70^\circ$ , it will be more convenient to find it true by the sines, as their differences are smaller in that part of the quadrant, than in the tangents or secants. Hence then, by the table of natural sines in Dr. Hutton's Tables, the length of the corresponding arc is 1.2309592; which multiplied by 10, the angle is found to be  $70^\circ 31' 43'' 34'''$ ; and then by table 12 of the same, the given radius, produces 12.309592 the length of the arc required.

*The same by Mr. John Blackwell.*

This question may be easily resolved by Trigonometry. But I choose rather to answer it by Dr. Hutton's Mathematical Tables. First, then, by tab. 10, the angle is found  $70^\circ 31' 43'' \frac{1}{2}$ , and by tab. 12, the corresponding length is found = 1.2309589 to radius 1; then as 1 : 1.2309589 :: 10 : 12.309589, the length of the arc as required.

*The same by Miss M. Castieau, Salop.*

By similar triangles as tangent : sine :: rad. 1 : cos.  $\frac{1}{3}$  of the angle  $70^\circ 31' 44'' = 70.528888$  degrees. Hence (by Dr. Hutton's Dicti. p. 135)  $.01745329 \times 70.528888 \times 10 = 12.3096$  nearly, is the length of the arc.

*The same by Master F. T. C. Rundell, Baldock Academy.*

As tangent : sine : radius (10) : cosine =  $\frac{1}{3}$ ; and, by right-angled triangles  $\sqrt{(10^2 - (\frac{10}{3})^2)} = \frac{20}{3}\sqrt{2}$  the sine to radius 10, or  $\frac{2}{3}\sqrt{2} = .9428$  to radius 1, answering to  $70^\circ 32'$ . But as 1 : 3.14159 :: 20 : 62.8318 the circumference; then as  $360^\circ : 62.8318 :: 70^\circ 32' : 12.309$  nearly, the answer.

## II. QUESTION 1060, by Mr. Isaac Rowbottom.

To a beautiful fair, of talents most rare,  
 My ardent addresses I paid;  
 But her answer was, that if from below,  
 I wou'd find out  $x, y,$  and  $z$ .  
 Then her hand and her heart, to me she'd impart,  
 And in wedlock's firm bands wou'd entwine:  
 So, ye generous fair, to you I repair,  
 To show how this nymph may be mine.

$$2z - \frac{1}{2}z^3 + \frac{1}{2}z^5 - \frac{1}{2}z^7, \&c. = 1.5704363 = a;$$

$$x^3\sqrt{z} + xy^2z^2 = 253750 = b;$$

$$x^2\sqrt{z} - y^2z^2 = 77x = cx.$$

*Answered by the Proposer, Mr. Isaac Rowbottom.*

Let  $z - \frac{1}{2}z^3 + \frac{1}{2}z^5 \&c. = d$ , be multiplied by  $z$ , then  $z^2 - \frac{1}{2}z^4 + \frac{1}{2}z^6 \&c. = dz$ , where  $d$  is the circular arc to the radius 1 and natural tangent  $z$ ; the fluxion of which is  $2z\dot{z} - \frac{1}{2}4z^3\dot{z} + \frac{1}{2}6z^5\dot{z} \&c. = d\dot{z} + \dot{d}z$ .

But  $\dot{d} = \frac{\dot{z}}{1+z^2}$ , therefore  $2z - \frac{1}{2}z^3 + \frac{1}{2}z^5, \&c. = d + \frac{z}{1+z^2} = a$  by the question. Hence by a few trials and a table of log. tangents, it is found that  $z = \frac{1}{2}$ .

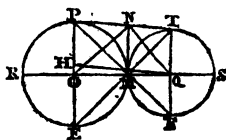
Again, from the 2d given equation  $xy^2z^2 = b - x^3\sqrt{z} = x^3\sqrt{z} - cx^2$  from the 3d; this in numbers and reduced, becomes  $x^3 - 21x^2 = 72500$ , in which  $x = 50$ ; and hence  $y = \sqrt{\frac{b - x^3\sqrt{z}}{xz^2}} = \frac{1}{2}$  as required.

III. QUESTION 1061, by Mr. Wm. Featherstonhaugh, *Lambton*.

If two circles touch each other; and a right line be drawn to touch both the circles, and terminate at the point of contact; then will the square of the said line, be equal to the rectangle of the diameters of the two circles. Required the demonstration?

*Answered by Mr. William Burdon, Acaster Malbis.*

Let  $o, q$ , be the centres of the two circles, and  $r, t$  the points of contact. Join  $or, qt$ , and draw  $qh$  parallel to  $rt$ . Then  $rt^2 = qh^2$  = (by Dr. Hutton's Geom.-Theor. 34)  $oq^2 - oh^2$  = (theorem 33)  $(oq + oh)(oq - oh) = rm \times ms$ .



*The same by Mr. T. S. Evans.*

Let  $PME, TMF$  be the two circles, touching each other in  $M$ ;  $PT$  the tangent, perpendicular to which, from the points of contact, draw  $PE, TF$ , and they will pass through the centres of the circles. Now, in the right-angled triangles  $PTE, PTF$ , the  $\angle EPT = PTF$ , being right angles, and the alternate  $\angle s PFT, EPF, PTE$  are equal, also  $PT$  is common to both triangles, and the triangles equiangular: therefore  $PE : PT :: PT : TF$ , and  $PT^2 = PE \times PF$ .

*Corol.* Hence it is evident that  $PF, TE$  are perpendicular to each other, and intersect in  $M$  the point of contact of the two circles.

*The same answered by Mr. William Middleton.*

Let  $RPM, MTS$  be the two circles, touching each other in  $M$ ;  $PT$  a tangent to both; and  $RS$  a right line passing through their centres and the point of contact. Erect the perpendicular  $MN$ , which (Euc. 3, 37) is equal to  $NP$  or  $NT$ , and join  $NO, NQ$ , which evidently bisect the angles  $MNP, MNT$ ; therefore the angle  $ONQ$  is a right angle, and in a semicircle. Hence the rectangle  $OMQ = MN^2 = PN^2$ , and 4 times the rectangle  $OMQ$  or rectangle  $RMS = 4PN^2 = PT^2$ .

IV. QUESTION 1062, by Mr. James Gale.

To find  $n$ , when  $n^2 + 13$  and  $n^2 - 13$  are both rational squares.

*Answered by Mr. James Adams, Stonehouse.*

Put  $n^2 + 13 = (x + d)^2$  } their dif.  $2dx + d^2 = 26$ ,  
and  $n^2 - 13 = x^2$  } ; therefore  $x = \frac{26 - d^2}{2d}$ ;

where  $d^2$  may be any square number less than 26, and conseq.  $d$  not greater than 5; hence the following table of values;

If	Then	And	Also
$d = 1$	$x = \frac{25}{2}$	$n = \frac{1}{2}\sqrt{677}$	$n^2 + 13 = (\frac{27}{2})^2$ ; and $n^2 - 13 = (\frac{25}{2})^2$
$d = 2$	$x = \frac{11}{2}$	$n = \frac{1}{2}\sqrt{173}$	$n^2 + 13 = (\frac{15}{2})^2$ ; and $n^2 - 13 = (\frac{11}{2})^2$
$d = 3$	$x = \frac{7}{2}$	$n = \frac{1}{2}\sqrt{757}$	$n^2 + 13 = (\frac{35}{2})^2$ ; and $n^2 - 13 = (\frac{29}{2})^2$
$d = 4$	$x = \frac{5}{4}$	$n = \frac{1}{4}\sqrt{233}$	$n^2 + 13 = (\frac{21}{4})^2$ ; and $n^2 - 13 = (\frac{17}{4})^2$
$d = 5$	$x = \frac{1}{10}$	$n = \frac{1}{10}\sqrt{1301}$	$n^2 + 13 = (\frac{11}{10})^2$ ; and $n^2 - 13 = (\frac{1}{10})^2$

By taking  $d$  less than unity, an infinite number of values may be had for  $n$ , which would answer the conditions of the question.

*The same answered by Amicus.*

This question requires that both  $n^2 + 13 = x^2$ , and  $n^2 - 13 = y^2$ , be rational squares; but since it does not require  $n^2$  to be a rational

square also, it will be sufficient if the other two be so, or  $n^2 = x^2 - 13 = y^2 + 13$ ,  $2n^2 = x^2 + y^2$ , and  $x^2 - y^2 = 26$ . Since 26 is not the difference of any two integral squares, let each of them be multiplied by a square number as 4, then  $4x^2 - 4y^2 = 104 = 225 - 121$ , where  $x^2 = 225$  and  $y^2 = 121$ , or  $x = 15$ , and  $y = 11$ , whence  $n^2 = \frac{1}{2}x^2 + \frac{1}{2}y^2 = \frac{1}{2}225 + \frac{1}{2}121 = 173 = 169 + 4$ . From which answer, an indefinite number of others may be found; because 104 may be divided in an infinite number of ways, so as to be the difference of two square numbers; as is taught in chap. 14, of the second part of Euler's Algebra, the French Edition. And in many other places.

Euler also resolves, at page 289, the general problem to find  $z$  when

$n^2 + z$ ,  $n^2 - z$ , and  $n^2$ , are all rational squares, putting  $n = \frac{p}{q}$ , and

finds  $z = \frac{4rs(r^2 - s^2)}{q^2}$ , where  $r$ ,  $s$  and  $q$  may be taken at pleasure;

and thus an infinity of values for  $z$  may be found. But it does not appear that 13 can be one of them; for  $\frac{1}{4}q^2$  is a square, and  $rs(r^2 - s^2)$  must be  $= 13$ , and is composed of the four factors  $r$ ,  $s$ ,  $r + s$ ,  $r - s$ , whereas 13 is a prime number, and can only have the four factors 13, 1, 1, 1; and since  $r$  is greater than  $s$ , and  $r + s$  than  $r - s$ , either  $r$  or  $r + s$  must be  $= 13$ ; now if  $r = 13$ ;  $s$  must be 1,  $r + s = 14$ , and  $r - s = 12$ , and their continual product  $= 13 \times 14 \times 12$ ; but  $14 \times 12 = 168$  is not a square number, and therefore cannot be divided out of  $q^2$  to give a square quotient, and the thing is impossible: and if  $r + s = 13$ , then  $r - s$  is 1,  $r = 7$ , and  $s = 6$ ,  $r^2 - s^2 = 13$ , and  $4rs = 158$ , just the same as before; and therefore  $z$  cannot be  $= 13$ .

*The answer by Mr. John Craggs, of Hilton.*

Put  $n^2 + 13 = x$ , and  $n^2 - 13 = (x - m)^2$ ; their difference is  $26 = 2mx - m^2$ ; and hence  $x = 13 \div m + \frac{1}{2}m^2$ . Here if  $m$  be  $= 1$ , then  $x = 13\frac{1}{2}$ , and  $n = \sqrt{169\frac{1}{4}}$ , and  $x - m = 12\frac{1}{2}$ . Again, if  $m = 2$ , then  $x = 7\frac{1}{2}$ ,  $x - m = 5\frac{1}{2}$ , and  $n = \sqrt{43\frac{1}{4}}$ . And so on.

#### *Additional Solution.*

The value of  $n$  found in the foregoing solutions is not rational, and as it is pretty clearly the proposer's meaning that it should be so, we have annexed the following method of effecting that purpose.

Put  $n^2 + 13 = (a + b)^2$ ,  $n^2 - 13 = (a - b)^2$ ; half the sum of the two expressions is  $n^2 = a^2 + b^2$ , and half their difference is

$2ab = 13$ . It is manifest from hence that we may take  $a = \frac{p^2 - q^2}{2r}$ ,

and  $b = \frac{2pq}{2r}$ , and then  $n = \frac{p^2 + q^2}{2r}$ ; also  $2ab = pq \times \frac{p^2 - q^2}{r^2}$

$= 13$ , or  $pq(p^2 - q^2) = 13r^3$ .

The factors of the first side of the preceding expression are  $p, q, p + q$ , and  $p - q$ , and the factors of the other side are  $13, r, r$ ; or  $13$  and  $r^2$ ; therefore take  $p = 13$  and write  $x^2$  in the place of  $q$ , and the expression  $pq(p^2 - q^2) = 13r^2$  will become  $13x^2 \times (13^2 - x^2) = 13r^2$ : dividing by  $13, x^2 \times (13^2 - x^2) = r^2$  a square; therefore  $13^2 - x^2$  must be a square. Now  $13^2 - x^2$  will obviously be a square when each of its factors  $13 + x^2$  and  $13 - x^2$  is a square. Put  $13 + x^2 = (c + d)^2$ , and  $13 - x^2 = (c - d)^2$ ; half the sum of the two equations is  $13 = c^2 + d^2$ , or  $2^2 + 3^2 = c^2 + d^2$ ; and half their difference is  $x^2 = 2cd$ .

By the method of resolving a number consisting of two squares into two other squares, we may take  $c = \frac{2(s^2 - 1) + 6s}{s^2 + 1}$  and

$$d = \frac{3(s^2 - 1) - 4s}{s^2 + 1}, \text{ and hence we shall have } x^2 = 2cd =$$

$$\frac{12(s^2 - 1)^2 + 20s(s^2 - 1) - 48s^2}{(s^2 + 1)^2} = \text{a square; therefore } 3(s^2 -$$

$1)^2 + 5s(s^2 - 1) - 12s^2 = \text{a square; and the last expression will be found to be a square when } s = 2, \text{ viz. } = 9. \text{ Therefore } c =$

$$\frac{2(s^2 - 1) + 6s}{s^2 + 1} = \frac{18}{5}; d = \frac{3(s^2 - 1) - 4s}{s^2 + 1} = \frac{1}{5}; x^2 = 2cd =$$

$$\frac{36}{25}; x = \frac{6}{5}; x^2(13^2 - x^2) = r^2, \text{ or } r = x\sqrt{(13^2 - x^2)} = \frac{1938}{125};$$

$$\text{and } n = \frac{p^2 + q^2}{2r} = \frac{13^2 + x^4}{2x\sqrt{(13^2 - x^2)}} = \frac{106981}{19380} = 5\frac{10021}{19380}.$$

From what has been done we obviously deduce the following conclusion, viz. if  $n$  is a given number, and we can find such a rational value for  $v$  as will make  $n + v^2$  and  $n - v^2$  both rational squares, we can, from thence, find such a rational value for  $x$  as will make  $x^2 + n$  and  $x^2 - n$  both rational squares.

Another value of  $s$ , which will make the expression  $3(s^2 - 1)^2 + 5s(s^2 - 1) - 12s^2$  a square, may be obtained as follows: Put  $s = 6t + 2$ , and then  $3(s^2 - 1)^2 + 5s(s^2 - 1) - 12s^2 = 3(36t^2 + 24t + 3)^2 + 5(6t + 2)(36t^2 + 24t + 3) - 12(6t + 2)^2 = 9 + 474t + 3024t^2 + 6264t^3 + 3888t^4 = \text{a square} = (3 + 79t - gt^2)^2 = 9 + 474t + 79^2t^2 - 6gt^2 - 158gt^3 + g^2t^4$ : Put  $79^2 - 6g = 3024$ , or  $g = \frac{1}{6}(79^2 - 3024) = 3217 \div 6$ , in order to take away the three leading terms on each side of the equation; and from the equality of the remaining terms, viz.  $6264t^3 + 3888t^4 = g^2t^3 - 158gt^3$ ,  $t = \frac{6264 + 158g}{g^2 - 3888} = (6264 + \frac{254143}{3}) \div (\frac{10349089}{36} - 3888) = \frac{36 \times 6264 + 12 \times 254143}{10349089 - 36 \times 3888} = \frac{3275220}{10209121}; s = 6t + 2 = \frac{40069562}{10209121}$ ; and hence the values of  $c, d, x$ , &c. may be derived

from what has been done, but we shall let the matter rest here at present. c.

v. QUESTION 1063, by Mr. David Henry.

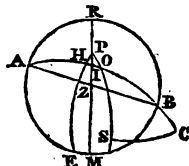
In latitude  $54^{\circ} 42'$  north there is a plane which declines from the south toward the west  $20^{\circ}$ , and reclines from the zenith  $30^{\circ}$ ; I desire to know what angle the rays of the sun made with that plane, on the 22d of December, 1798, at ten o'clock in the morning, solar time; regard being had to refraction?

*Answered by the Proposer, Mr. David Henry.*

Having given the latitude of the place, the sun's declination  $= 23^{\circ} 28'$ , and hour of the day; the sun's true altitude is found  $= 7^{\circ} 41' 16''$ , and azimuth  $= 27^{\circ} 33' 52''$ ; and adding  $6' 20''$  to the altitude for refraction, gives  $7^{\circ} 47' 36''$  for the apparent altitude. Then, with the given latitude, the azimuth and apparent altitude, by spherics, find a new declination  $23^{\circ} 22' 55''$ , and hour angle  $= 29^{\circ} 58'$ , both of which are corrected for refraction.

Hence, in the annexed figure let ARBE represent the horizon, MR the meridian, P the pole, Z the zenith, AOB the given plane, PHE a great circle at right angles to it, pos the 10 o'clock hour circle, s the place of the sun, and sc a part of a great circle perpendicular to the given plane.

Then, in the triangle ARI, are given the angle  $A = 60^{\circ}$ , and side  $AR = 70^{\circ}$  (from the situation of the plane), to find  $IR = 58^{\circ} 26'$ , and angle  $i = 72^{\circ} 46' 14''$ ; then  $58^{\circ} 26' - 54^{\circ} 42' = 3^{\circ} 44' = IP$ . Next, in the triangle IPH, there is found  $HP = 3^{\circ} 33' 56''$  = the height of the pole above the plane, and angle  $IPH = 17^{\circ} 15' 50''$  = the plane's difference of longitude, to which add the hour angle  $mps = 29^{\circ} 58'$ , gives  $47^{\circ} 13' 50'' = EPS$ . Also in the triangle OPH (having  $HP$  and angle  $P$  given) is found  $OP = 5^{\circ} 14' 34''$ , and angle  $o = 42^{\circ} 53' 21''$ ; then from  $SP = 90^{\circ} + 23^{\circ} 22' 55''$ , take  $OP$ , and there remains  $so = 108^{\circ} 8' 21''$ . Lastly, in the triangle soc, having given  $so$ , and angle  $o$ , there is found  $sc = 40^{\circ} 17' 42''$  = the measure of the angle made by the rays of the sun and the given plane at the time proposed.

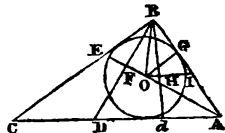


vi. QUESTION 1064, by Mr. Wm. Burdon, Acaster Malbis.

The three sides of a triangular field being 3, 6, and 7 chains, now it is required to draw a right line from the greatest angle to its opposite side, so that it may divide the inscribed circle into two parts, which shall have to one another the ratio of 3 to 1; and to find the areas of the parts the triangle is divided into by the said line?

*Answered by Mr. Newton Bosworth, Cambridge.*

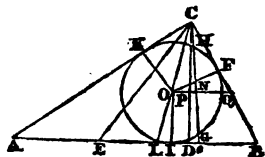
Let  $ABC$  represent the given triangular field, whose area is easily found, from the three given sides, to be 8.9442719 chains. This area divided by 8 (the half perimeter) gives 1.118034 for the radius  $OE$  or  $OG$  of the inscribed circle, the double of which, or 2.236068 is the diameter. Now it is required to cut off a segment of this circle equal to one-fourth of the area, by a line drawn from the angle  $B$  to the base  $AC$ . The versed sine of this segment may be easily found from the very useful tables at the end of Dr. Hutton's *Mensuration*, or in his new course, thus: as 1 : 2980136 (versed sine of the tabular similar segments, the diameter being 1) :: 2.236068 (diameter above found) : .666378 = versed sine  $EF$ , which taken from the radius  $EO$ , leaves .451656 for  $OF$ . With this distance and centre  $O$ , describe a small concentric circle; to touch which draw  $BD$  (or  $bd$ ), and it will be the line required, giving two solutions.



The areas of the two parts into which the triangle is divided by this line, may be found as follows: First,  $BC = (AB + BC - AC) \div 2 = 1$ , and the angle at  $C$  is right, whence  $BO = 1\frac{1}{2}$ ; then as  $BO$  : radius ::  $OG$  :  $\sin. \angle OBG = 48^\circ 11' 23''$  and ::  $OF$  :  $\sin. \angle OBF$  or  $OBD = 17^\circ 31' 26''$ ; the sum of these gives the  $\angle ABD = 65^\circ 42' 49''$ , and their difference the  $\angle CBD = 30^\circ 39' 57''$ . Again, in the right-angled triangle  $AOG$ , as  $AG$  (2) : radius ::  $OG$  :  $\tan. \angle OAG$  or  $OAC = 29^\circ 12' 21\frac{1}{2}''$ , the double of which gives the  $\angle BAD = 58^\circ 24' 43''$ , consequently  $\angle D = 55^\circ 52' 28''$  and  $\angle d = 90^\circ 55' 20''$ ; hence in the two triangles  $ABD$ ,  $ABd$ , are given all the angles and the common side  $AB = 3$ , to find  $AD = 3.303305$ , and  $Ad = 1.530288$ . Then, since the areas of triangles between the same parallels are as their bases, we have as  $AC$  : area of  $ABC$  ::  $AD$  : area of  $ABD = 4.220807$  and ::  $Ad$  : area of  $ABd = 1.955331$  chains, answering to the two cases of the part next to the less side  $AB$ ; and if these be taken from the whole triangle  $ABC$ , the remainders will give the two corresponding parts next to the other side  $CB$ , viz.  $CBD = 4.723465$ , and  $cbd = 6.988941$ .

*The same answered by Mr. Wm. Middleton.*

Let  $ABC$  be the triangle, whose sides are 3, 6, and 7;  $CE$  or  $ce$  the dividing line,  $OE$ ,  $OG$ , radii of the inscribed circle;  $CD$  the perpendicular and  $COL$  the line bisecting the vertical angle; also  $ON$  perpendicular on  $CHG$ , and  $OP$  on  $CD$ . Now the area of the triangle is  $\sqrt{80} = 4\sqrt{5}$ , and  $4\sqrt{5} \div 8 = \frac{1}{2}\sqrt{5} = OE$  the radius of the inscribed circle, whose area therefore is  $5 \times .7854 = 3.927$ , hence the chord dividing it in the ratio of 3 to 1 is found to be  $2.04549 = GH$ , and the versed sine  $NQ = .666378$ ; hence  $ON = OQ$



—  $NQ = .451656$ ,  $CF$  or  $CK = \frac{1}{2}$  sum of sides — base  $= 1$ ,  $CH = .307642$ ,  $CN (= CH + \frac{1}{2}GH) = 1.430387$ ,  $CO = \frac{1}{2}$ ,  $ON = .451656$ ,  $CD = 4\sqrt{5} \div \frac{7}{2} = \frac{8}{7}\sqrt{5}$ . Again,  $AC + CB : AB :: AC : AL = 4\frac{2}{3} :: BC : BL = 2\frac{1}{3}$ ; but  $AI = AK = 5$ , so  $IL = \frac{1}{3}$ ,  $LD$  (by similar triangles)  $= \frac{1}{3}\frac{8}{7}$ , and  $ID = OP = \frac{1}{7}$ . Now the places of  $E$  and  $e$  may be computed, either by trigonometrical tables, from the quantities of the angles, or otherwise, but as it is probable you will have many solutions by the former, I shall here give one or two by the latter.

And first, (by Euc. 3, 36, and trial and error),  $LE$  is found  $= .47$  nearly; hence easily follows  $Ae$ ,  $ne$ , the bases of the two triangles  $Ace$ ,  $bce$ , and thence their areas, having the common perpendicular  $CD$ .

Or thus, still more accurate and scientific: Since  $OP = \frac{1}{7}$  and  $ON = .451656$  are the sines of the angles  $OCF$  and  $OCN$  or  $OCE$ , of which the cosines are  $CF = \frac{9}{14}\sqrt{5}$  and  $CN = 1.430387$ , to the common radius  $OC = 1\frac{1}{2}$ , therefore  $ON \cdot CF - OP \cdot CN = \text{sine of their difference } Dce$ , and  $ON \cdot OP + CN \cdot CF = \cos.$  of the same  $\angle Dce$ ; consequently the

$$\text{tang. of the said difference } Dce = \frac{ON \cdot CF - OP \cdot CN}{ON \cdot OP + CN \cdot CF} = \text{tang. diff.}$$

$$Dce, \text{ and } \frac{ON \cdot CF + OP \cdot CN}{ON \cdot OP + CN \cdot CF} = \text{tang. } DCE \text{ the sum of the same}$$

angles; then multiply these by the common perpendicular  $CD$ ,  $De = \frac{15ON - CN \cdot 2\sqrt{5}}{2ON + CN \cdot 3\sqrt{5}} \times \frac{8}{7} = .041144$ , and  $DE = \frac{15ON + CN \cdot 2\sqrt{5}}{-2ON + CN \cdot 3\sqrt{5}} \times \frac{8}{7} = 1.731784$ , these added and subtracted with  $LD = \frac{1}{3}\frac{8}{7}$ , gives  $be = .803049$ , and  $LE = .969879$ ; and these added and subtracted with  $BL = 2\frac{1}{3}$ , gives  $Be = 1.530284$ , and  $BE = 3.303212$ , the distances from  $B$  in the two positions of the dividing line; and, by multiplying them by  $\frac{1}{2}CD$ , the corresponding areas are found, viz.  $bce = 1.955325$ , and  $BCE = 4.221689$ . Then these taken from the whole triangle  $ABC$ , leaves the other two corresponding triangles  $Ace$ ,  $ACE$ , all agreeing nearly with the first solution above.

#### VII. QUESTION 1065, by Mr. Wm. Middleton, Holland.

Three men,  $A$ ,  $B$ ,  $C$ , bought a field, in the form of an ellipsis, the greatest length being 1260 links, and breadth 840, for the sum of 150*l*. of which  $A$  paid 40*l*.  $B$  50*l*. and  $C$  60*l*. Now  $A$  is to make the fence separating his own ground from that of  $B$ , and  $B$  the fence between his own ground and that of  $C$ ; for which each is to be allowed after the rate of 4*d*. per yard running, from the joint estate. Hence it is required to divide the field equitably among them, by fences parallel to the conjugate axis;  $B$ 's share lying in the middle?

*Answered by Mr. John Ryley, of Leeds.*

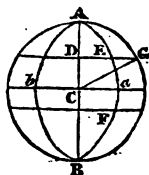
Here are given, transverse axe  $= 1260$  links  $= 277\frac{2}{3}$  yards, and

AA 3

the conjugate = 840 links = 184·8 yards, which are in the proportion of 3 to 2. Therefore the area of a similar ellipse is 4·7124; then, as 15 : 4·7124 :: 4 : 1·25664 A's part :: 5 : 1·5708 B's part :: 6 : 1·88496 C's part of the same. Hence,  $1·25664 \div 6 = .20944$  the tabular area corresponding to A's part; whose versed sine is .31222887; and  $1·88496 \div 6 = .31416$  C's tabular area, whose versed sine is .42113265; these versed sines being severally multiplied by 277·2, give the abscissa of A's part = 86·5498, and that of C's = 116·73798; therefore as 3 : 2 ::  $2\sqrt{86·5498} \times 190·6502$  : 171·2736 the length of A's fence ::  $2\sqrt{116·73798} \times 160·46202$  : 182·4862 the length of B's fence, when the field is divided in proportion to the money first laid out, or as the numbers 4, 5, and 6. But one yard running is worth 4·47 square yards of the surface; then, by a few trials, the length of A's fence is found 172·27 yards, and that of B's = 181·73 yards, very nearly. Hence the expence of A's fence is 2l. 17s. 5d. and that of B's 3l. 0s. 7d.; therefore the ground and expences together cost A 42l. 17s. 5d. B 53s. 0s. 7d. and C 60l. which are in proportion to the numbers 10289, 12727, and 14400; now as the sum of these numbers, viz. 37416 : 40283·34 the area of the whole field :: 10289 : 11063·74 yards = 2 ac. 1 ro. 6 per. A's part :: 12727 : 13685·31 yards = 2 ac. 3 r. 12 p. B's part :: 14400 : 15484·29 yards = 3 ac. 0 r. 32 p. = C's part of the field, the sum of which is 8 ac. 1 r. 10 p. the content of the whole. And, from what is done above, the length of A's part along the transverse axe is found = 88·4 yards, B's = 75·4 yards, and C's = 113·4, dividing the field as required.

*The same by the Rev. Mr. T. Scurr, of Hexham.*

Let AC, ca be the two semiaxes, and the other lines as in the figure. Then  $AB^2 \times .7854 = 1246901·04$  links, the area of the circumscribing circle. Theref. as 150 : 1246901·04 :: 40 and 50 and 60 respectively to 33250·94 and 415633·68 and 498760·416, the areas of the shares. Hence  $332506·94 \div 1260^2 = 20944$ , which corresponds nearly with .312 in the table of circular segments in Dr. Hutton's Mensuration. Therefore  $.312 \times 1260 = 393·12$  links, the height of the first segment. In like manner the height of the third segment is found = 530·46; and the sum of these two taken from 1260 leaves 336·42 for the height of the middle part. Now, (by Dr. Hutton's Conic Sections page 10)  $ca^2 : ca^2 :: ca^2 - cd^2 : de^2$ , and  $ca^2 : cb^2 :: cb^2 - ch^2 : hf^2$ ; hence  $2de = 77·36$  the length of A's fence, and  $2hf = 829·44$  B's fence, the charges of which, at 4 pence the yard, is 2l. 17s. 1d. and 3l. 0s. 9d.



But if the proposer mean (for his meaning is not clearly expressed) that A and B are to have ground according to their expences in the first proportion, we ought to consider the money it cost them to be 155l. 17s. 10d. and their three sums as 42l. 17s. 1d. and 53l. 0s. 9d. and

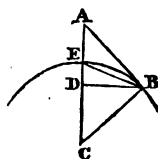
601. Then, by a similar process, the height of A's segment comes out 401.94 links, B's 342.72, and C's 515.34.

VIII. QUESTION 1066, by Mr. John Hawkes, *Finedon*.

Admitting the earth to be a perfect sphere, whose circumference is 25000 miles, which it is very nearly; how many acres of its surface may a person view by walking 100 miles on a great circle of the sphere; supposing the height of the eye to be  $5\frac{1}{2}$  feet above the path, and having nothing but the earth's convexity to obstruct his view?

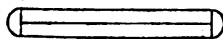
*Answered by Mr. Wm. Francis, jun.*

Here are given the height  $AE = 5\frac{1}{2}$  feet, and the whole circumference 25000 miles. Now the tangent  $AB$  is a mean proportional between  $AC$  and  $AD$ ; hence  $AC - CB^2 \div AC = AD = 11$  feet; or indeed  $AD = 2AE = 11$ , because  $AE = ED$  in small arcs. Therefore  $25000 \times 5\frac{1}{2} \times 1760 \times 3 = 726000000$  square feet, the circular surface, or segment visible from the given height. Again,  $DB$  is a mean proportional between  $AD$  and  $DC$ , or  $\sqrt{(AD \cdot DC)} = DB = 15201.74047$ , and hence  $EB = 15201.74147$  the chord, and the length of double the arc  $EB$ , visible at one view, is 30403.4836 feet. Therefore  $30403.4836 \times 100 \times 1760 \times 3 = 16053039340.8$  square feet, is the area of the parallelogram, or part of the zone seen in travelling. And the sum of the two is  $16779039340.8$  square feet = 385193.7406 acres, the surface as required.



*The same by Mr. Green, Academy, Deptford.*

Let 25000 miles be divided by 3.14159 &c. it gives the diameter of the earth  $7957\frac{1}{2}$ ; then, by a well known theorem (putting  $d$  = the diameter and  $a$  the height of the eye)  $2\sqrt{((d + a) \times a)}$  gives the diam. of the sensible horizon = 5.7582568 miles; therefore  $5.7582568 \times 100 \times 640 = 368528.4352$  acres, the area of the space travelled; to which adding the area of the two semicircles, at the ends, or the boundary of sight,  $5.758^2 \times .7854 \times 640$  nearly = 16665.3405, gives 385193.7757 acres in the whole.



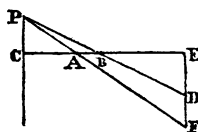
IX. QUESTION 1067, by Mr. Thomas Coultherd.

I observed a rectangular looking glass, 18 inches in length, and 12 in breadth, by the light of a candle, to reflect a quadrilateral figure on the side of the room, at right angles with that against which the glass was suspended by the middle of the upper frame. Now supposing the

nearer edge of the glass to be 18 inches from the intersection of the sides of the room, the candle at 72 inches distance from the corner of the other edge, but in a horizontal plane 15 inches below the said corner, and 36 inches from the wall on which it hung; I desire to know how many square inches the reflected figure contained, admitting the glass to be level with the paper of the room?

*Answered by Mr. Wm. Burdon, Acaster Malbis.*

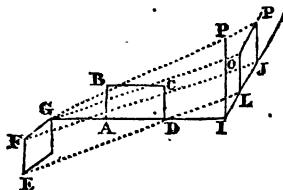
Let  $PCBEF$  be a plane at right angles to the plane of the mirror, in which the candle was placed;  $P$  the point beyond the glass where the image of the candle appears:  $AB$  the points where the two longest sides of the glass meet that plane, and  $DF$  the point where the two parallel sides (produced) of the reflected quadrilateral figure on the wall meet that plane. Then (47 Eucl. 1)  $PA = \sqrt{(72^2 - 15^2)} = 70.4201675$ ,  $CA = \sqrt{(PA^2 - PC^2)} = 60.522723$ , and  $PB = \sqrt{(PC^2 + CB^2)} = 80.966322$ . By similar triangles  $CA : CP :: AE : EF = 17.844537$ , and  $CB : CP :: BE : ED = 8.93513$ ; therefore  $DF = EF - ED = 8.909407$  inches is the perpendicular distance of the parallel sides of the figure. Also  $AF = \sqrt{(AE^2 + EF^2)} = 34.904548$ , and  $BD = \sqrt{(BE^2 + ED^2)} = 20.095684$ . Moreover, lines drawn from  $P$  to the corners of the mirror, and produced to the wall, will give the four corners of the figure, therefore by similar triangles  $PA : 15 :: PF : 22.4349$  = the height of the lowest corner of the farthest side, and  $PA : (15 + 18) 33 :: PF : 49.3568$  = the height of the highest corner of the same side; therefore  $49.3568 - 22.4349 = 26.9219$  inches is the length of that side of the figure. Exactly in the same manner the length of the nearest side will be found to be 22.4675 inches. Consequently by mensuration the area is found to be 220.015 square inches.



*The same by Mr. T. Swanwick, Teacher of Mathematics, Derby.*

As the angles of incidence and reflection are equal, the same effect will be produced as if there were a hole in the wall of the same dimensions as the glass, and the candle similarly situated behind its surface, as by the question it is before it.

Let  $ABCD$  be the glass,  $FJ$  the wall on which the luminous figure is formed, and  $E$  the place of the candle. Then  $AE = 72$ ,  $EF = 15$ , and  $FG = 36$ ; to find  $GA$ , thus  $AE^2 - EF^2 = AF^2$ , and  $AF^2 - FG^2 = GA^2$ , therefore  $\sqrt{(72^2 - 15^2 - 36^2)} = 60.522 = GA$ .



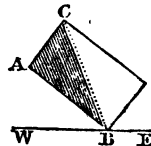
Let the flame be now removed to F, which will not alter the breadth of the figure; then, as  $GA : GF :: AI : AJ = 17.8447$ , and as  $GD : GF :: DI : IL = 8.9352$ ; hence,  $17.8447 - 8.9352 = 8.9095$  is the breadth of the spectrum. For the ease of calculation, let the candle now be placed at G, which will not alter the lengths of the sides of the spectrum: then, as  $GA : AB :: GI : IF = 26.922$ , and as  $GD : DC :: GI : IO = 22.468$ ; then  $\frac{1}{2} (26.922 + 22.468) = 24.695$  the mean length of the spectrum, and  $24.695 \times 8.9095 = 220$  inches nearly, is the area required.

X. QUESTION 1068, by Mr. Tho. Farnell, *Norwich*.

My garden is of a rectangular form, containing  $112\frac{1}{2}$  square yards, the sides of which are in the ratio of 2 to 1, and the longer side declines from the south towards the west 15 degrees, being inclosed on all sides except at the end towards the east. Now, I, one day, observed the shadow of the wall on the southern side (which is  $11\frac{1}{2}$  feet high) to make a complete diagonal to the garden. Required the true time of observation; the latitude of the place being  $52^{\circ} 45'$  north?

*Answered by Mr. John Barron, Schoolmaster, Spilsby.*

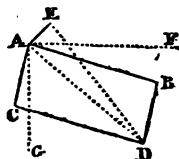
The sides AB, AC, of the rectangle, being denoted by  $2x$  and  $x$ , their product  $2x^2 = 112\frac{1}{2}$  square yards, hence  $x = \sqrt{56\frac{1}{4}} = 7\frac{1}{2}$  yards = AC, and  $AB = 2x = 15$  yards. Now let EW be an east and west line, AB the south side of the garden, declining  $15^{\circ}$  from the west, and BC the shadow of the wall, or diagonal of the garden  $= \sqrt{(AC^2 + AB^2)} = 16.77$  yards, or 50.31 feet. Then, as  $AB : \text{radius} :: AC : \text{tangent } ABC = 26^{\circ} 33' 54''$ ; hence  $26^{\circ} 33' 54'' + 15^{\circ} = 41^{\circ} 33' 54''$  = the angle WBC; also  $41^{\circ} 33' 54'' + 90^{\circ} = 131^{\circ} 33' 54''$  the sun's azimuth from the north. Again, as  $BC : \text{radius} :: 11\frac{1}{2}$  (height of the wall) : tangent  $12^{\circ} 52' 32''$  the apparent altitude of the sun's upper limb, which corrected for semidiameter, refraction and parallax, gives  $12^{\circ} 32'$  for the true altitude of the sun's centre. Hence, by spherics, having given two sides and the included angle, viz.  $37^{\circ} 15'$  the colat. and  $77^{\circ} 28'$  the coaltitude, also  $131^{\circ} 33' 54''$  the azimuth angle; to find the hour angle and the polar distance, and thence the declination, viz. the hour angle  $48^{\circ} 17' 53''$ , answering to 3h. 13m. 11 sec. before noon, and the declination  $12^{\circ} 45' 10''$  south, answering to the 15th of Feb. or 27th of October.



*The same answered by Mr. O. G. Gregory, of Cambridge.*

The ratio of the sides, and their rectangle, or the area of the garden being known, those sides are soon found to be 15 and  $7\frac{1}{2}$  yards; hence is found the diagonal  $= 7\frac{1}{2} \sqrt{5} = 16.705098$  yards. Now, if

As, in the annexed figure, represent a line running east and west, and as a meridian line, the angle  $FAB = 15^\circ = CAG$ , and as the three sides of the right-angled triangle  $ACD$  are known, the angle  $CAD$  is thence discovered  $= 63^\circ 26' \frac{1}{2}$ , being the angle whose sine is double the cosine. Hence,  $CAD - CAG = EAD = 48^\circ 26' \frac{1}{2}$ , the supp. of which is  $131^\circ 33' \frac{1}{2}$ , the sun's azimuth from the north. Having  $AE = 11\frac{1}{2}$  feet,  $= 3\frac{3}{4}$  yards, the height of the wall, we get  $ED = \frac{1}{2}\sqrt{10654} = 17.203036$  yards: then the sides of the right-angled triangle  $AED$  being known, the angle  $ADE$  is determined  $= 12^\circ 52' \frac{1}{2}$ , the apparent altitude of the sun's upper limb; this,  $- 4' 6''$  refraction,  $- 16' 1''$  semidiameter  $+ 7''$  parallax, gives  $12^\circ 32' \frac{1}{2}$ , for the correct altitude of the sun's centre. Then, in an oblique spherical triangle, we have two sides, viz.  $77^\circ 27' \frac{1}{2}$  coaltitude, and  $37^\circ 15'$  colatitude, and the angle between them  $= 131^\circ 33' \frac{1}{2}$  the azimuth; whence the third side is found  $= 102^\circ 39' 35''$ , which lessened by  $90^\circ$ , leaves  $12^\circ 39' 35''$  for the declination south, agreeing with February 15 or October 26; also the hour angle  $48^\circ 27' 52''$  from noon, or in time 3h. 13m.  $51\frac{1}{2}$ sec. So that the time of the observation was at 8h. 46m.  $8\frac{1}{2}$ sec. in the morning, either February 15 or October 26,



XI. QUESTION 1069, by Mr. W. Marrat, of Boston.

Suppose a ball of cast iron, having fallen from an infinite height, in air of the same density as at the surface of the earth, had acquired a uniform velocity of 7.285 feet per second, it is required to find the diameter of the ball; the specific gravity of cast iron, and air, being 7425, and  $\frac{1}{13}$ ?

Answered by the Rev. Mr. J. Ewbank, Vicar of Thornton Steward.

Let  $n = 7425$ ,  $n = \frac{1}{13}$ ,  $v = 7.285$  feet,  $g = 16\frac{1}{13}$  feet,  $p = 3.1416$ , and  $x =$  diameter of the ball in feet. Then, by Dr. Hutton's Course of Math. vol. 2, page 254,  $\frac{1}{8}px^3(n - n)$  is the force or weight by which the ball is urged; and  $\frac{pnv^3x^3}{32g}$  is the resistance it meets with; but these being equal, by the question, reduction gives  $x = \frac{3nv}{16g(n - n)}$   $= .0001$  of a foot, the diameter of the ball required.

The same by Mr. Thomas Hornby, Land Surveyor, Wimbleton.

It is well known that the greatest velocity acquired by a globe descending in a fluid medium, is when its relative weight in the fluid is equal to the force that resists it; which force, by art. 5, prob. 30, page 225, Dr. Hutton's Conics, is  $= pnv^3d^3 \div 32g$ ; also its relative

weight =  $\frac{1}{8}pd^3(N-n)$ ;  $p$  being = 3.1416,  $v = 7.285$ ,  $g = 16\frac{1}{12}$ ,  $N = 7425$ ,  $n = 1\frac{1}{2}$ , and  $d =$  the diameter. Hence  $pnv^3d^3 \div 32g = pd^3(N-n) \div 6$ , and  $d = 3nv^3 \div 16g(N-n) = .0001$  feet nearly.

*The same answered by Mr. John Ryley, of Leeds.*

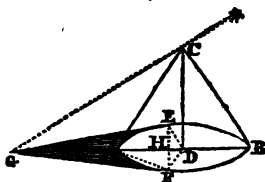
Put  $v = 7.285$  the given uniform velocity,  $g = 16\frac{1}{12}$  feet the force of gravity,  $N = 7.425$  the density of the ball,  $n = 1\frac{1}{2}$  the density of the air at the earth's surface, and  $d =$  the diameter of the ball. Then by prob. 32 Dr. Hutton's Practical Exercises, or prob. 22, of his course of Mathematics, we have  $2g \times 8d \times (N-n) \div 3n = v^2$ ; hence  $d = 3nv^2 \div 16g(N-n) = \frac{1}{10000}$  part of a foot.

XII. QUESTION 1070, by Mr. Wm. Newby.

On September 4, 1799, in the latitude of  $54^\circ 32'$  north, at 9 o'clock in the forenoon, the area of the shadow of an upright cone, on a horizontal plane, was 8 feet. Required the dimensions and content of the cone, the diameter of the base being double the height?

*Answered by Mr. Burdon, Acaster Malbis.*

Let  $D$  be the centre of the cone's base, and  $a$  the vertex or extremity of the shadow. Draw the tangents  $GE$ ,  $GF$ , and the rest of the lines as in the figure; then the area of the curvilinear space  $GEAFG$ , is to be 8 square feet. Put  $DA = DE = DC = x$ ; then, as radius to cotangent  $30^\circ 49'\frac{1}{2}$  (the apparent altitude of the sun's upper limb)  $\therefore x : 1.675852x = DG$ ; and by sim. tri.  $DG : DE :: DE : DH = .5967112x$ ; hence  $EH$  or  $HF = .802456x$ , and  $AF = \sqrt{(2AD \cdot AH)} = .8780966x$ . Now the area of the triangle  $GEF$  is  $.8659632x^2$  and that of the segment  $AEF$  (by rule 5, page 139, of Dr. Hutton's Mensuration) is  $4.20665x^2$ , and difference of these, or the space  $GEAF$ , is  $.4138967x^2 = 8$ , by the question, hence  $x = 4.3964$  feet, height of the cone, or radius of its base; consequently its solidity is 89 feet very nearly.



*The same answered by the Rev. L. Evans.*

Let  $AD = DC = 1$ , be the radius of the base and height of a cone similar to the one required; and let  $GEAFG$  represent the shadow of this cone at the given time. Then by spherics, the  $\angle DGC = 30^\circ 48' 56''\frac{1}{2}$ , the sun's semidiameter, parallax and refraction accounted. Hence  $GD$  is known  $= 1.676729$ , and  $EDF = \text{arc } EAF = 106^\circ 46' 30''\frac{1}{2} =$

106°77516 degrees; then  $360^\circ : 3 \cdot 14159$  area of base of assumed cone  $:: 106 \cdot 77516^\circ : \cdot 931788 =$  area of sector AEDF; which subtracted from area GEDF  $= 1 \cdot 344891$  (found from data DE and DG) leaves  $\cdot 413103 =$  area GEAFG; then, by theor. 89, Dr. Hutton's Geom. as  $\cdot 413103 : 1^2$  the square of its height  $:: 8$  the given shadow  $: 4 \cdot 400639$  the height of the cone required: hence its solidity is  $89 \cdot 24337$  feet.

XIII. QUESTION 1071, *by the Rev. Mr. J. Furnass, Heddon-on-the-Wall.*

If a regular polygon of six sides, and area 72 feet be supposed to turn round the produced diameter of its inscribed circle, or the greatest or central diagonal of the hexagon, as an axis; Quere the superficies and solidity of the solid generated by one revolution of the polygon?

*Answered by Mr. John Houlgate, Horsforth Academy, near Leeds.*

Suppose ABCEFG be the hexagon, and BF the axis on which it revolves, then will the two cones ABC, EFG, and the cylinder ACEG be the solids generated by the revolution. Now, the area being given  $= 72$ , we have  $\sqrt{(72 \div 2 \cdot 5080762)} = 5 \cdot 264296 =$  the side of the hexagon AB or BC or CE, &c. the half of which is  $2 \cdot 652148 =$  CI or BD or HF; also, by right-angled triangles,  $CD = BA = 4 \cdot 559$ , which doubled, gives  $9 \cdot 118 =$  AC or EG the common diameter of the base of both the cylinder and the cones. Then  $AC \times AB$  (or  $AG$ )  $\times 3 \cdot 1416 = 150 \cdot 7963316 =$  the superficial content of the two cones, which is evidently half the content of the whole figure; hence  $150 \cdot 7963316 \times 2 = 301 \cdot 5926632$  the whole convex surface required. Again,  $9 \cdot 118^2 \times \cdot 7854 \times 5 \cdot 264296$  is the solidity of the cylinder ACEG, which is evidently equal to the triple of the two cones; therefore  $9 \cdot 118^2 \times \cdot 7854 \times 5 \cdot 264296 \times \frac{1}{3} = 458 \cdot 32031$  is the whole solidity.



*The same answered by Index, Free School, Alnwick.*

Let the annexed figure represent the generated solid; the middle part of which, viz. CG is the cylinder, and parts at the ends two equal cones. Then,  $\sqrt{(72 \div 2 \cdot 5298076)} = 5 \cdot 264$  feet  $=$  BC or CE or DH or  $\frac{1}{2}$  BF the length of each side of the hexagon, or  $10 \cdot 528 =$  BF the central diagonal; which multiplied by  $\frac{1}{3}$  of  $2 \cdot 598076$ , gives  $9 \cdot 1175 =$  AC or EG; then  $AC^2 \times 1 \cdot 0472$  ( $= \frac{1}{3}$  of  $\cdot 7854$ )  $\times$  DH  $= 458 \cdot 244285$  the whole solidity of the figure, and  $AC \times 8 \cdot 1416 \times 2DH = 301 \cdot 559168$  the convex superficies.

XIV. QUESTION 1072, *by Mr. John Surtees, Wearmouth.*

Mr. Emerson, at page 26 of his Trigonometry, has determined the

sine of  $18^\circ$  by a cubic equation. Now the same may be done by quadratics. Quere how?

*Answered by Mr. Wm. Burdon, Acaster Malbis.*

In the introduction to Dr. Hutton's Logarithms, it is shewn that "By dividing the radius in extreme-and-mean ratio is obtained the sine of  $18^\circ$ ." Therefore if  $x$  denote the sine, and  $r$  the radius, it will be  $(r - 2x)r = 4x^2$ , or  $x^2 + \frac{1}{2}rx = \frac{1}{4}r^2$ , a quadratic equation, in which the value of  $x$  is  $(\sqrt{5} - 1) \frac{1}{4}r$  the same as found by Mr. Emerson by a cubic equation.

*The same, by Mr. John Craggs, of Hilton.*

Put  $x = \sin 18^\circ$ ; then  $2x\sqrt{1-x^2} = \sin 36^\circ$ , and  $1 - 2x^2$  its cosine; hence, by Emerson's Trigonometry, prop. 5, cor. 1,  $\sqrt{(1-x^2)} \times (1-2x^2) - 2x\sqrt{1-x^2} = 2x\sqrt{1-x^2}$ ; this reduces to  $x^2 + \frac{1}{2}x = \frac{1}{4}$ , which gives  $x = \sqrt{(\frac{1}{4} + \frac{1}{16})} - \frac{1}{4} = \frac{1}{4}\sqrt{5} - \frac{1}{4} = .3090169943$  as required.

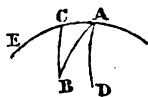
PRIZE QUESTION, by Captain Mudge, Royal Artillery.

In the Trigonometrical survey now carrying on, it is highly necessary that the values of all the elliptic arcs, on the earth's surface, in directions oblique to the meridian, should be known. It is, therefore, proposed to find some convenient expressions, by which the several values of their degrees may be determined: and, by way of example, let the length of the oblique arc which passes through Dunnose and Beachy Head be found, the point on the arc being that where it cuts the meridian at the former place, the latitude of Dunnose being  $50^\circ 37' 7''$ , and the length of the degree of the meridian and great circle perpendicular to it 60850 and 61182 fathoms respectively. And the angle which Beachy Head makes with the pole  $= 81^\circ 56' 53''$ .—Vide Philosophical Transactions, 1795, part 2, page 519.

*Answered by Verus.*

Let ACE be an arch of the meridian, AD the curve at right angles to it, and AB an indefinitely small portion of a section of the terrestrial spheroid, making with the meridian the angle EAB  $= z$ .

1. To find the radius of curvature of AB, supposing the radius of curvature of the meridian, and of the perpendicular to it at A, to be given. Let the radius of curvature of the meridian at A be  $= r$ ; that of the curve AD  $= r + d$ , and of the curve AB  $= y$ . Let AB, supposed to be indefinitely small,  $= L$ ; from B draw the arch BC perpendicular to AE. Then, AB being indefinitely small, AC  $= L \cos. z$  and BC  $= L \sin. z$ . Also, from the nature of the circle, the



depression of the point *c* below the horizontal plane touching the earth in *A*, is  $= \frac{L^2 \times \cos. z^2}{2r}$ ; in like manner  $\frac{L^2 \times \sin. z^2}{2(r+d)}$  is the de-

pression of *B* below the horizontal plane touching the earth in *c*, for the radius of curvature of the arch *BC*, is ultimately the same with that of the arch *AB*, or is  $= r + d$ . The sum of these two depressions gives the total depression of *B* below the horizontal plane at *A*  $= \frac{L^2 \cdot \cos. z^2}{2r} + \frac{L^2 \cdot \sin. z^2}{2(r+d)} = \frac{L^2}{2} \left( \frac{\cos. z^2}{r} + \frac{\sin. z^2}{r+d} \right) = \frac{L^2}{2} \left( \frac{r+d \cdot \cos. z^2}{r(r+d)} \right)$ ,

supposing always that the point *B* is indefinitely near to *A*.

Now, from the nature of the circle, it is again evident, that if *L'* be divided by the quantity last found, or the depression of *B* below *A*, it will give *2y*, or the diameter of the arch equicurve with the arch *AB*.

Therefore  $y = \frac{r(r+d)}{r+d \cdot \cos. z^2}$ .

2. Next, to compare the degrees of the circles of which the radii are *r*, *r+d*, and *y*; let the degree of the first, or of the meridian at *A*, be  $= D$ ; of the second, or the arch *AD*, let the degree  $= D + E$ ; and of the third, or of the oblique circle *AB*, let the degree  $= x$ ; then, by inserting *D* for *r*, *E* for *d*, *x* for *y*, in the preceding formula, we have

$$x = \frac{D(D+E)}{D+E \cos. z^2}, \text{ or } x = \frac{D+E}{1 + \frac{E}{D} \cos. z^2}.$$

Therefore, if *D*, *E*, and *z* are given, *x* is found. Q. E. I.

*Corollary.* When *E* is small in comparison of *D*, this formula may be rendered more simple; for dividing by  $1 + \frac{E}{D} \cos. z^2$ , gives  $x = (D + E) \left( 1 - \frac{E}{D} \cos. z^2 + \frac{E^2}{D^2} \cos. z^4 - \&c. \right) = D + E \sin. z^2 - \frac{E^2}{D} \cos. z^2 \cdot \sin. z^2$  nearly. The first two terms of this expression will give the value of *x* very near the truth; for, in the degrees measured in the south of England,  $\frac{E}{D}$  is not greater than  $\frac{1}{180}$ . If, however, the third term is required, it is easily computed by multiplying the second, or  $E \sin. z^2$ , by  $\frac{E}{D} \cos. z^2$ . The small fraction that results, must be subtracted from the sum of the other two terms.

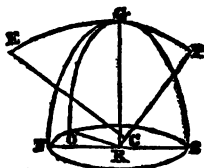
In captain Mudge's example,  $z = 81^\circ 56' 53''$ ;  $D = 60870$ ,  $D + E = 61182$ ,  $E = 332$ ;  $E \cdot \sin. z^2 = 325.48$  fathoms, the second term; and  $\frac{E^2}{D} \cos. z^2 \cdot \sin. z^2 = .04$ , therefore  $60850 + 325.48 - .04 = 61175.44 = x$ , the length required.

In this instance I presume the third term might safely have been neglected. That term will be greatest when the azimuth is  $= 45^\circ$ .

*Additional Solution by Mr. Dalby, taken from the Philosophical Transactions for 1803.*

**PROBLEM.** [Having the length of the degree on the meridian, and also that of the degree perpendicular to it, at the same point; to find the length of a degree in any other given direction, supposing the earth to be an ellipsoid.

Let  $EP$  be one fourth of the elliptic meridian;  $c$  the centre of the earth;  $CE$ ,  $CP$ , the equatorial and polar semiaxes;  $G$  a given point on the meridian  $EP$ . Draw  $GR$  perpendicular to the meridian at  $G$ , meeting  $EP$  in  $R$ ; then  $RG$  is the radius of curvature of the ellipse, which is perpendicular to the meridian at  $G$ .



Conceive another ellipsoid to touch the given one in the point  $G$ . Then, it is evident, that if the curvature be respectively the same in the direction of the meridian, and the perpendicular on both ellipsoids at the point  $G$ , the curvature will also be equal on both figures, in any other direction at that point. And the like is manifest in spheroids of any other kind.

Let  $m$  be the radius of curvature of the meridian at the point  $G$ ; then because  $RG$  is the radius of curvature in the perpendicular direction, if we take  $RS$  (at right angles to  $RG$ )  $= 2\sqrt{(RG \times m)}$ , and about  $RS$ , the axis to the semidiameter  $RG$ , describe the ellipsoid  $RSO$ , it will be that having the curvature at  $G$  the same as on the other ellipsoid at that point.

Let  $OGR$  be the plane of an ellipse, inclined to the meridian  $EP$ , or to the plane  $FGS$ , in a given angle  $FRO$ , whose *sine* and *cosine* are  $s$  and  $c$ . Then, since  $RG$ , or rather its equal, is a semitransverse, in the plane  $ROSK$ , (which is perpendicular to  $RG$ ) to the semiconjugate  $RF$ , we shall

have  $\frac{RG^2 \times RF^2}{RG^2 c^2 + RF^2 s^2} = RO$ , which divided by  $RG$ , ( $RG$  being the semi-

transverse to  $RO$  in the perpendicular plane  $ROG$ ,) gives  $\frac{RG \times RF^2}{RG^2 c^2 + RF^2 s^2}$ , for

the radius of curvature of the inclined ellipse  $OG$  at the point  $G$ . But because the lengths of the degrees are proportional to their radii of curvature, if we put  $m$  and  $p$  for the meridional and perpendicular degrees, then  $RF$  or  $\sqrt{(RG \times m)}$  and  $RG$  may be expounded by  $\sqrt{pm}$

and  $p$ ; hence the expression will become  $\frac{pm}{pc^2 + ms^2}$ , for the length of the degree oblique to the meridian; or putting  $1 - s^2$  for  $c^2$ , and  $r$  for  $p - m$ , it will be  $\frac{pm}{p - rs}$ .

*Cor.* If  $d$  be the length of the oblique degree, then, since  $d = \frac{pm}{pc^2 + ms}$ , we have  $p = \frac{s^2 dm}{m - c^2 d}$ , and  $m = \frac{c^2 dp}{p - s^2 d}$ . And if  $n$  be put for the length of another oblique degree at the same point, and  $s$  and  $c$  the sine and cosine of its inclination to the meridian, we shall get  $m = \frac{s^2 c^2 - c^2 s^2}{s^2 d - s^2 d} \times nd$ , and  $p = \frac{s^2 c^2 - c^2 s^2}{c^2 d - c^2 d} \times nd$ , the meridional and perpendicular degrees, exhibited in terms of the oblique degrees combined with the sines and cosines of their inclinations to the meridian. Therefore, an ellipsoid may be determined from the lengths of two oblique degrees in the same latitude.

We may also remark that  $\frac{pm}{p - rs^2}$  will give the oblique degree on different spheroids, because the expression is in terms of the meridional and perpendicular degrees and the sine of the obliquity only.

*Remark.* In the question proposed in the diary,  $p = 61182$ ,  $m = 60850$ , and  $s = \text{sine of } 81^\circ 56' 53''$ ; theref.  $\frac{pm}{p - rs^2} = 61175.4$ . L.



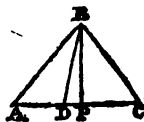
*Questions proposed in 1801, and answered in 1802.*

1. QUESTION 1074, by the Rev. J. Furnass, of Ponteland.

There is a field, in form of an isosceles triangle, through which runs a rivulet from the vertical angle towards the base, which, after a direct course of twelve chains, terminates in the said base, dividing it in such sort that the rectangle made of the two segments is equal to 48 chains, and the rivulet making an angle of  $25^\circ 20'$  with one of the equal sides. Required the areas of the two parts of the field on each side of the rivulet?

*Answered by Mr. Wm. Bewley, Hawkeshead.*

Let  $ABC$  be the triangle,  $BD$  the rivulet, and  $BF$  the perpendicular to the base  $AC$ . Then because  $AB = BC$ , we have  $AB^2 + AD \times DC = AB^2 = 144 + 48 = 192$ , and  $AB = 13.8564$ . And  $AB + BD : AB - BD :: \text{tangent } \frac{1}{2} \text{ sup. } \angle ABD : \text{tangent } 17^\circ 42' .97 = \frac{1}{2} \angle D - \frac{1}{2} \angle A$ ; hence  $\angle A = 59^\circ 37' .03$ . Then, as  $\text{sine } \angle A : BD :: \text{sine } \angle ABD : AD = 5.952$ ; consequently  $48 \div 5.952 = 8.064 = DC$ , and as  $\text{radius} : AB :: \text{sine } \angle A : BF = 11.953$ .



Therefore  $11.953 \times 8.064 \div 2 = 48.1844$  the area of the part BDC,  
and  $11.953 \times 5.952 \div 2 = 35.5721$  the area of the part BDA.

*The same by Mr. Gregory, Cambridge.*

In the figure,  $BD = 12$  chains,  $AD \times DC = 48$ ,  $AB = BC$ , and  $\angle ABD = 25^\circ 20'$ . It has been proved (see Dr. Hutton's Geom. Theor. 39) that  $AB^2 = BD^2 + AD \times DC$ , that is  $= 144 + 48 = 192$ , therefore  $AB = \sqrt{192} = 13.8564065$ . Here then we know  $AB$ ,  $BD$ , and the included  $\angle ABD$ , whence we find  $\angle D = 95^\circ 2' 58''$ ,  $\angle A = 59^\circ 37' 2''$ , and the side  $AD = 5.95202$ ; then  $48 \div 5.95202 = 8.064489 = DC$ ; hence  $AB \times BD \times \frac{1}{2} \sin \angle ABD = 35.57359$  chains, or 3 ac. 2 ro. 9.177 perches, the area of the triangle  $ABD$ ; and, since triangles of equal altitudes are as their bases, we have  $AD : DC :: \text{area of } ABD : \text{area of } BDC = 48.19922$  chains  $= 4$  ac. 3 ro. 11.187 perches.

*The same by Mr. Wm. Green, Academy, Deptford.*

By Theor. 39, Dr. Hutton's Geom.  $AD \times DC + BD^2 = AB^2$ , hence  $AB = \sqrt{192} = 13.8564 = BC$ , and  $\frac{1}{2} AB \times BD \times \sin \angle ABD = 3$  ac. 2 ro. 10 perches nearly. As  $AB + BD : AB - BD :: \tan \frac{1}{2} \angle D + \frac{1}{2} \angle A : \tan \frac{1}{2} \angle D - \frac{1}{2} \angle A$ , hence the  $\angle A$  or  $C = 59^\circ 37'$  and therefore  $\angle ABC = 60^\circ 46'$ ; hence  $\frac{1}{2} BC^2 \times \sin \angle ABC = \text{area of triangle } ABC = 8$  ac. 1 ro. 20p. from which take the triangle  $ABD = 3$  ac. 2 ro. 9 p. leaves the area of the triangle  $BDC = 4$  ac. 3 ro. 11 p.

II. QUESTION 1075, by Mr. Alex. Rowe, *Reginnis*.

Required to find  $x$ , when  $x^2 + 87$  and  $x^2 - 87$  are both rational squares.

*Answered by Mr. J. Barron, Schoolmaster Spilsby.*

Put  $x^2 + 87 = (z + d)^2$ , and  $x^2 - 87 = z^2$ ; their difference is  $2dz + d^2 = 174$ ; hence  $z = (174 - d^2) \div 2d$ , where  $d^2$  may be taken any square number less than 174, and therefore  $d$  not greater than 13. If  $d = 1$ , then  $z = \frac{173}{2}$ , and  $x = \sqrt{\frac{39277}{4}}$ ; hence  $x^2 + 87 = \left(\frac{173}{2}\right)^2$ , and  $x^2 - 87 = \left(\frac{173}{2}\right)^2$ .

*The same answered by Mr. Wm. Cole, Colchester.*

To obtain a general solution to this question put  $n = 87$ . Then, by the question  $x^2 + n$  and  $x^2 - n$  must be rational squares. Let  $x^2 + n = y^2$ , then will  $x^2 - n = y^2 - 2n$ . Substitute  $y^2 - 2n = (y - m)^2 = y^2 - 2ym + m^2$ , hence  $y = \frac{2n + m^2}{2m}$ , where  $m$  and  $n$  may

be taken equal to any numbers whatever; and the values of the two squares will be  $y^2 = ((2n + m^2) \div 2m)^2$ , and  $y^2 - 2n = ((2n + m^2) \div 2m)^2 - 2n$ . In the present case we have  $n = 87$ , and taking  $m = 1$ , we have  $x^2 + n = \frac{30625}{4}$ , and  $x^2 - n = \frac{9929}{4}$ , whose roots are  $\frac{175}{2}$  and  $\frac{173}{2}$  respectively; and the value of  $x = \sqrt{(((2n + m^2) \div 2m)^2 - n)} = \sqrt{\frac{30677}{4}}$ , an irrational number, which it must be in all cases, because  $x^2$  is an arithmetical mean between two rational squares. Suppose  $n = 87$  and  $m = 2$ , then we have  $y^2 = \frac{7991}{4}$ , and  $y^2 - 2n = \frac{7745}{4}$ , whose roots are  $\frac{89}{2}$  and  $\frac{85}{2}$ . Or, supposing  $n = 1$ , and  $m = 1$ , we have  $y^2 = 2$  and  $y^2 - 2n = \frac{1}{4}$ , whose roots are  $\frac{2}{2}$  and  $\frac{1}{2}$ .

*The same answered by Mr. Alexander Rowe, Reginnis.*

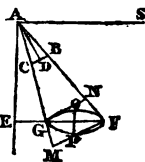
In the latter expression, as I sent it, it was  $x^2 - 85$ , not  $x^2 - 87$ . This being corrected let  $x^2 + 87 = (x + 1)^2 = x^2 + 2x + 1$ , and  $x^2 - 85 = (x - 1)^2 = x^2 - 2x + 1$ ; the difference of these two gives  $172 = 4x$ , and hence  $x = 43$ . Consequently  $43^2 + 87 = 1936$ , and  $43^2 - 85 = 1764$ , are the square numbers, the roots of which are 44 and 42, as required.

### III. QUESTION 1076, by Mr. J. Hefford, Dronfield Academy.

If rays of light be emitted from the vertex of a right cone, (the diameter of the base of which is 6, and altitude 12 inches), on a table; it is required to find the area of the shadow of the cone on the table, when the axis makes an angle of  $60^\circ$  with a plane parallel to the table, and the nearest or perpendicular distance of the cone's vertex above the table being 5 feet?

*Answered by Mr. G. Buffham, Boston.*

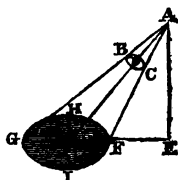
Let  $EF$  represent the plane of the table, as parallel to it,  $ABC$  a section of the cone, whose perpendicular  $AD$  makes the  $\angle DAS = 60^\circ$ ; and  $GOP$  the shadow on the table, which is an ellipse by the property of the cone, whose transverse is  $GF$ , and conjugate  $OP = \sqrt{(GN \times FM)}$ , by Dr. Hutton's Conics, page 6; also draw the lines as per figure. In the right-angled triangle  $DAB$ , there are given  $AD = 12$  inches, and  $DB = 3$  inches, to find the angle  $DAB = 14^\circ 2' 10''\frac{1}{2}$ ; in the right-angled triangle  $AGE$ , there are given  $AE = 60$  inches, and the  $\angle GAE =$



$15^{\circ} 57' 49''\frac{1}{2}$ , to find  $AG = 62.4067$ . Then, in the triangle  $AOE$ , are given the angles and the side  $AG$ , to find  $GF = 40.85108$ , and  $AF = 83.46082$ . In the same manner, or by similar triangles are found  $GN = 30.2717$ , and  $FM = 40.48447$ ; theref.  $OP = \sqrt{(GN \times FM)} = 35.00762$ ; conseq. the area of the shadow  $= GF \times OP \times .7854 = 1123.2$  inches, or 7.8 square feet.

*The same by Mr. Wm. Francis, jun. Maidenhead.*

$\sqrt{(12^2 + 3^2)} = 12.3693 = AC$ , the cone's slant height; and as  $AC : \text{radius} :: DC : \sin. \angle DAC = 14^{\circ} 2' 30'' = \angle BAD$ . Also as  $\sin. \angle AOE : AE :: \sin. \angle OAE : OE = 34.6411$  and  $:: \text{radius} : OA = 69.282$ . Again, as  $\sin. \angle AFE : AE :: \sin. \angle FAE : EF = 17.1575$ ; and as  $\sin. \angle AGE : AE :: \sin. \angle GAE : GE = 58.026$ . Theref.  $GF = GE - EF = 40.8085$ , the transverse diameter of the elliptical shadow. And as  $AD : DB :: AO : OH = 17.3205$  an ordinate to the absciss  $OF$ . Hence as  $\sqrt{(GO \times OF)} : OH :: GF : 35$  the conjugate diameter. Therefore the area is  $1123.4342$  square inches.

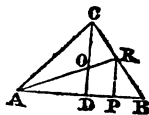


IV. QUESTION 1077, by Mr. T. Hornby, *Land Surveyor, Wimbleton.*

A triangular field consists of two different values or qualities of land, the line of partition, or quality line, being the perpendicular drawn from the greatest angle to its opposite side: it is required to find a point in the shortest side, so that a line drawn from thence to its opposite angle may lay off next to the base or longest side a piece of land worth seven pounds: the value between the point and perpendicular being 20 shillings, and that on the other side of the perpendicular 25 shillings the acre; the area of the field being 12 acres, and the segments of the base 9 and 11 chains respectively.

*Answered by Mr. John Cavill, of Beighton.*

Here are given  $AB = 20$ ,  $CD = 12$ ,  $AD = 11$ ,  $BD = 9$ . Therefore since  $BD : CD :: 3 : 4$ ,  $BP$  to  $PR$  are in the same ratio. Put  $3x = BP$ ,  $4x = PR$ ,  $a = AD$ ,  $b = AB$ ,  $7 \times 20 = 140$  shillings  $= s$ , then  $b - 3x = AP$ . Now by similar triangles  $b - 3x : 4x :: a : 4ax \div (b - 3x) = DO$ , and  $2a^2x \div (b - 3x) = \text{area of the triangle } ADO$ ; but  $2bx = \text{area of the triangle } ABR$ , therefore  $2bx - 2a^2x \div (b - 3x) = \text{area of the trapezium } DOBR$ ; consequently  $2 \times (2bx - 2a^2x \div (b - 3x)) + 2\frac{1}{2} \times 2a^2x \div (b - 3x) = s$ , that is  $4bx + a^2x \div (b - 3x) = s = 7b$ ; this reduced, and in numbers, is  $x^2 - (2141 \div 240)x = -$



$140 \div 12$ ; hence  $x = 1.591855$ , therefore  $BR = 4.775565$ , and  $PR = 6.36742$ , also  $BR = 7.959275$ , which gives the point  $R$ .

*The same answered by Mr. A. Corse, Royal Artillery.*

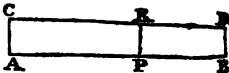
Let  $ABC$  represent the triangular field,  $AOR$  the dividing line, also  $CD$  and  $RP$  perpendicular to  $AB$ . Now  $AD = 11$ ,  $BD = 9$ , and, the area being given, the perpendicular  $CD$  and side  $BC$  are found  $= 12$  and  $15$  respectively. Now put  $x = BR$ ; then by sim. tri.  $15 : 12 :: x : \frac{2}{3}x = PR$ , and  $15 : 9 :: x : \frac{3}{4}x = PB$ , hence  $20 - \frac{2}{3}x = AP$ , and as  $20 - \frac{2}{3}x : \frac{2}{3}x :: 11 : 44x \div (100 - 3x) = DO$ . Then  $242x \div (100 - 3x) = \frac{1}{2} DO \times 11 = \text{area of the tri. } ADO$ , or quantity of land at 25 shillings per acre, and  $\frac{1}{2} AB \times PR = 8x$  the area of the tri.  $ABR$ , therefore  $3x - 242x \div (100 - 3x) = DOBR$  the quantity of land at 20 shillings the acre. Hence  $16x + 121x \div (100 - 3x) = 140$ , which reduced gives  $x^2 - (2141 \div 48)x = -14000 \div 48$ , which gives  $x = 7.95927$  the distance of the required point from the angle at the base.

V. QUESTION 1078, by Mr. Henry Hunter, of *Alnwick*.

If a piece of timber, 24 feet long, 10 inches deep at the greater end, 4 inches at the less, and of equal breadth or thickness throughout be supported at both ends, and a weight be dispersed uniformly throughout the length of it; it is required to determine under what part of the beam a prop ought to be put, so as it may be made the strongest possible?

*Answered by Mr. Bewley, Colthouse Academy.*

Rejecting the weight of the beam, as not material to the present question; and calling  $AB = 24 \text{ feet} = 288 \text{ inches} = a$ ,  $AC = 10 = b$ ,  $BD = 4 = d$ ,  $AP = x$ . As the weight is equally distributed on the whole length of the beam, when propped at  $P$ , the part  $AP$  will be weakest at  $P$ , and the part  $PB$  will be weakest at  $B$ . Now the strength at  $P$  is as the square of the depth  $PR$ , and the strength at  $B$  as the square of the depth  $BD$ , also the momentum or stress on the part  $AP$  is as  $AP^3$ , and that on  $PB$  as  $PB^3$ , therefore the comparative resistances against breaking at  $P$  and  $B$ , are as  $PR^3 \div AP^3$ , and  $BD^3 \div PB^3$ ; but when the beam is the strongest, these two will be equal to each other, or  $PR \div AP = BD \div PB$ , or  $PR \times PB = AP \times BD$ . Now, by similar figures,  $AB : AP :: AC - BD : AC - PR$ , that is,  $a : x :: b - c : b - PR$ ; hence  $PR = b - (b - c)x \div a$ ; hence  $PB \times PR = AP \times BD$  gives the equation  $(a - x) \times (b - (b - c)x \div a) = cx$ ; which reduces to  $x^2 - 2abx \div (b - c) = -a^2b \div (b - c)$ , or in numbers,  $x^2 - 80x = -960$ , when the numbers are brought



to feet; then the root  $x = 14.70178$  feet, the distance from the larger end of the beam where the prop must be set, to make the beam the strongest under the given circumstances.

*The same by Mr. Henry Hunter, Alnwick.*

Put  $b =$  the depth at the greater end,  $c =$  that at the less,  $a =$  the length of the beam, and  $x =$  the depth at the prop. Then  $b - c : a :: x - c : a(x - c) \div (b - c) =$  the distance of the prop from the less end, consequently  $a - a(x - c) \div (b - c) = a(b - x) \div (b - c)$  the distance from the greater end. Hence the stress on each fibre at the prop will be as  $(x - c) a \times (b - x) a \div (b - c)(b - c) \times 2x$ , which is to be a minimum, or  $(x - c) \times (b - x) \div x$  a min. or  $-x - bc \div x$  a minimum; its fluxion  $-\dot{x} + bc\dot{x} \div x^2 = 0$ , hence  $x^2 = bc$ , and  $x = \sqrt{bc} = \sqrt{40} = 6.324555$  the depth of the beam at the prop, which is a mean proportional between the thickness at the two ends. Hence  $(x - c) a \div (b - c) = 9.29822$  feet, the distance of the prop from the smaller end.

VI. QUESTION 1079, by Mr. John Sowerby, of Dudley.

Having procured a vessel of considerable depth, into which, after filling it with water, I inverted a glass barometer tube, of  $3\frac{1}{2}$  feet in the length, and pushed it down till the lower or open end was  $9\frac{1}{4}$  feet below the surface: it is required to determine the height to which the water will rise within the tube, the pressure of the atmosphere at the same time being equal to that of  $33\frac{1}{2}$  feet of the water?

*Answered by Mr. Geo. Barrett.*

Put  $x$  for the number of feet the water will rise in the tube. Then, as the product arising from the space occupied by the air in its compressed state multiplied by its compressing force, will always amount to the same, we therefore have  $(33\frac{1}{2} + 9\frac{1}{4} - x) \times (3\frac{1}{2} - x) = 33\frac{1}{2} \times 3\frac{1}{2}$ ; an equation which reduces to  $x^2 - 47x = -34\frac{1}{8}$ : Hence  $x = .7376407$  feet, or  $8.8516884$  inches, the height sought.

*The same by Mr. J. Collins, Kensington.*

First,  $w =$  weight of the atmosphere  $= 33\frac{1}{2}$ ,  $AF = 9\frac{1}{4}$ ,  $ce = 3\frac{1}{2}$ , and  $BF = x$ . Then by Dr. Hutton's new Course, p. 241, as  $33\frac{1}{2} + AF - BF : 33\frac{1}{2} :: 3\frac{1}{2} : 3\frac{1}{2} - x$ , that is as  $43\frac{1}{4} : 33\frac{1}{2} :: 3\frac{1}{2} : 3\frac{1}{2} - x$ , by multiplying extremes and means, &c.  $x^2 - 47x = -34\frac{1}{8}$ ; this solved gives  $x = .73765$  feet  $= 8.85180$  inches.



*The same by Mr. Tho. Croudace, and Mr. Rob. Surtees, Lanchester.*

As  $9\frac{1}{4}$  feet is the given depth the lower end of the tube is immersed,  $33\frac{1}{2}$  the given pressure of the atmosphere at the same time, and  $3\frac{1}{2}$  the length of the tube. Let  $x$  = the space occupied by water in it; then  $3\frac{1}{2} - x$  = the space occupied by air; but by vol. 1 page 390, Dr. Hutton's Dictionary, the space occupied by air, is to the space filled with water, as the pressure of the atmosphere, is to the depth of the surface of the water in the tube below the common surface of it, viz. as  $3\frac{1}{2} - x : x :: 33\frac{1}{2} : 9\frac{1}{4} - x$ , hence  $34\frac{1}{8} - 13\frac{1}{4}x + x^2 = 33\frac{1}{2}x$ , or  $x^2 - 47x = -34\frac{1}{8}$ , and  $x = .7376541$  of a foot, or 8.8518492 inches, the height required.

*The same by Mr. Alex. Rowe, Reginnis, near Penzance.*

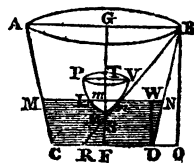
Put  $a = 3\frac{1}{2}$  feet, the length of the barometer tube;  $b = 9\frac{1}{4}$  feet, its depth below the surface of the water,  $p = 33\frac{1}{2}$  feet, the pressure of the atmosphere, and  $x$  = the height to which the water will rise in the tube. Then, by Dr. Hutton's Course of Mathematics, vol. 2, page 241, as  $b + p - x : p :: a : a - x$ , or in numbers,  $43\frac{1}{2} - x : 33\frac{1}{2} :: 3\frac{1}{2} : 3\frac{1}{2} - x$ ; hence, mult. extremes and means,  $152\frac{1}{4} - 47x + x^2 = 118\frac{1}{8}$ , and the root is  $x = .73764$  feet = 8.85168 inches, as required.

VII. QUESTION 1080, by Mr. Isaac Rowbottom, West-Hallam.

The perpendicular depth of a vessel, in form of the frustum of a cone, is 108 inches, and the top and bottom diameters 56 and 36 inches respectively, being placed with its lesser end on an horizontal plane, and containing a certain quantity of common water. Now if a solid of cork, in form of a semispheroid, the axis of which are 8 and 6 inches, be put into it, with its vertex downward, and the semitransverse coinciding with the axis of the vessel, then a right line drawn from any point in the circumference of the top of the vessel, will just touch the solid at the surface of the fluid. From hence it is required to determine the quantity of water in the vessel: supposing the weight of a cubic inch of cork to be .298888, and that of water .578697 ounces?

*Answered by Mr. Tho. Croudace, and Mr. R. Surtees:*

Let ABCD represent the vessel, PEY the half spheroid, MLEN the surface of the water after the cork is put in. The solidity of PEY is 603.18578 inches, which mult. and divided by the weight of a cubic inch of cork and water, gives 311.527733 inches, the quantity of water displaced by the cork, or the solidity of LEN; then by Dr. Hutton's Mensuration for finding the segment



of a spheroid, the height  $em = 5.32104$ , the ordinate  $mn = 5.653586$ , and by Conics as  $im : ie :: ie : is = 23.889678$ ; but by similar tri. as  $mn : ms :: gb : gs = 105.048391$ , and  $mf = 24.162327$ ; also as  $fg : gb :: fm$  or  $dw : wn = 2.2372525$ , and  $mn = 40.474505$ , hence the solidity of  $mcn$  less solidity of  $len$ , gives  $27467.499249$  inches, or  $97.402$  gallons, the quantity of water in the vessel.

*The same by Mr. Isaac Rowbottom, West-Hallum.*

Let  $ABDE$  represent the vessel,  $AB$  and  $CD$  the two diameters,  $GF$  the axis,  $FVE$  the semispheroid,  $MN$  the surface of the water after the solid is put in,  $LEN$  the part of the solid immersed in the water,  $BN$  the line touching the solid at  $n$ , which will be a tangent to the curve at  $n$ , produce  $BN$  to cut  $GF$  in  $s$  and  $CD$  in  $r$ , and  $CD$  produced to  $o$  where a perpendicular from  $B$  meets it.  $FV = 2c = 12$ ,  $IE = t = 8$ ,  $m = .298888$  of an ounce the weight of a cubic inch of cork,  $w = .578697$  oz. that of water,  $p = 3.1416$ , and  $x = em$ . Then  $\frac{2}{3}ptc^2$  is the solidity of the semispheroid, and  $\frac{2}{3}ptc^2m$  its weight, consequently  $2ptc^2m \div 3w$  the cubic inches of water removed by the solid, which must be equal to the part  $LEN$  immersed, hence, by the property of the spheroid,  $pc^2(tx^2 - \frac{1}{3}x^3) \div t^2 = 2ptc^2m \div 3w$ , in numbers  $24x^2 - x^3 = 528.880073$ , hence  $x = 5.3211219$ , and  $mn = 5.65360785$ , consequently the sub-tangent  $sm = 21.21172$ . Then, by similar triangles,  $sm : mn :: BO : RO = 28.785485$ , and  $RO - OF = RF = .785485$ . Again, by sim. tri.  $RO : OB :: RF : fs = 2.94705$ ; then  $sm + sf = fm = 24.15877$ , the height of the water in the vessel; consequently  $MN = 40.473846$ , hence the solidity of  $MNDC$  — that of  $LEN = 97.38197$  gallons, the quantity of water in the vessel.

VIII. QUESTION 1081, by Mr. John Blackwell, Hungerford.

After observing a flash of lightning I counted 20 vibrations of a pendulum which measured  $9.540869$  inches from the point of suspension to the surface of its spherical bob, before I heard the thunder; what was my distance from the cloud, supposing the surface of the bob to be in proportion to its solidity, as 5 to 2?

*Answered by Mr. Wm. Burdon, of Acaster Malbis.*

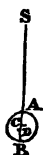
Put  $x$  = the diameter of the globe; then  $5 : 2 :: 3.1416x^2 : .5236x^3 :: 6 : x = 2.4$ . And, from Dr. Hutton's Dictionary, vol. 1, page 268, the distance of the centre of oscillation from the point of suspension will be found =  $9.8$  inches. Therefore  $\sqrt{9.8} : \sqrt{39\frac{1}{4}} :: 60 : 120$  the number of vibrations made in one minute, and  $120 : 60'' :: 20 : 10''$  the time of making 20 vibrations. Consequently  $1'' : 1142$  (the velocity of sound per second)  $:: 10'' : 11420$  feet, =  $.2$  miles and  $862$  feet, the distance of the cloud.

*The same answered by Mr. William Richards, London.*

Put  $x$  = the diameter of the ball, and  $p = 3.1416$ . Then  $px^2 =$  the superficies, and  $\frac{1}{2}px^3$  the solidity of the sphere: therefore by the question  $px^2 : \frac{1}{2}px^3 :: 5 : 2$ , hence  $x = 2.4$ , and the radius of the sphere = 1.2 inches. Put  $r = 1.2$ ,  $a = 8.540869$ , and  $g = a + r$ , then, by Dr. Hutton's Dictionary, Vol. 1, page 268,  $g + 2r^2 \div 5g = 9.8$  inches nearly is the distance of the centre of oscillation or the length of a pendulum isochronal to the given one. Therefore as  $39\frac{1}{8} : 9.8 :: 20^2 : 100.1917$  the square of the vibrations of a second's pendulum, and the square root  $10.00957$  is the number of seconds; then  $10.00957 \times 1142 = 11430.9289$  feet, or 2 miles 290 yards 1 foot nearly, is the distance required.

*The same by Mr. Wm. Watkins, Haddon-on-the-Wall.*

Put  $a = AS = 8.540869$ ,  $t = .5236$ ,  $p = 3.14159$ ,  $AB = x$ . Then as  $px^2 : tx^3 :: 5 : 2$ , therefore  $x = 2p \div 5t = 2.4$  the axis of the spherical bob. But, by p. 268, Dr. Hutton's Mathematical Dictionary, the length of the pendulum is  $SD = g + 2r^2 \div 5g = 9.8$  inches very nearly. Again, by the same, the lengths of pendulums are reciprocally as the squares of the number of vibrations, therefore as  $\sqrt{9.8} : 1'' :: \sqrt{39\frac{1}{8}} : 1.988$  vibrations in a second, and  $20 \div 1.988 = 10.06$  seconds nearly; then, page 472 above, sound flies 13 miles in a minute nearly; hence  $60'' : 13 :: 10.06 : 2.1796$  miles, the distance.

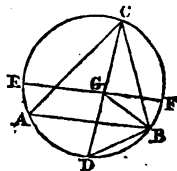


IX. QUESTION 1082, by Mr. J. Collins, Kensington.

Given the vertical angle of a triangle, with the radii of the circumscribing and inscribed circles, to construct it.

*Answered by Mr. George Barrett.*

**Construction.** With the given radius describe the circumscribing circle; from any point  $D$  take the arcs  $DA$ ,  $DB$  each equal to the measure of the given angle, and join  $AB$ , parallel to which draw  $EF$  at a distance from it equal to the given radius of the inscribed circle; join  $BD$ , and from  $D$  to  $EF$  apply  $DG = DB$ , which produce to the circumference at  $C$ , joining  $AC$  and  $BC$ ; then  $ABC$  shall be the triangle that was to be constructed.

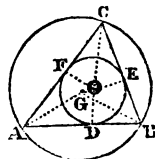


**Demonstration.** The  $\angle ACB$  will be equal to the given angle, having the arc  $AD$  or  $DB$  for its measure by construction and Euclid 20, 3; also the  $\angle ACD = \angle DCB$  by Euclid 27, 3; consequently the line  $CD$  bisects the  $\angle ACB$ . Join  $BG$ ; then, the  $\angle DGB = \angle DBG$ , for  $DB =$

ng; and also = the angles gcn + gno by Euclid 32, 1; but because the  $\angle DCB = \angle ABD$  on the equal arcs DA, DB; the  $\angle GBC$  shall be = the  $\angle ABG$ ; consequently the line ng will bisect the  $\angle ABC$ ; and therefore as g is the point of intersection of the two bisecting lines cd and ag, it will be the centre of the inscribed circle of the triangle abc.

*The same by Mr. John Craggs, Hilton, near Sunderland.*

*Geomet. Analysis.* Suppose abc to be the triangle required, and g and o the centres of the inscribed and circumscribed circles. Now the  $\angle ACB = \frac{1}{2} \angle AGB$ , and  $AG = GB =$  radius of the circumscribed circle are both given, hence the base AB of the triangle ABC is also given; likewise  $OE = OF =$  radius of the inscribed circle, the  $\angle OCE = \angle OCF = \frac{1}{2}$  vertical angle, and  $CE = CF$  are also given, lastly  $2CE + AB = AC + CB$  are also given. Therefore the question is reduced to this, given the base, sum of the sides, and the vertical angle, to construct the triangle. See the construction at p. 315, Simpson's Algebra, 6th Edition.



X. QUESTION 1083, by the Rev. Thomas Scurr, Hexham.

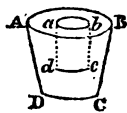
There is a circular fortification, which occupies a quarter of an acre of ground, surrounded by a ditch, coinciding with the circumference, 24 feet wide at bottom, 26 at top, and 12 deep: How much water will fill the ditch; and how many feet square must the sluice be to fill it in one hour; the head of water above the sluice being 10 feet?

*Answered by Mr. John Barron, Schoolmaster Spilsby.*

The area of the fortification is given = 10890 feet, from which the diameter is found = 117.752. And since the width of the ditch is 24 at bottom and 26 at top, the mean width is 25, consequently  $117.752 + 50 = 167.752$  is the diameter of the fort and ditch together; hence, by a well known property of circles, as  $117.752^2 : 167.752^2 :: 10890 : 22101.7523$  the area including the ditch, consequently  $22101.7523 - 10890 = 11211.7523$  is the mean area of the ditch, and  $11211.7523 \times 12 = 134541.0276$  cubic feet is the quantity of water required to fill the ditch. Then by Dr. Hutton's Math. vol. 2, page 341, putting  $g = 16\frac{1}{2}$ , and  $x =$  the area of the sluice, as  $\sqrt{g} : \sqrt{10} :: 2g : 2\sqrt{10}g$  the velocity of the water through the sluice, consequently  $2x\sqrt{10}g$  is the quantity of water per second running through it, and  $3600 \times 2x\sqrt{10}g$  the quantity running in 3600'' or one hour thereof.  $3600 \times 2x\sqrt{10}g = 134541.0276$ , hence  $x = 13454.10276 \div 7200\sqrt{10}g = 1.47344692$ , the area of the square sluice, and  $\sqrt{x} = 1.21385$  its side required.

*The same by Mr. Wm. Francis, jun. Maidenhead.*

Put  $x$  = the diameter of the fortification. Then  $.7855x^2 = 2.5$  square chains, and  $x = 1.784$  chains = 117.744 feet. Now the ditch in fortification slopes equally on each side, therefore the conical frustum  $abcd$  deducted from the conical frustum  $ABCD$ , will be the water's content = 135474.84512. Then, by Dr. Hutton's Course, the Hydraulics prop. 61,  $2a\sqrt{16\frac{1}{2}} \times 10 = 135474.84512 \div 60'$ , hence  $a = 1.483673$  feet the area of the sluice, and  $\sqrt{a} = 1.218$  feet, its side.



*The same by Mr. Gregory, Cambridge.*

A quarter of an acre is equal to 10890 feet, the ground occupied by the fortification, which being circular,  $\sqrt{(10890 \div .785398)} = 117.7522$  its diameter. Then, 26 feet being the breadth of the ditch at the top,  $(117.7522 + 26) \times 3.141593 = 451.6109$  the length of the ditch measuring along the middle; and the area of a vertical section of the ditch will be  $\frac{1}{2}(26 + 24) \times 12 = 300$ ; therefore  $451.6109 \times 300 = 135483.27$  feet of water required to fill the ditch. Then, the head of water being 10 feet above the sluice we have, by prob. 24, Dr. Hutton's Conic Sections and Select Exercises,  $\sqrt{g} : \sqrt{10} :: 2g : 2\sqrt{10}g$ , velocity of the water per second through the sluice,  $g$  being  $16\frac{1}{2}$ ; and, if  $x$  denote the area of the sluice,  $2x\sqrt{10}g$  will be the quantity of water running through it per second. Hence, 3600 being the seconds in one hour, we have  $3600 \times 2x\sqrt{10}g = 135483.27$ ; whence  $x$  is found to be  $135483.27 \div 7200 \sqrt{160\frac{1}{2}} = 1.483765$  feet, area of the square sluice, and  $\sqrt{x} = 1.2181$  feet very nearly, the side of the sluice.

XI. QUESTION 1084, by Mr. Wm. Burdon, *Acaster Malbis.*

June the 21st, in latitude  $70^\circ$  north, it was observed that the shadow of a staff passed over a space, during the sun's apparent revolution, of 1000 square yards, on a horizontal plane. Required what was the length of the staff, supposing it was erected perpendicular to the plane?

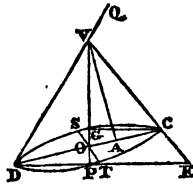
*Answered by Mr. Wm. Burdon, the Proposer.*

The latitude ( $70^\circ$ ) and declination ( $23^\circ 28'$ ) being given, the two meridian altitudes on that day are easily found to be  $3^\circ 28'$  and  $43^\circ 28'$ . Now let  $av$  represent the staff,  $nsctro$  the plane of the horizon. Then the rays intercepted by  $v$ , in one revolution, will form a cone,

the axis of which produced passes through the poles of the world, and consequently, when the sun does not set, the path of the shadow will be an ellipsis, suppose  $pscr$ : draw  $cg$  and  $dpe$  perpendicular to  $vc$  the cone's axis,  $vc$  being produced to meet  $de$  in  $e$ .

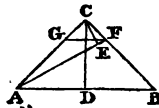
Put  $f = \text{sine of } 46^{\circ} 56' = \angle qvc = \text{double the sun's declination } vdp \text{ or } vep$ ,  $g = \text{sine of } 43^{\circ} 28' = \angle vcn$  his greatest meridian altitude,  $h = \text{sine of } 3^{\circ} 28' = \angle vnc$  his least meridian altitude,  $p = \text{sine of } 66^{\circ} 32' = \angle dvp \text{ or } evp$  his co-declination, and  $x = av$  the height of the staff. Then, by trigonometry, as  $g : x :: 1$  (radius) :  $x \div g = cv$ , and  $h : cv :: f : fx \div gh = dc$ ; as  $1 : cv :: p : px \div g = cg$ ; as  $h : x :: 1 : x \div h = dv$ ; as  $1 : dv :: p : px \div h = dp$ . Now by Dr. Hutton's Conics, the conjugate diameter  $st = 2\sqrt{(cg \times dp)} = 2px \div \sqrt{gh}$ , hence  $dc \times st \times .7854 = 2px \times fx \times .7854 \div gh \sqrt{gh} = 1000$  the area of the ellipsis  $dsct$ ; therefore  $x = \sqrt{(1000gh \sqrt{gh} \div 1.5708pf)} = 2.839$  yards the required height of the staff.

N. B. If an allowance be made for semidiameter, &c. the answer will come out a little different.



*The same by Mr. George Barrett.*

The sun being in the tropic on the 21st of June, will be at its greatest declination North, which is nearly  $23^{\circ} 28'$ ; and if his apparent semidiameter at that time be  $16'$ ; the northern part of his disc will decline  $23^{\circ} 44'$  from the equator. Let ABC represent the section of an upright cone, through its axis cd. Supposing the angles A and B of the section to be each  $= 23^{\circ} 44'$ , then half the vertical  $\angle ACB$ , will be the  $\angle ACD$  or  $BCD = 66^{\circ} 16'$ ; consequently if cd be the earth's axis produced, a line drawn from the northern part of the sun's disc to the point c will, if produced, pass through the circumference of the cone's base; and the shadow (of the axis produced) cd would pass over a space, on a plane coinciding with the cone's base, during the sun's apparent revolution, equal to the area of the cone's base; and AF be a plane parallel to the horizon, where the staff is erected, its inclination to cd will be  $70^{\circ}$ ; and the section of the cone by this plane, passing through A will be an ellipsis, similar to that which the shadow of the staff will make on the horizontal plane, the greater axis of this ellipsis will be AF, and the less a mean proportion between AB and its parallel FG. Draw CE perpendicular to AF; then will CE be parallel to the staff, and the  $\angle DCE$  will be  $20^{\circ}$ . Let CE = 1; then as the  $\angle ACE = 86^{\circ} 16'$ , its tangent will be AE = 15.325358, and its secant AC = 15.357949; and as the  $\angle ECF = 46^{\circ} 16'$ , its tangent will be EF = 1.0452221, and its secant CF = 1.4465439; consequently AE + EF = AF = 16.370580 the transverse axis. To find AB, say as radius : sine  $\angle ACD$  :: 2AC :



$AB = 28.1182115$ ; and to find  $FG$ , say as radius : sine  $\angle ACD :: 2cr$  :  $FG = 2.6484153$ ; therefore  $\sqrt{(AB \times FG)} = 8.629525$  the conjugate axis. Hence the area  $AF \times 8.629525 \times .78539 \&c. = 110.9534584$ . Then  $\sqrt{(1000 \div 110.9534584)} = 3.00213$  yards, the length of the staff as required.

XXX. QUESTION 1085, by Mr. John Surtees, *Sunderland*.

If, by experiment, a piece of square timber, whose side  $= s$ , length  $= l$ , and weight  $= w$ , bears a weight  $= p$  on its middle part, when supported at each end: Quere the length of a beam of timber, of the same matter and quality, whose side  $= m$ , and likewise supported at each end, that will but just bear its own weight?

*Answered.*

The general or algebraic solution to this question, will admit of several different forms, viz. according as the ratio or comparison of the contents and weights of the similar beams, is taken from that of the cube of their sides, or the cube of their lengths, or as the product of their length and square of their sides, &c. and yet all of these would yield the same conclusion in numbers. Accordingly all those modes of solution are used by our ingenious correspondents, and we shall therefore endeavour to insert a specimen of each, as far as our room will allow.

*The Solution by Mr. Wm. Burdon.*

Most authors, who have written, on the strength and stress of timber, shew that when the prism  $x$  will just support itself,  $swx^3 = mxl^3 + 2mpl^2$ , and consequently  $x = l \sqrt{(m(w + 2p) \div sw)}$ . For example, a piece of oak 3 feet long, and 1 inch square, will bear in the middle 330 pounds (Dr. Hutton's Dictionary, page 533, vol. 2), and consequently a beam 1 foot square, will break of itself if its length exceed  $243\frac{1}{2}$  feet.

*The same by Mr. D. Henry, Preston.*

Let  $x$   $\doteq$  the length required, and the rest as given in the question. Then, the weights of similar bodies being as their magnitudes, as  $s^3 : w :: m^3x : m^3xw \div s^3l$  = the weight of the piece wanted. And, in square timber, the strength being as the cube of the depth, and the stress as the length and weight to be born,  $s^3 : (\frac{1}{2}w + p) b :: m^3 : m^3xw \div 2s^3l$ ; this turned into an equation, and reduced, gives  $x = l \sqrt{(m(w + 2p) \div sw)}$ . Example, if  $l = 3$  feet,  $s = 1$  inch,  $w = 21b$ ,  $p = 210b$ , and  $m = 12$  inches, the above conclusion gives  $x = 150.957$  feet.

## XIII. QUESTION 1086, by Mr. O. G. Gregory, Cambridge.

There is a musical string, the length of which is 20 inches, and diameter  $\frac{1}{30}$  of an inch, which is stretched, by a weight of nine pounds. It is required to find the lengths, diameters, and tensions, of three other strings, one to sound a greater third, another a fifth, and the last an octave more acute than the first, and these so adjusted, that the length, diameter, and tension of the first, or given string, shall have one and the same ratio to the length, diameter, and tension of the second; that the length, diameter, and tension of the second, shall bear one and the same ratio to those of the third; and that the length, diameter, and tension of the third, shall have one and the same ratio to those of the fourth.

Answered by Mr. Geo. Barrett.

As the time of vibration, on which the tone depends, is proportional to the diameter and length of the string directly, and as the square root of the tending force inversely; we have  $20 \times \frac{1}{30} \div \sqrt{9} = \frac{1}{3}$  for a standard measure for the times of vibration of all the strings; consequently  $\frac{1}{3} \times \frac{4}{3} = \frac{4}{9}$  for the proportionate time of vibration of the 2d string to sound a *greater third* more acute than the 1st;  $\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$  ditto of the 3d string to sound a *fifth*; and  $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$  ditto of the 4th string to sound an *octave*. Hence the times of vibration of the several strings will be to each other in order as  $\frac{1}{3}, \frac{4}{9}, \frac{1}{4},$  and  $\frac{1}{6}$ , or as 1,  $\frac{4}{3}, \frac{3}{4},$  and  $\frac{2}{3}$ , agreeable to the times set forth in the diatonic scale. Then if we put  $20x, 20y,$  and  $20z,$  for the lengths of the 2d, 3d, and 4th strings, their several diameters will be  $\frac{1}{30}x, \frac{1}{30}y, \frac{1}{30}z$ ; and also their tending forces  $9x, 9y, 9z$ . Then, by proceeding in the same manner as in finding the standard measure, we get  $20x \times \frac{1}{30}x \div \sqrt{9x}$  or  $x \div \sqrt{9x} = \frac{4}{9}$ ; in like manner  $y \div \sqrt{9y} = \frac{3}{4}$ ; and  $z \div \sqrt{9z} = \frac{2}{3}$ ; these three equations, being resolved, give  $x = \cdot 8617738, y = \cdot 7631354,$  and  $z = \cdot 6299604$ .

Hence the annexed table of the dimensions of the several strings.

	Length.	Diameter.	Tension.
1st. string	20	·05	9
2d. string	17·235476	·04308869	7·7559642
3d. string	15·262708	·30815677	6·8682186
4th. string	12·599208	·03149802	5·6696436

## XIV. QUESTION 1087, by Amicus.

At an equal perp. distance from the horizontal line, which is the common intersec. of two equally inclined smooth planes, are laid two equal globes, of one inch radius, to touch the planes, and have their centres

in the same vertical plane, which is perpendicular to both the inclined ones, and resting on these two globes with their centres in the same plane with the first, are laid two others of the same bulk with them; then, to touch these, are laid on, two other equal ones, and so forth; till on each side there are five so laid on, with their centres in the same vertical plane, each of the two uppermost touching one and the same globe at top, of equal bulk with the others; these eleven globes thus laid resting in equilibrio on the inclined planes, and forming an arch. Now two lines joining the centres of the three uppermost form at the centre of the highest, an angle of  $120^\circ$ . Hence the height and span of the arch, and the inclination of the two supporting planes may be found, and are here required.

*Answered by Amicus, the Proposer.*

By what has been done on the subject by Mr. Stirling, it appears that the co-tangents of the respective angles made by the equilibrium forces, at the respective centres of the given globes, beginning at the uppermost or key one, are to one another as the numbers 1, 3, 5, 7, 9, 11, 13, &c. And the angle made at the top by the question being  $= 120^\circ$ , the cotangent of its half, to radius 1, must be  $= \sqrt{\frac{1}{3}}$ , consequently the said cotangents are  $\sqrt{\frac{1}{3}}$ ,  $3\sqrt{\frac{1}{3}}$ ,  $5\sqrt{\frac{1}{3}}$ ,  $7\sqrt{\frac{1}{3}}$ , &c. And as there are 5 globes on each side, besides the key-globe; the two inclined planes serve instead of adding on each side a 6th globe, and therefore the cotangent of their inclination to their horizon, must be the 6th in order here,  $= 11\sqrt{\frac{1}{3}} = 6.350850$  the tangent of  $8^\circ 57' 6''$ , the inclination of the planes to the horizon. And since the distance of the centres of every two is given  $= 2$ , and the angles made there, it is easily found by trigonometry that the distance of the centres of the two bottom ones is  $2\sqrt{3} + 2 + 2\sqrt{\frac{7}{3}} + 2\sqrt{\frac{13}{3}} + 2\sqrt{\frac{19}{3}}$ , and the height of the centre of the key or top one, above the line joining the centres of the bottom ones,  $= 1 + \sqrt{3} + 5\sqrt{\frac{1}{3}} + 7\sqrt{\frac{1}{3}} + 9\sqrt{\frac{1}{3}}$  as required.

*The same by Mr. Geo. Barrett, Petworth. (Suppl.)*

*Construction.* As the positions of the three uppermost balls are given, the positions of all the rest may be found as follows. From A, the centre of the uppermost ball draw the indefinite line AY bisecting the line joining the centres of the balls b and b; draw BO parallel to Ab, and BP parallel and  $= AO$ ; join AP, and parallel to it draw AC  $= AB$ , then c will be the centre of the ball next below the ball B. In like manner, draw CQ parallel to AB, and draw CN parallel and  $= BQ$ ; join BN, and parallel to it draw CD  $= CB$ , then D will be the centre of the next lower ball. By proceeding in this manner, the positions of all the remaining balls may be found on both sides of the arch, as also the directions of the semidiameters Rg and fr of the two lowest balls touch-

ing the inclined planes  $gs$  and  $ts$  at  $g$  and  $t$ . Now join  $gt$ , which will be the span of the arch, and consequently  $sa$  its height. Also, if  $wx$  be drawn through  $s$  and parallel to  $gt$ , the  $\angle wsg$  will be the inclination of the planes with the horizon, and which is evidently equal to the  $\angle sgv$ , as well as to the  $\angle gfn$ , which will be known by the following calculation.

*Calculat.* Let  $ci$ ,  $dk$ ,  $el$ ,  $fm$  and  $gn$  be drawn parallel to  $ab$ , each meeting the perps. from the centres  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$  respectively in the points  $i$ ,  $k$ ,  $l$ ,  $m$ , and  $n$ . Produce  $ya$ ,  $pb$ ,  $kc$ ,  $dc$ , and  $cb$  to the respective points  $g$ ,  $h$ ,  $k$ ,  $n$ , and  $m$ . Then (by prop. 66, Emerson's Mechanics, or by Hutton's Principles of Bridges, p. 12 and 15, second edit.)  $B : A ::$

$$\frac{\sin. CBA}{\sin. hbc} : \frac{\sin. BAb}{\sin. gAb} ; \text{ and, since } A \text{ and } B \text{ are } =, \frac{\sin. CBA}{\sin. hbc} = \frac{\sin. BAb}{\sin. gAb},$$

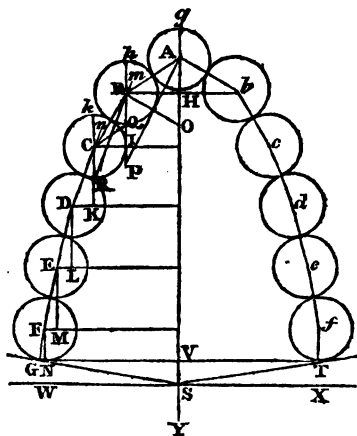
and since  $\sin. CBA$  is  $= \sin. ABm$  or of the equal alternate  $\angle BAp$ , also

$$\sin. hbc = \sin. hbm \text{ or of the equal } \angle BPA ; \text{ conseq. } \frac{\sin. BAp}{\sin. BPA} =$$

$$\frac{\sin. BAb}{\sin. gAb} = \frac{BP}{BA}, \text{ the sides having the same ratio as the sines of their}$$

opposite angles; hence will be found  $BP = AB \times \sin. BAb \div \sin. gAb$ , because  $AB$  and those angles are given; then, as the sides  $BA$  and  $BP$  as well as the included  $\angle ABP$  are given, we may find  $BPA =$  the alternate  $\angle CBP$ . By a like process we shall find that  $CR = BC \times \sin. ABC \div \sin. hba$ , conseq. the  $\angle DCK$  may be found, as well as the angles  $EDL$ ,  $FEM$ , and  $GFN$ . Or, in the present case, in which the  $\angle BAb = 120^\circ$  by the quest. the process will be easier and simpler, because the triangle  $ABP$  will be identical with the given triangle  $ABb$ , being mutually equal and similar in all respects. Hence the

$\angle BAH = 60^\circ 0' 0''$	$BH = 1.7320508$	$AH = 1.0000000$
$\angle CBI = 30 \ 0 \ 0$	$CI = 1.0000000$	$BI = 1.7320508$
$\angle DCK = 19 \ 6 \ 23.7$	$DK = .6546536$	$CK = 1.8898224$
$\angle EDL = 13 \ 52 \ 34.6$	$EL = .4796526$	$DL = 1.9419406$
$\angle FEM = 10 \ 51 \ 55.3$	$FM = .3770032$	$EM = 1.9641457$
$\angle GFN = 8 \ 55 \ 12.8$	$GN = .1550594$	$FN = .9879051$
the inclin. of the	$GV = 4.3984196$	$VS = .6903662$
planes with the	$VT = 4.3984196$	
horizon. }	$GT = 8.7968392$	$As. = 10.2062308$
	span of the arch. }	height of the arch.



PRIZE QUESTION, *by Terricola.*

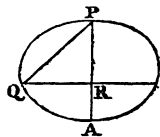
A quantity of matter being given, it is proposed to determine the figure of a solid of rotation made up of it, which shall have the greatest possible attraction on a point at its surface.

*Answered by Amicus.*

If  $x$  = the abscissa,  $y$  = the corresponding semiordinate of the generating curve, and  $p = 3.14159$ , &c. Then, by Simpson's Flux. art. 375, the fluent of  $py^2x$  must be, by the quest. a given quantity, and the attraction = the fluent of  $2px(1 - x(x^2 + y^2)^{\frac{1}{2}})$  a maximum. If  $x^2 + y^2 = w^2$ , then, by the general method for isoperimetrical problems, as  $x$  is the quantity, and its flux. to be made invariable (by Simp. Math. Tracts, p. 101)  $yy' = w'w$ , and  $2pyx' + 2peyx'w^{-3} = 0$ , consequently,  $w^3 = -ex$ ,  $-w = \sqrt[3]{ex} = -\sqrt[3]{(x^2 + y^2)}$ ,  $\sqrt[3]{e^2x^3} = x^2 + y^2$ , or  $y^2 = \sqrt[3]{(e^2x^3)} - x^2$ , is the required equation of the curve. Or, making  $e = c^3$ ,  $y^2 = \sqrt[3]{(c^3x^3)} - x^2$ , the fluent of  $py^2x$  or of  $pc^{\frac{4}{3}}x^{\frac{5}{3}} - px^3$  is  $\frac{3}{5}pc^{\frac{4}{3}}x^{\frac{8}{3}} - \frac{1}{4}px^4$  = the solidity, which when  $x = c$  becomes  $(\frac{3}{5} - \frac{1}{4})pc^{\frac{4}{3}} = \frac{1}{20}pc^{\frac{4}{3}}$  or  $\frac{8}{9}$  of the sphere of the same axis  $c$ . From the equation of the curve,  $y = 0$  both when  $x = 0$  and  $x = c$ , and  $c$  is the greatest abscissa or axis. And  $y$  is a max. when  $c^{\frac{4}{3}}x^{\frac{5}{3}} - x^4$  is so,  $\frac{4}{3}c^{\frac{4}{3}}x^{\frac{2}{3}} - 4x^3 = 2x$ ,  $\frac{4}{3}c^{\frac{4}{3}} = x^{\frac{8}{3}}$ ,  $x = c(\frac{1}{2})^{\frac{3}{8}} = c \times .4532063$ ; and thus both the form and content of the solid becomes known.

*The same by Mr. John Ryley, Leeds.*

Let PA be the axe of the solid, made up of the given matter  $m$ , and let QR be perp. to PA, and join PQ. Then the force of a particle at Q acting on P is as  $PQ^{-2}$ , and  $QR : PR :: PQ^{-2} : PR \div PQ^3$  = the force of Q in the direction PA, which must be a constant quantity, or equal to that at A, when the whole attraction is the greatest, that is  $PR \div PQ^3 = PA^{-2}$ , or  $PR \times PA^3 = PQ^3$ .



Now let  $PR = x$ ,  $QR = y$ ,  $PA = a$ ; then  $a^3x = (x^2 + y^2)^{\frac{3}{2}}$  or  $y^2 = a^{\frac{4}{3}}x^{\frac{2}{3}} - x^2$ , which expresses the nature of the curve PQA. To find the content of the solid, put  $c = 3.1416$ , then its flux. =  $cy^2x' = ca^{\frac{4}{3}}x^{\frac{2}{3}}x' - cx^2x'$ , whose flu. is  $\frac{3}{5}ca^{\frac{4}{3}}x^{\frac{5}{3}} - \frac{1}{4}cx^4$ , which, when  $x = a$ , becomes  $\frac{1}{20}ca^{\frac{4}{3}} = m$  the given quantity of matter; or  $\frac{8}{9}$  of the sphere of the same axis  $a$ , therefore  $a = \sqrt[3]{(15m \div 4c)}$ .

END OF THE THIRD VOLUME.





